# 11/13/2007

Recall the flow past a sphere. In spherical polar coordinates we have $(r,\theta,\phi)$ and we have axis-symmetric flow so that $\frac{\partial}{\partial\phi}=0$ and we have $u_r$, $u_\theta$ only, where the flow is moving along the $z$ axis. We expect that

$$u_i = \Phi_{ij}(\vec{x})u_j^{(0)}$$

and

$$p = \frac{1}{a}\mu\Psi(\vec{x})u_j^{(0)}$$

where the force on the sphere is given by

$$F_i = \alpha\mu a u_i^{(0)}$$

for some $\alpha$. We will use a stream function $\psi(r,\theta)$ where

$$u = \frac{1}{r^2\sin\theta}\begin{vmatrix} e^{(r)} & re^{(\theta)} & r\sin\theta e^{(\phi)} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & 0 \\ 0 & 0 & \psi \end{vmatrix}$$

since $u=\nabla\times(\frac{\psi}{r\sin\theta}e^{(\phi)})$, where the $\frac{1}{r\sin\theta}$ is to make sure that $\psi$ is constant along the streamlines. Then, we get that

$$\begin{aligned} u_r &= \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \\ u_\theta &= -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \end{aligned}$$

What are the boundary conditions? We have no normal flow at $r=a$ so that $u_r|_{r=a}=0$ and no slip so that $u_\theta|_{r=a}=0$. Hence,

$$\left.\frac{\partial\psi}{\partial r},\frac{\partial\psi}{\partial\theta}\right|_{r=a} = 0$$

(for all $\theta$ on $r=a$). This means that $\psi$ is not varying on the boundary of the sphere so that $\psi=$ constant on $r=a$ for all $\theta$. We also need the far field condition as $r\to\infty$. In the far field, we have that

$$u = u_0e^{(z)}.$$

Then, if we project $u$ in the far field onto $e^{(r)}$ and $e^{(\theta)}$ we get

$$\begin{aligned} u_r &\sim u_0\cos\theta \\ u_\theta &\sim -u_0\sin\theta. \end{aligned}$$

Now, using our equations for $\boldsymbol{u}$ in terms of $\psi$ we get that for $u_\theta$

$$\begin{aligned} -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} &\sim -u_0\sin\theta \\ \Rightarrow\frac{\partial\psi}{\partial r} &\sim u_0r\sin^2\theta \\ \psi &\sim \frac{1}{2}u_0r^2\sin^2\theta. \end{aligned}$$

This is also the stream function for uniform form in spherical polars.

This suggests looking for a solution of $\psi$ of the form $f(r)\sin^2\theta$. But before we try this solutoin, we check to see if this form satisfies the far field condition for $u_r$.

$$\begin{aligned}\frac{\partial\psi}{\partial\theta} &= u_0 r^2 \sin\theta\\ \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} &= u_0\cos\theta.\end{aligned}$$

So we expect that $\psi \sim \frac{1}{2}u_0 r^2 \sin^2\theta$ in the far field. Remember that for Stokes flows we have

$$\nabla^2\omega = 0$$

since we have that $0 = -\nabla p + \mu\nabla^2 u$ and taking the curl gives us the above relationship. How do we write Laplace's equation in spherical polar coordinates? We have that

$$\begin{aligned}u &= \nabla\times\left(\frac{\psi}{r\sin\theta}e^{(\phi)}\right)\\ \omega &= \nabla\times u\end{aligned}$$

which gives us

$$\omega = \frac{1}{r^2\sin\theta}\begin{vmatrix} e^{(r)} & re^{(\theta)} & r\sin\theta e^{(\phi)}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & 0\\ u_r & ru_\theta & 0\end{vmatrix}.$$

The only nonzero vorticity is in the $e^{(\phi)}$ direction (i.e. $\boldsymbol{\omega} = \omega_\phi e^{(\phi)}$). Hence,

$$\begin{aligned}\omega_\phi &= \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right]\\ &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(-\frac{1}{\sin\theta}\frac{\partial\psi}{\partial r}\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right]\\ &= -\frac{1}{r\sin\theta}\left[\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right].\end{aligned}$$

That is,

$$\omega_\phi = -\frac{1}{r\sin\theta}\left[\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right].$$

