9/18/07

Let us discuss Cartesian Tensors:

We have position vectors, velocity vectors and force vectors, which each have 3 components. We also have procedures for direction, magnitude, adding vectors, and coefficient multiplying. For example, we can write

$$m\frac{d\boldsymbol{V}}{dt} = \boldsymbol{F} \text{ (Newton's Law)}$$

This equation is independent of coordinate axes. Looking at other quantities, $x_i x_j$ where $i,\, j = 1, 2, 3$ and we have 9 different components. These nine components do not constitute a vector in the conventional sense. Instead, this more aptly can be viewed as a matrix. Also, we could take $x_i x_j x_k$ which has 27 components! Both are examples of *tensors*. $x_i x_j$ is rank two and $x_i x_j x_k$ is a rank three tensor (a vector is a rank one tensor and the vector dot product is a rank zero tensor). So a tensor generalizes the vector concept.

Now let us focus on changing coordinate axes. In the original axes, $x_i = \boldsymbol{x} \cdot e^{(i)}$ where $\boldsymbol{x} = (x,\, y,\, z)$. In the new axes, we have that $\begin{cases} x_i' = x \cdot e^{(i)'} \\ x = \sum_{i=1}^{3} x_i' e^{(i)'} \end{cases}$. How do we interchange between the two? Call $Q_{ij} = e^{(i)'} \cdot e^{(j)}$ for $i\,, j = 1, 2, 3$. Then, $e^{(i)'} = \sum_{j=1}^{3} Q_{ij}\, e^{(j)}$ (think of treating $e^{(i)\prime}$ as $\boldsymbol{x}$). So

$$x_i' = x \cdot e^{(i)'} = x \cdot \sum_{j=1}^{3} Q_{ij}\, e^{(j)} = \sum_{j=1}^{3} Q_{ij}\, x \cdot e^{(j)} = \sum_{j=1}^{3} Q_{ij}\, x_j.$$

So we define tensor of a rank of rank 1 by its transformation rule,

$$x_i' = \sum_{j=1}^{3} Q_{ij}\, x_j,$$

for the components. What does the matrix Q look like? $e^{(1)'} = Q_{11}e^{(1)} + Q_{12}e^{(2)} + Q_{13}e^{(3)}$. Using $e^{(1)}$ as the basis then $e^{(1)'} = Q_{11}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + Q_{12}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + Q_{13}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Then,

$$Q = \begin{pmatrix} e^{(1)'}e^{(1)} & e^{(1)\prime}e^{(2)} & e^{(1)\prime}e^{(3)} \\ e^{(2)'} \rightarrow & \cdots & \\ e^{(3)'} \rightarrow & \cdots & \end{pmatrix}$$

We have orthogonal, unit vectors. So for instance, If we take $e^{(i)'} \cdot e^{(j)'} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. So we get that

$$e^{(i)'} \cdot e^{(j)'} = \left(\sum_{k=1}^{3} Q_{ik}e^{(k)}\right) \cdot \left(\sum_{k=1}^{3} Q_{jm}e^{(m)}\right)$$

where the free indices are k and m, where we use a unique *dummy* index for each summation. Then ,

$$\begin{aligned} \left(\sum Q_{ik}e^{(k)}\right) \cdot \left(\sum Q_{jm}e^{(m)}\right) &= \sum_k \sum_m Q_{ik}Q_{jm}e^{(k)'} \cdot e^{(m)'} \\ &= \sum_{k=1}^{3} Q_{ik}Q_{jk} \\ &= (QQ^T)_{ij} \end{aligned}$$

since $Q_{jk}=(Q^T)_{kj}$. Since $e^{(i)'}\cdot e^{(j)'}$ gives the identity matrix, then we know that $QQ^T=I$ or $Q^TQ=I$. Hence, Q is an orthogonal matrix since $Q^{-1}=Q^T$ (this means that $Q^TQ=I$ so that $(Q^TQ)^T=I^T\Rightarrow QQ^T=I$).

