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A posteriori estimates for the Bubble Stabilized Discontinuous Galerkin Method

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In this paper two reliable and efficient *a posteriori* error estimators for the Bubble Stabilized Discontinuous Galerkin (BSDG) method for diffusion-reaction problems in two and three dimensions are derived. The theory is followed by some numerical illustrations.

Keywords: Discontinuous Galerkin method, a posteriori error estimates, diffusion-reaction equation

1. Introduction

In this paper we consider the following diffusion-reaction equation: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\nabla \cdot \varepsilon \nabla u + \tau u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with $f \in L^2(\Omega)$, a reaction coefficient $\tau \geqslant 0$ and a diffusion coefficient that is piecewise constant on each element and satisfies $\varepsilon(x) > \varepsilon_0 > 0$. We assume that there exists a constant $\rho > 0$ such that $\varepsilon|_{\kappa_1} \leqslant \rho \varepsilon|_{\kappa_2}$ for two elements satisfying $\partial\kappa_1 \cap \partial\kappa_2 \neq \emptyset$, i.e. in other words that ε is of bounded variation from one element to the other.

The Bubble Stabilized Discontinuous Galerkin (BSDG) was first developed for Poissons's problem by Brezzi and Marini (2006) for the non-symmetric formulation and by Burman and Stamm (2008c) for both the symmetric and non-symmetric variants. A more refined analysis was then presented by Burman and Stamm (2008b). In Burman and Stamm (2008a) the method was extended to the diffusion-reaction problem as described by equation (1.1) and to time dependent problems. Further, superconvergence of some residual quantities, that play an important role in the upcoming *a posteriori* analysis, are pointed out. In addition, the BSDG-method has a close relation to the classical mixed lowest order Raviart-Thomas method.

A posteriori estimations for discontinuous Galerkin methods is a recent and fast developing research area. First results were published by Karakashian and Pascal (2003); Rivière and Wheeler (2003) and Becker et al. (2003). A posteriori estimates are mostly used for problems with lower regularity of the exact solution, i.e. $u \in H^1(\Omega)$ in order to solve problems where a local refinement strategy is really needed. Therefore, the theory of a posteriori estimates was further developed in (Ainsworth, 2007; Ern et al., 2008; Houston et al., 2007, 2008; Stephansen, 2007) to provide estimates that are firstly build on the assumption of $u \in H^1(\Omega)$, instead of $u \in H^2(\Omega)$ as in some of the earliest works. Secondly, attention is given to have a better and if possible an explicit control of the constants. A posteriori estimates with strongly variable diffusions coefficients are discussed by Ern and Stephansen (2008) using the technique of weighted averages. Based on a posteriori estimates, adaptive refinement strategies were designed by Hoppe et al. (2008); Karakashian and Pascal (2007); Bonito and Nochetto (2008) and global convergence towards the exact solution can be proven.

For a large class of non-conforming approximations, a posteriori error estimates were developed by Carstensen et al. (2002). A posteriori error estimates for the LDG-method can be found in (Bustinza et al., 2005) and for the Stokes problem in (Houston et al., 2005).

In this paper we develop two a posteriori error estimators for the Bubble Stabilized Discontinuous Galerkin (BSDG) method for diffusion-reaction problems in two and three dimensions. In favour of a simple presentation of the main arguments of the a posteriori error estimations, the theory is first developed for the pure diffusion equation. The first error estimator is given by the classical quantities such as the residual of equation (1.1), the jump of the approximation and the jump of the flux over faces of the mesh. The main result of its effectivity and reliability is given by Theorem 5.1 and 5.2. Due to the particular properties of the BSDG-method, some of the above local quantities are bounded by the local oscillation of the data. Out of this conclusion we derive the second error estimator which is only based on the oscillation of the right hand side and the ”oscillation” of the jump of the approximation over faces. This error estimator is shown to be effective and reliable in Theorem 5.3 and 5.4. Numerical test verify the close relation between the two estimators and show a very stable behaviour for smooth and non-smooth functions. Once, the mechanisms of the estimators for the diffusion equation are analysed, the theory is then extended to the reaction-diffusion equation and the upper and lower bounds of the estimators are given by the Theorems 6.1 resp. 6.2. The arguments of the extension are mostly based on the fact that the reaction term is a low order term. Therefore, if the reaction is sufficiently resolved by the mesh size h, i.e. $h \approx \sqrt{\varepsilon/\tau}$, similar results as in the pure diffusion equation are theoretically expected and also were numerically shown.