Now we have similarity between $\omega_\phi$ and $u$. Define $D^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)$. Then,

$$\omega = -\frac{1}{r\sin\theta}D^2\psi e^{(\phi)}.$$

So we have that

$$\begin{aligned}\omega &= \nabla\times\left(\nabla\times\left\{\frac{\psi}{r\sin\theta}e^{(\phi)}\right\}\right)\\ &= -\frac{1}{r\sin\theta}D^2\psi e^{(\phi)}.\end{aligned}$$

We still need to find $\nabla^2\omega$. Let us take

$$\nabla\times(\nabla\times\omega) = \nabla(\nabla\cdot\omega) - \nabla^2\omega$$

and note that $\nabla\cdot\omega = \nabla\cdot(\nabla\times u) \equiv 0$ so that

$$\nabla\times(\nabla\times\omega) = -\nabla^2\omega.$$

So we want to show that $\nabla\times(\nabla\times\omega) = 0$. $\omega$ is only in the $e^{(\phi)}$ direction. If we replace $\psi$ by the term $(-D^2\psi)$ we get

$$\begin{aligned}\nabla\times(\nabla\times\omega) &= -\nabla\times\left(\nabla\times\frac{D^2\psi}{r\sin\theta}e^{(\phi)}\right)\\ &= \frac{1}{r\sin\theta}D^2(D^2\psi)e^{(\phi)}.\end{aligned}$$

Hence, we solve

$$D^2(D^2\psi) = 0.$$

This is now a scalar PDE. First, how do we solve $D^2 g = 0$? Let us consider $g(r,\theta) = G(r)\times f(\theta)$. We know that the sphere is isotropic so it not angularly dependent. However, in the far field $\psi$ is angularly dependent. So let us try $\psi = f(r)\sin^2\theta$ since this is the form of $\psi$ in the far field. Then,

$$\begin{aligned}D^2\psi &= \frac{d^2 f}{dr^2}\sin^2\theta + \frac{\sin\theta}{r^2}f(r)\frac{d}{d\theta}\left(\frac{1}{\sin\theta}2\sin\theta\cos\theta\right)\\ &= \frac{d^2 f}{dr^2}\sin^2\theta - \frac{2\sin^2\theta}{r^2}f(r)\\ &= \left\{\frac{d^2}{dr^2} - \frac{2}{r^2}\right\}f(r)\sin^2\theta.\end{aligned}$$

So we set $F(r) = \left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)f(r)$ so that

$$\begin{aligned}D^2(D^2\psi) &= \left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)F(r)\sin\theta\\ &= 0.\end{aligned}$$

This is a second order ODE for $F(r)$. That is, we need to solve

$$F''(r) - \frac{2}{r^2}F(r) = 0.$$

Note that this is a homogeneous equation with the same dimensionality for $F''$ and $F/r^2$. Hence, $F\propto r^\alpha$. Then, $F''\propto\alpha^2 r^{\alpha-2}$ and

$$\begin{aligned}\alpha(\alpha-1)r^{\alpha-2} - 2r^{\alpha-2} &= 0\\ \alpha^2 - \alpha - 2 &= 0\\ (\alpha-2)(\alpha+1) &= 0.\end{aligned}$$

We get that $\alpha = 2, -1$. Then,

$$F(r) \propto r^2, r^{-1}.$$

Next, we find $\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)f = Pr^2 + Qr^{-1}$. We can try $f\propto r^\alpha$. Again, we get that

$$(\alpha^2 - \alpha - 2)r^{\alpha-2} \propto r^2, r^{-1}.$$

If we set RHS $=0$ then we get $\alpha=2,-1$. If we want an $r^2$ term, then $\alpha=4$ and if we want $r^{-1}$ we need $\alpha=1$. This is the method of undetermined coefficients (we guess at the solution from the RHS). We look for the solution of the form

$$f(r) = Ar^4+Br^2+Cr+Dr^{-1}.$$

We get that for $Ar^4$ we have $A(12r^2-2r^2)=Pr^2$ so that $10A=P$. Also we have that $Cr$ becomes $C(-2r^{-1})=Qr^{-1}$ so that $-2C=Q$. So we have that