Other types of matrices gives us orthognal matrices: Rotations and Reflections. For example, we can have a rotation θ about $e^{(3)}$ so that $e^{(3)}=e^{(3)'}$, $e^{(1)'}=\cos\theta e^{(1)}+\sin\theta e^{(2)}$, and $e^{(2)'}=-\sin\theta e^{(1)}+\cos\theta e^{(2)}$. Then

$$Q=\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Remember that the physical vector x is NOT changing. We are just changing the basis. We have that the $\det Q=+1$ (this is a property of a rotation matrix).

Reflections:

Reflection over the xy plane gives us $Q=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$ where $\det Q=-1$. Basically all Q's are products of successive rotations or reflections. Recall that $QQ^T=I$ and $\det(QQ^T)=\det Q\det Q^T=(\det Q)^2=1$. Thus, the $\det Q=\pm1$.

For a scalar: $\phi'=\phi$ (tensor or rank 0)

For a vector, we have $x_i'=\sum_{j=1}^3 Q_{ij}x_j$ (tensor of rank 1)

Prototype rank-2 tensor: $x_ix_j=\sum_{k=1}^3\sum_{m=1}^3 Q_{ik}x_kQ_{jm}x_m$ (tensor product of x).

So we adopt T_{ij} as a tensor of rank 2. Then,

$$T_{ij}=\sum_{k=1}^3\sum_{m=1}^3 Q_{ik}Q_{jm}T_{km}.$$

Then, a tensor of rank 3 would be

$$T'_{ijk}=\sum_{p=1}^3\sum_{q=1}^3\sum_{r=1}^3 Q_{ip}Q_{jq}Q_{kr}T_{pqr}$$

and so on... In a rank-3 tensor we have 3 free indices and thus 3 dummy indices.

(say we want $x_j=x\cdot e^{(j)}=\sum_i x_i'e^{(i)'}\cdot e^{(j)}=\sum_i x_i'(e^{(i)'}\cdot e^{(j)})=\sum_i x_i'Q_{ij}=\sum_i (Q^T)_{ji}x_i'$ where $(Q^T)_{ij}\mathrel{\hat{=}}Q_{ji}$)

Summation convention: If an index is *repeated once* in a simple product, we imply a summation over that index:

$x\cdot y=x_iy_i$.

$x_iy_iz_i$ is not defined (since the summation index is repeated more than once in a product term!).

If we want $x_iy_i=(x_1y_1,\ldots,)$ we would write: x_iy_i (no sum).

Note that free indices must match in an equation. For instance,

$$x_i=b_j$$

has no meaning because the free indices do not match. But

$$x_i=b_i+a_i$$

is okay (here there is no product). So we can have

$$x_i = T_{ij} b_j + S_{ik} a_k$$

is okay since the free indices match. Then we sum on j and sum on k. We could also write this as $x_i = T_{ij} b_j + S_{ij} a_j$ where the dummy index is used twice.

Let us look at Kronecker's Delta function $\delta_{ij} = \left\{ \begin{smallmatrix} 1 & i=j \\ 0 & i \neq j \end{smallmatrix} \right.$. One can think of this as the components of the identity matrix. Consider $x_i \delta_{ij} y_j = \sum_i \sum_j x_i \delta_{ij} y_j$ which is a rank zero tensor since $x_i \delta_{ij} y_j = x_i y_i = x \cdot y$. This is the usual inner product where $\delta_{ij} y_j = y_i$. We also have that $Q_{ij} Q_{kj} = \delta_{ik}$ since $Q Q^T = I$.

Now let us consider the permutation symbol, in 3D, $\varepsilon_{ijk} = \begin{cases} 0 & \text{if any 2 indiced are equal} \\ 1 & (i,j,k) = (1,2,3), (3,1,2), (2,3,1) \\ -1 & (i,j,k) = (2,1,3), (1,3,2), (3,2,1) \end{cases}$.

This is also known as the Levi-Cita Tensor Density. If we take a vector product: $a = b \times c$ then we can rewrite this as $a_i = \varepsilon_{ijk} b_j c_k$. We get $a_1 = \varepsilon_{1jk} b_j c_k = 1 \cdot b_2 c_3 + (-1) \cdot b_3 c_2$ and so on where we get the usual cross product. We can also use this notation to fine the determinant of Q.

$$\det Q = \varepsilon_{pqr} Q_{ip} Q_{2q} Q_{3r}$$

for a 3×3 matrix.