This paper is organized as follows. In Section 2, we first introduce the notation used in this paper whereas in Section 3, the BSDG-method is presented together with a statement of the *a priori* estimates developed by Burman and Stamm (2008a). In Section 4 we establish the splitting of the bubble enriched discontinuous finite element space in a conforming and a non-conforming part. This splitting is then used in Section 5 to derive the *a posteriori* estimates for the diffusion equation and in Section 6 for the diffusion-reaction equation including numerical results. Section 7 is left for the conclusions.

2. Notation

Let Ω be a polygonal domain (polyhedron in three space dimensions) in $\mathbb{R}^d$, $d = 2,3$, with outer normal n. Let $\mathscr{K}$ be a subdivision of $\Omega \subset \mathbb{R}^d$ into non-overlapping d-simplices κ. Suppose that each $\kappa \in \mathscr{K}$ is an affine image of the reference element $\widehat{\kappa}$, i.e. for each element κ there exists an affine transformation $T_\kappa : \widehat{\kappa} \to \kappa$.

Let $\mathscr{F}_i$ denote the set of interior faces ($(d-1)$-manifolds) of the mesh, i.e. the set of faces that are not included in the boundary $\partial\Omega$. The set $\mathscr{F}_e$ denotes the faces that are included in $\partial\Omega$ and define $\mathscr{F} = \mathscr{F}_i \cup \mathscr{F}_e$. Denote by Γ the skeleton of the mesh, i.e. the set of points belonging to faces, $\Gamma = \left\{x \in \overline{\Omega} \,\middle|\, \exists F \in \mathscr{F} \text{ s.t. } x \in F\right\}$.

Assume that $\mathscr{K}$ is shape-regular, does not contain any hanging node and covers $\overline{\Omega}$ exactly. For an element $\kappa \in \mathscr{K}$, h_κ denotes its diameter and for a face $F \in \mathscr{F}$, h_F denotes the diameter of F. Set $h = \max_{\kappa\in\mathscr{K}} h_\kappa$ and let $\mathtt{h}$ be the function such that $\mathtt{h}|_{\overset{\circ}{\kappa}} = h_\kappa$ and $\mathtt{h}|_{\overset{\circ}{F}} = h_F$ for all $\kappa \in \mathscr{K}$ and $F \in \mathscr{F}$.

For a subset $R \subset \Omega$ or $R \subset \mathscr{F}$, $(\cdot,\cdot)_R$ denotes the $L^2(R)$–scalar product, $\|\cdot\|_R = (\cdot,\cdot)_R^{1/2}$ the corresponding norm, and $\|\cdot\|_{s,R}$ the $H^s(R)$–norm. The element-wise counterparts will be distinguished using the discrete partition as subscript, for example $(\cdot,\cdot)_{\mathscr{K}} = \sum_{\kappa\in\mathscr{K}} (\cdot,\cdot)_K$. For $s \geqslant 1$, let $H^s(\mathscr{K})$ be the space of piecewise Sobolev H^s–functions and denote its norm by $\|\cdot\|_{s,\mathscr{K}}$.

In this paper $c > 0$ denotes a generic constant and can change at each occurrence, while an indexed constant stays fix. Any constant is independent of the mesh size h, ε and τ, but possibly dependent of ρ.