$$f(r) = Ar^4+Br^2+Cr+Dr^{-1},$$

and $\psi = f(r)\,\sin^2\theta$ and $\psi \sim \frac{1}{2}r^2\sin^2\theta u_0$ as $r\to\infty$. As $r\to\infty$ $A$ must be identically zero. Matching the $r^2$ term, we also get that $B=\frac{1}{2}u_0$. Then,

$$f(r) = \frac{1}{2}u_0r^2+Cr+\frac{D}{r}.$$

For $r=a$, remember that $\psi=$ constant for all $\theta$ and $\frac{\partial\psi}{\partial r}\Big|_{r=a}=0$. Hence, $f(a)=0$ and $\frac{df}{dr}=0$ on $r=a$ otherwise we have a $\sin^2$ variation for $\psi$ if we dont let $f(a)=0$. Then

$$\begin{aligned} 0 &= \frac{1}{2}u_0a^2+Ca+\frac{D}{a} \\ \Rightarrow 0 &= \frac{1}{2}u_0a+C+\frac{D}{a^2}. \end{aligned}$$

After some algebra, we get that $C=-\frac{3}{4}au_0$ and $D=\frac{1}{4}u_0a^3$. Then,

$$\psi(r,\theta) = \frac{1}{2}u_0r^2\sin^2\theta\left\{1-\frac{3}{2}\frac{a}{r}+\frac{1}{2}\frac{a^3}{r^3}\right\}.$$

Note that $Pr^2+Qr^{-1}$ come from the vorticity so that $A$, $C$ are linked with the vorticity. Whereas $B$ and $D$ have no vorticity (irrotational or potential flows) (but $C$ has vorticity).

What is the vorticity? We showed that

$$\begin{aligned} \omega_\phi &= -\frac{1}{r\sin\theta}D^2\psi, \\ \psi &= f(r)\sin^2\theta \\ \Rightarrow \omega_\phi &= -\frac{1}{r\sin\theta}\left\{\frac{d^2}{dr^2}-\frac{2}{r^2}\right\}f(r)\sin^2\theta \\ &= -\frac{1}{r}\left\{\frac{d^2}{dr^2}-\frac{2}{r^2}\right\}f(r)\sin\theta \\ &= \frac{2C\sin\theta}{r^2}. \end{aligned}$$

This peaks at $\theta=\frac{\pi}{2}$ so that $\omega_\phi=\frac{2C}{r^2}=-\frac{3}{2}u_0\frac{a}{r^2}$. Hence, the maximum vorticity is at $\theta=\pi/2, r=a$ and we get $\|\omega_\phi\|_\infty=-\frac{3}{2}u_0\frac{1}{a}$.

As for the pressure term, we have that

$$\begin{aligned} \nabla p &= \mu\nabla^2u \\ &= -\mu(\nabla\times\omega) \end{aligned}$$

since $\nabla\cdot u=0$ because $u=\nabla\times\psi$ and the flow is incompressible. That is, we are using

$$\nabla\times(\nabla\times u) = \nabla(\nabla\cdot u)-\nabla^2u.$$

So the pressure is determined by the $C$ term. One can find

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = \frac{\mu}{r\sin\theta}\frac{\partial}{\partial r}(r\sin\theta\omega_\phi)$$

and so

$$\begin{aligned} p &= p_0 + 2\frac{\mu C}{r^2}\cos\theta \\ &= p_0 - \frac{3}{2}\mu u_0 \frac{a}{r^2}\cos\theta, \end{aligned}$$

where $p_0$ is the ambient pressure. The $r$ dependence on $p$ vanishes because $\omega$ is only in the $\hat{\phi}$ direction. So on the $\theta = \pi$ axis, we have a higher pressure as compared with $\theta = 0$. So we have a net pressure force. Let us look at the pressure term a little more closely.

$$p - p_0 = -\frac{3}{2}\mu u_0 \frac{a}{r^2}\cos\theta.$$

Recall that $u^{(0)} = u_0 e^{(z)}$. If we take $x \cdot u^{(0)} = u_0 r\cos\theta$. Then, the above equation for the pressure becomes

$$p - p_0 = -\frac{3}{2}\mu a \frac{x \cdot u^{(0)}}{r^3}.$$

Now note that $\frac{x \cdot u^{(0)}}{r^3}$ is a particular solution to Laplace's equation. Note that we have $\nabla^2 p = 0$ and we have a potential dipole term (recall that $\nabla p = -\mu(\nabla \times \omega)$ and if take the divergence of both sides we get $\nabla^2 p = 0$).