9/20/07

(Room 104, 37 manning at 4pm for Homework discussion).

Cartesian Tensors (Kundu Ch. 2, Riley et al. Ch. 26):

We discussed that tensors generalize a vector and we use this as tranformation between axes.

$$e^{(i)} = Q_{ij} e^{(j)}$$

where the summation over j is implied (summation notation). For a vector x, we write

$$x_i' = Q_{ij} x_j.$$

Recall that a vector is a rank 1 tensor. Rank-2 tensors were defined as

$$T_{ij}' = Q_{ip} Q_{jq} T_{pq},$$

and so on for rank-r tensors.

Let us look at some of the standard physical vectors: u, x, f. Suppose we have a velocity field $u_i(x,t)$ which represents the flow. We want to know how u varies locally. Consider

$$T_{ij} = \frac{\partial u_i}{\partial x_j}.$$

Is this a tensor? We have to verify the transformation properties.

$$\begin{aligned} T_{ij}' &= \frac{\partial u_i'}{\partial x_j}. \\ &= \frac{\partial}{\partial x_j} (Q_{ip} u_p) \\ &= Q_{ip} \frac{\partial x_q}{\partial x_j'} \frac{\partial u_p}{\partial x_q} \end{aligned}$$

using that $\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_q}\frac{\partial x_q}{\partial x'_j}$ where this is a sum over q. Since $x'_i = Q_{ij}x_j$ we have that

$$\begin{aligned}(Q^T)_{ki}x'_i &= (Q^T)_{ki}Q_{ij}x_j \,(\text{sum on } i, j)\\ Q_{ik}x'_i &= \delta_{kj}x_j \;\; (Q^TQ = I)\end{aligned}$$

Then, we have

$$\begin{aligned}\frac{\partial x_k}{\partial x'_j} &= \frac{\partial}{\partial x'_j}(Q_{ik}x'_i)\\ &= Q_{ik}\frac{\partial x'_i}{\partial x'_j}\\ &= Q_{ik}\delta_{ij}\\ &= Q_{jk}.\end{aligned}$$

Then,

$$\begin{aligned}T'_{ij} &= Q_{ip}\left(\frac{\partial x_q}{\partial x'_j}\right)\frac{\partial u_p}{\partial x_q}\\ &= Q_{ip}Q_{jq}\frac{\partial u_p}{\partial x_q}\\ &= Q_{ip}Q_{jq}T_{pq}\end{aligned}$$

so that T_{ij} is indeed a tensor of rank 2. Now notice that we have

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

where $S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is symmetric and $R_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ is anti-symmetric. We call S_{ij} the rate of strain tensor and R_{ij} the rotation tensor. Similarly, we can show that

$$\frac{\partial^2 u}{\partial x_j \partial x_k}$$

is a rank 3 tensor.

We can also talk about hte contraction of tensors. For example, take a rank 3 tensor T_{ijk}. Then T_{iik} is a contraction over i, s.t. $T_{iik} = A_k$ which is a rank 1 tensor. Let us check that this is indeed true. We have that

$$\begin{aligned}T'_{ijk} &= Q_{ip}Q_{jq}Q_{kr}T_{pqr}\\ &\Rightarrow\\ T'_{iik} &= Q_{ip}Q_{iq}Q_{kr}T_{pqr}\\ &= (Q^T)_{pi}Q_{iq}Q_{kr}T_{pqr}\\ &= \delta_{pq}Q_{kr}T_{pqr}\\ &= Q_{kr}T_{ppr}\end{aligned}$$

so this is a rank 1 tensor.

We also have what is known as the *Tensor Detection Theorem*: If $T_{ij\ldots k}a_i = S_{j\ldots k}$ and a_i is an arbitrary vector and $S_{j\ldots k}$ is a tensor, then so is $T_{ij\ldots k}$.

Recall $\delta_{ij}$and ϵ_{ijk} (δ_{ij} is in fact a tensor). Set $T_{ij} = \delta_{ij}$ in the original axes. Let us assume the tensor tranformation rule. Then

$$\begin{aligned} T'_{ij} &= Q_{ip} Q_{jq} \delta_{pq} \\ &= Q_{ip} Q_{jp} \\ &= \delta_{ij}. \end{aligned}$$

This means that every time we write the components of this delta tensor, it will always have the same values! In summary, δ_{ij} is a rank 2 tensor and the component values are the same in all coordinate axes. We call this an *isotropic tensor* of rank 2. What is the general isotropic tensor of rank 2 ($T'_{ij} = T_{ij}$)? We know that

$$T'_{ij} = Q_{ip} Q_{jq} T_{pq}.$$

Let us look at some simple tranformations:

$$Q_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is a rotation 180° (det = 1). Also, consider

$$Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is a 90° rotation. Let us use Q_1 as the Q for T'_{ij}. Then,

$$\begin{aligned} T'_{11} &= (-1)^2 T_{11} \\ &= T_{11}. \end{aligned}$$

So this doesn't tell us much.

$$\begin{aligned} T'_{12} &= (-1)(-1) T_{12} \\ &= T_{12}. \end{aligned}$$

Finally, let us try

$$\begin{aligned} T'_{13} &= (-1)(+1) T_{13} \\ &= -T_{13} \end{aligned}$$

but this should equal T_{13} since we assume T_{ij} is isotropic. This only happens is if $T_{13} = 0$! We will also see that $T_{23} = 0$, $T_{31} = 0$ and $T_{32} = 0$. So far we have

$$T_{ij} = \begin{pmatrix} T_{11} & & 0 \\ & T_{22} & 0 \\ 0 & 0 & T_{33} \end{pmatrix}.$$

If we take a 180° rotation about the $e^{(2)}$ axis, we also get that $T_{12} = T_{21} = 0$ and $T_{23} = T_{32} = 0$. Then, we get that

$$T_{ij} = \begin{pmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & T_{33} \end{pmatrix}$$

so that the tensor MUST be diagonal. Finally, let us take 90° degree rotations to obtain information about the diagonal entries. Consider

$$
\begin{aligned}
T'_{11} &= Q_{1p}Q_{1q}T_{pq} \\
&= (1)(1)T_{22}
\end{aligned}
$$

so that $T_{11} = T_{22}$. Now we take another 90° rotation about $e^{(2)}$ axes which tells us that $T_{11} = T_{33}$. Therefore, we have that $T_{ij} = c\delta_{ij}$. This is called cubic symmetry since we only need symmetry in ninety degree rotations. Therefore, cubic summetry is equivalent to isotropy.

We can say that there are no isotropic vectors (rank 1 tensors). A scalar is automatically istropic. A rank 3 isotropic tensor is ϵ_{ijk}. Moreover, ϵ_{ijk} is an istropic *pseudo-tensor*. Under rotations all tensors are *equal*. Under a reflection Q a proper tensor will transform as $T'_{ij\ldots k} = Q_{ip}Q_{jq}\ldots Q_{kr}T_{pq\ldots r}$. A pseudo tensor will transform as $R'_{ij\ldots k} = (\det\ Q)Q_{ip}Q_{jq}\ldots Q_{kr}R_{pq\ldots r}$, where $\det Q = 1$ for a rotation and $\det Q = -1$ for a reflection. Consider ϵ_{ijk}: We have

$$
\epsilon_{ijk}A_{1i}A_{2j}A_{3k} = \det A.
$$

We claim that

$$
\epsilon_{ijk}\det A = \epsilon_{pqr}A_{ip}A_{jq}A_{kr}.
$$

This is a general result in linear algebra, based on teh symmetry rules of the determinants. for example, $\det A = \epsilon_{123} \det A = \epsilon_{pqr}A_{1p}A_{2q}A_{3r}$. Now we take the tranformation matrix Q. If we define $R_{ijk} = \epsilon_{ijk}$ in the original axes and apply the transformation rule, we get $R'_{ijk} = Q_{ip}Q_{jq}Q_{kr}\epsilon_{pqr} = \epsilon_{pqr}\,Q_{ip}Q_{jq}Q_{kr} = \epsilon_{ijk}(\det Q)$. So as $\det Q = \pm 1$ then