The pressure force term is

$$\begin{aligned} F_z^{(p)} &= -\oint_S p n \cdot e^{(z)} dS \\ &= \frac{3}{2}\mu u_0 \oint_S \frac{a}{r^2}\cos^2\theta dS \\ &= \frac{3}{2}\mu u_0 \frac{1}{a} 2\pi a^2 \int_0^\pi \cos^2\theta\sin\theta d\theta \\ &= \frac{3}{2}\mu u_0 \frac{1}{a} 2\pi a^2 \left(-\frac{1}{3}\cos^3\theta\right) \end{aligned}$$

By symmetry we know that the force must be in the streamwise direction. Thus,

$$F_z^{(p)} = 2\mu\pi a u_0.$$

We also have that

$$F_i = \oint_S \sigma_{ij} n_j dS$$

and $n = e^{(r)}$ so we need $\sigma_{rr} e^{(r)} + \sigma_{\theta r} e^{(\theta)}$. Then,

$$\begin{aligned} \sigma_{rr} &= -p + 2\mu\frac{\partial u_r}{\partial r} \\ \sigma_{\theta r} &= \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right). \end{aligned}$$

Finally, we get that

$$F_z^{(v)} = 4\pi\mu a u_0.$$

Hence, the total force is

$$F = 6\pi a\mu u_0 e^{(z)}.$$

The $B$ and $D$ terms do not contribute to the force. Only the $C$ term matters. The $B$ term corresponds to uniform flow so we did not expect that to contribute a force. As for the $D$ term, recall that $\psi \sim D/r$ so that $u \sim D/r^3$ and the stresses will look like $D/r^4$. So the stresses vanish like $1/r^4$ while the surface area increases as $r^2$ so these terms vanish.

## 11/15/2007

Multipole Methods:

Suppose we have a sphere of radius $a$ moving with velocity $v$ centered at the origin. Then,

$$\begin{aligned} u_i(x) &= \frac{3}{4}a\left[\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right]v_j + \frac{1}{4}a^3\left[\frac{\delta_{ij}}{r^3} - 3\frac{x_i x_j}{r^5}\right]v_j \\ p(x) &= \frac{3}{2}a\mu\frac{v\cdot x}{r^3}. \end{aligned}$$

The nice thing here is that we have the formula in terms of regular cartesian coordinates. For the $u_i$ term we call the first term on the RHS as the Stokeslet and the second term as the Potential flow-dipole. Recall that

$$\psi = \frac{1}{2}u_0 r^2 \sin^2\theta\left\{1 - \frac{3}{2}\frac{a}{r} + \frac{1}{2}\frac{a^3}{r^3}\right\},$$

where the last term $\frac{1}{2}\frac{a^3}{r^3}$ has a zero vorticity. Similarly, $\frac{\delta_{ij}}{r^3} - 3\frac{x_i x_j}{r^5}$ has zero vorticity. We have that

$$\begin{aligned} u_i^{(2)} &= \left[\frac{\delta_{ij}}{r^3} - 3\frac{x_i x_j}{r^5}\right]B_j \\ &= \frac{\partial}{\partial x_i}\left(\frac{x_j B_j}{r^3}\right) \end{aligned}$$

and we have that

$$u^{(2)} = \nabla\left(\frac{x\cdot B}{r^3}\right)$$

and we have a potential flow: $\nabla \times u^{(2)}$ (the superscript only refers to the respective term and does not denote a derivative). For the first term,

$$u_i^{(1)} = \left[\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right]A_j.$$

Then,

$$\begin{aligned}\omega_i &= \epsilon_{ijk}\frac{\partial}{\partial x_j}u_k^{(1)}\\ &= \epsilon_{ijk}\frac{\partial}{\partial x_j}\bigg[\frac{\delta_{km}}{r}+\frac{x_k x_m}{r^3}\bigg]A_m\\ &= \epsilon_{ijk}\bigg\{\delta_{km}\bigg(-\frac{1}{r^2}\bigg)\frac{x_j}{r}+\frac{\delta_{kj}x_m}{r^3}+\frac{\delta_{mj}x_k}{r^3}-3\frac{x_k x_m x_j}{r^5}\bigg\}A_m.\end{aligned}$$