$$
R'_{ijk} = (\det Q)\,R_{ijk}
$$

Thus, we have verified that $R_{ijk} \equiv \epsilon_{ijk}$ is an isotropic, pseudo-tensor of rank 3 meaning that under rotations, ε_{ijk} will remain unchanged, but under reflections ε_{ijk} will change sign. Here are some examples:

$$
\begin{aligned}
c &= a \times b \\
&\Rightarrow \\
c_i &= \epsilon_{ijk}a_jb_k \text{ (sum over } j \text{ and } k).
\end{aligned}
$$

If a, b are proper vectors, then c is a pseudo-vector. When we define a vector cross product, we use the so called right hand rule. If we reflect the system, we switch from a right handed system to a left handed system (as in a reflection). Thus,

$$
\begin{aligned}
c'_i &= \epsilon_{ijk}a'_jb'_k \\
&= \epsilon_{ijk}\,Q_{jp}Q_{kr}\,a_p\,b_r.
\end{aligned}
$$

Then,

$$
Q_{im}c'_i = (\epsilon_{ijk}\,Q_{im}Q_{jp}Q_{kr})a_pb_r
$$

and using $\epsilon_{ijk}\det A = \epsilon_{pqr}A_{ip}A_{jq}A_{kr}$ or equivalently $\epsilon_{pqr}\det A = \epsilon_{ijk}A_{ip}A_{jq}A_{kr}$ we get

$$
\begin{aligned}
Q_{im}c'_i &= (\det Q)\epsilon_{mpr}\,a_pb_r \\
&= (\det Q)\,c_m
\end{aligned}
$$

and again we multiply by Q_{nm} to get

$$\begin{aligned}
Q_{nm}Q_{im}c'_i &= (\det Q)Q_{nm}c_m \\
&\downarrow \\
\delta_{ni}c'_i &= (\det Q)Q_{nm}c_m \\
&\downarrow \\
c'_n &= (\det Q)Q_{nm}c_m.
\end{aligned}$$

What happens when we have $\epsilon_{ijk}\epsilon_{pqr}$. We get back a proper vector. For example,

$$a \times (b \times c) = (a \cdot b)c - (a \cdot c)b.$$

If a, b, and c are proper, then we get a proper vector. Let us show that we obtain a proper vector. Let us call $f_k = (b \times c)_k = \epsilon_{kqr}b_qc_r$ and then $e_i = \epsilon_{ijk}a_jf_k$. Then,

$$\begin{aligned}
e'_i &= \epsilon_{ijk}a'_jf'_k \\
&= \epsilon_{ijk}a'_j(\epsilon_{kqr}b'_qc'_r) \\
&= \epsilon_{ijk}(Q_{jp}a_p)(\det(Q)Q_{km}f_m) \\
&= \det(Q)\epsilon_{ijk}Q_{jp}Q_{km}a_pf_m.
\end{aligned}$$

Multiplying both sides by Q_{ir} we get

$$\begin{aligned}
Q_{ir}e'_i &= \det(Q)\{\epsilon_{ijk}Q_{ir}Q_{jp}Q_{km}\}a_pf_m \\
&= \det(Q)\{\det(Q)\epsilon_{rpm}\}a_pf_m \\
&= \det(Q)^2\epsilon_{rpm}a_pf_m \\
&= \det(Q)^2e_r
\end{aligned}$$

Now, multiplying both sides by Q_{nr} and rewriting Q_{ir} as $(Q^T)_{ri}$ we get

$$\begin{aligned}
Q_{nr}(Q^T)_{ri}e'_i &= \det(Q)^2Q_{nr}e_r \\
&\downarrow \\
\delta_{ni}e'_i &= \det(Q)^2Q_{nr}e_r \\
&\downarrow \\
e'_n &= \det(Q)^2Q_{nr}e_r.
\end{aligned}$$

Note that $\det(Q)^2 \equiv 1$ so that

$$e'_n = Q_{nr}e_r,$$

and $e = a \times (b \times c)$ is thus a proper vector.