Note that the last term is symmetric is $j$, $k$ and $\epsilon_{ijk}$ is antisymmetric in $j$, $k$ so this term vanishes after summation. Similarly, with $\frac{\delta_{kj}x_m}{r^3}$. Hence,

$$\begin{aligned}\omega_i &= \epsilon_{ijk}\bigg\{\delta_{km}\bigg(-\frac{1}{r^2}\bigg)\frac{x_j}{r}+\frac{\delta_{mj}x_k}{r^3}\bigg\}A_m\\ &= 2\epsilon_{ijk}\bigg(\frac{A_j x_k}{r^3}\bigg),\end{aligned}$$

by switching $j, k$ is the latter term. Furthermore,

$$\omega = 2\frac{A\times x}{r^3}.$$

$$\begin{aligned}A &= \frac{3}{4}a\vec{V}\\ &= \frac{3}{4}aVe^{(z)}\end{aligned}$$

and

$$e^{(z)}\times x = e^{(\phi)}r\sin\theta$$

so that

$$\omega_\phi = \frac{3}{2}\frac{aV\sin\theta}{r^3}.$$

We also need to verify that $\nabla\cdot u^{(1)}=0$ $(r\neq 0)$. Last time we showed that the fluid force on the sphere is given by

$$\vec{F} = -6\pi a\mu\vec{V}.$$

On the other hand, the force of the sphere on the fluid is

$$\begin{aligned}\hat{\vec{F}} &= -\vec{F}\\ &= 6\pi a\mu\vec{V}.\end{aligned}$$

Comparing this to $A=\frac{3}{4}a\vec{V}$, we get that

$$\vec{A} = \frac{\hat{\vec{F}}}{8\pi\mu}.$$

As for the pressure term,

$$p = \frac{1}{4\pi}\frac{\hat{F}\cdot x}{r^3}.$$

All of this comes from the Stokeslet term, which we can write as

$$
\begin{aligned}
u &= \frac{1}{8\pi\mu}\left\{\frac{\delta_{ij}}{r}+\frac{x_i x_j}{r^3}\right\}F_j \\
p &= \frac{F_j x_j}{4\pi r^3}.
\end{aligned}
$$

One can verify that

$$
\begin{aligned}
0 &= -\nabla p+\mu\nabla^2 u+F\delta(x) \\
0 &= \nabla\cdot u.
\end{aligned}
$$

So if we take any sphere, radius $R$, and evaluate the force on the fluid, it always equals $F$. Note that $F\delta(x)$ is merely notation.

**Boundary Conditions:**

As $r\to\infty$, $u\to 0$. At $r=a$, we have

$$
\begin{aligned}
u_i(x) &= \frac{3}{4}\left[\delta_{ij}+\frac{x_i x_j}{a^2}\right]v_j+\frac{1}{4}\left[\delta_{ij}-3\frac{x_i x_j}{a^2}\right]v_j \\
&= v_i,
\end{aligned}
$$

so that we satisfy the no slip boundary conditions.

If we are dealing with a clean gas bubble with radius $a$ we have that

$$
u\cdot n = v\cdot n, \text{ on } r=a.
$$

or $(v-u)\cdot n=0$. Also, the shear stress is $\cong 0$ and so

$$
\begin{aligned}
\epsilon_{kmi}n_m(\sigma_{ij}n_j) &= 0 \\
n\times(\sigma\cdot n) &= 0.
\end{aligned}
$$

Then,

$$
u_i(x) = \left[\frac{\delta_{ij}}{r}+\frac{x_i x_j}{r^3}\right]A_j+\left[\frac{\delta_{ij}}{r^3}-3\frac{x_i x_j}{r^5}\right]B_j,
$$

for some appropriate $A,B$. After some calculation, we find that

$$
\begin{aligned}
B &= 0, \\
A &= \frac{4\pi a\mu V}{8\pi\mu} \\
&= \frac{1}{2}aV.
\end{aligned}
$$

For other example: liquid sphere, external $(r>a)$ flow is same format. In general, the force multipole expression can be written as

$$
0 = -\frac{\partial p}{\partial x_i}+\mu\nabla^2 u_i+\left\{F_i^{(0)}\delta(x)+F_{ij}^{(1)}\frac{\partial}{\partial x_j}\delta(x)+F_{ijk}^{(2)}\frac{\partial^2}{\partial x_j\partial x_k}\delta(x)+\cdots\right\}.
$$

Can we always neglect fluid intertia? In Stokes flow, $u \propto \frac{Va}{r}$. This is when we have a moving sphere, where the fluid is at rest at $\infty$. Recall that our equation of motion is

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i$$

and we have that $\mathrm{Re} = \frac{Va}{r} \ll 1$ so we can ignore the intertia term. Does this hold away from the sphere? The intertia term is $\rho u \cdot \nabla u$. Some quick dimenional analysis gives us

$$\begin{aligned} \rho u \cdot \nabla u &\to \rho \frac{V^2 a^2}{r^2} \frac{1}{r} \\ \mu \nabla^2 u &\to \mu \frac{Va}{r} \frac{1}{r^2}. \end{aligned}$$

This gives us that the ratio of $Va$ to $\mu/p$ is $\frac{Va}{\nu}\!: 1$.

A flow past a fixed sphere, we have that $u = u^{(0)}$ as $r \to \infty$ and we have that $u$ behaves like $u^{(0)} - \frac{u^{(0)} a}{r}$. Hence, the intertial term is given by

$$u \cdot \nabla u \sim u^{(0)} \cdot \frac{u^{(0)} a}{r^2}$$

for $r \gg a$. Then,

$$\nu \nabla^2 u \sim \nu \frac{u^{(0)} a}{r^3}.$$

If we look at the ratio of terms, we get $\frac{u_0 r}{\nu}\!: 1$. For a fixed sphere, we require that $\frac{u_0 r}{v} \sim \mathcal{O}(1)$ everntually. So we should correct for this because we need this to be small. Hence, we introduce the Oseen correction to drag force:

$$F = 6\pi a \mu \, \vec{u} \left\{ 1 + \frac{3}{8} \mathrm{Re} \right\}.$$

In 3D, consistent stokes solutions are possible. In 2D, flow past a cylinder is a problem.

**Hele – Shaw Cell**:

We have a flow between glass plates, narrow gap $h$. For the length scales, we have $h, D, L$ where $h \ll D \ll L$. In a steady flow, we have

$$\begin{aligned} u \cdot \nabla u &= -\frac{1}{\rho} \nabla p + \nu \nabla^2 u \\ \nabla \cdot u &= 0 \end{aligned}$$

where

$$\mu \nabla^2 u = \nu \left\{ \frac{u_0}{D^2} + \frac{u_0}{D^2} + \frac{u_0}{h^2} \right\}$$

where the RHS refers to $x_1$, $x_2$ and $x_3$ terms. For $u_1$, $u_2$ we have $u_0 \frac{u_0}{D}$. The nonlinear terms versus the viscous terms are

$$\begin{array}{c} \frac{u_0^2}{D} \text{ vs } \nu \frac{u_0}{h^2} \\ \downarrow \\ \frac{u_0 h}{\nu} \frac{h}{D} \; : \; 1 \end{array}$$

where $\frac{u_0 h}{\nu}$ is small and $\frac{h}{D}$ is very small. Hence the effect of the nonlinear terms are neglible. So, for $u_1, u_2$ we have that

$$0 = -\frac{1}{\rho}\nabla p + \nu \nabla^2 u.$$

Hence, we would get a parabolic velocity profile for $u_1, u_2$ in $x_3$ and $u_1, u_2 \propto \nabla p$. Also,

$$0 = \frac{\partial p}{\partial x_1} + \mu \frac{\partial^2 u_1}{\partial x_3^2}$$

and $u_1 = 0$ at $x_3 = 0, h$. Also, $\frac{\partial}{\partial x_3}(\frac{\partial p}{\partial x_1})$ is negligible. In summary,

$$\begin{aligned} (u_1, u_2) &\propto \left( -\frac{\partial p}{\partial x_1}, -\frac{\partial p}{\partial x_2} \right) \\ \nabla \cdot u &= 0 \\ \nabla^2 p &= 0. \end{aligned}$$

So we actually observe 2D potential flow past a circle.