Further let us define the jump and average operators. Fix $F \in \mathscr{F}_i$ and thus $F = \kappa_1 \cap \kappa_2$ with $\kappa_1, \kappa_2 \in \mathscr{K}$. Let $v \in H^2(\mathscr{K})$ and denote by v_1, v_2 the restriction of v to the element κ_1, κ_2, i.e. $v_1 = v|_{\kappa_1}$ resp. $v_2 = v|_{\kappa_2}$ and denote by n_1, n_2 the exterior normal of κ_1 resp. κ_2. Then, we define the standard average and jump operators by

$$\{v\} = \tfrac{1}{2}(v_1 + v_2), \qquad [v] = v_1 n_1 + v_2 n_2,$$
$$\{\nabla v\} = \tfrac{1}{2}(\nabla v_1 + \nabla v_2), \qquad [\nabla v] = \nabla v_1 \cdot n_1 + \nabla v_2 \cdot n_2.$$

Still for inner faces $F \in \mathscr{F}_i$, let $n_F \in \{n_1, n_2\}$ be arbitrarily chosen but fixed. Then, observe that

$$[v] \cdot \{\nabla w\} = [v] \cdot n_F \, \{\nabla w\} \cdot n_F \tag{2.1}$$

for all $v, w \in H^2(\mathscr{K})$. On outer faces $F \in \mathscr{F}_e$ we define the average and jump operators by

$$\{v\} = v, \qquad [v] = vn, \qquad \{\nabla v\} = \nabla v, \qquad [\nabla v] = \nabla v \cdot n$$

where n is the outer normal of the domain Ω. The following integration by part holds.

LEMMA 2.1 (Integration by parts) Let $v, w \in H^2(\mathscr{K})$, then

$$(\varepsilon \nabla v, \nabla w)_{\mathscr{K}} = -(\nabla \cdot \varepsilon \nabla v, w)_{\mathscr{K}} + (\{\varepsilon \nabla v\}, [w])_{\mathscr{F}} + ([\varepsilon \nabla v], \{w\})_{\mathscr{F}_i}. \tag{2.2}$$

Proof. Equality (2.2) results from element-wise integration by parts and applying the definitions of the standard jump and average operators. □

2.1 *Finite element spaces*

Let us denote by V_h^p the standard discontinuous finite element space of degree $p \geqslant 0$ defined by

$$V_h^p = \left\{ v_h \in L^2(\Omega) \,\middle|\, v_h|_\kappa \in \mathbb{P}_p(\kappa), \ \forall \kappa \in \mathscr{K} \right\},$$

where $\mathbb{P}_p(\kappa)$ denotes the set of polynomials of maximum degree p on κ. Consider the enriched finite element space, which will use for the discontinuous Galerkin scheme, defined by

$$V_{bs} = V_h^1 \oplus V_h^b,$$

with

$$V_h^b = \left\{ v_h \in L^2(\Omega) \,\middle|\, v_h(x) = \alpha\, x \cdot x, \ \alpha \in V_h^0 \right\}$$

and where $x = (x_1, \ldots, x_d) \in \Omega$ denotes the physical variable. Observe that $\nabla \cdot \varepsilon \nabla v_h \in V_h^0$ and that $\nabla v_h \cdot n_F$ is constant along faces for all $v_h \in V_{bs}$. For details of V_{bs} and proofs we refer to (Burman and Stamm, 2008a,b). By $V_{h,c}^1$, we denote the piecewise linear continuous finite element space defined by

$$V_{h,c}^1 = \left\{ v_h \in C^0(\overline{\Omega}) \,\middle|\, v_h|_\kappa \in \mathbb{P}_1(\kappa), \ \forall \kappa \in \mathscr{K} \right\}.$$

Let us additionally define some functional space that consists of functions only defined on the skeleton Γ of the mesh:

$$W_h^0 = \left\{ v_h \in L^2(\Gamma) \,\middle|\, v_h|_F \in \mathbb{P}_0(F), \ \forall F \in \mathscr{F} \right\}.$$

Let $v \in H^1(\mathscr{K})$ and define by $\overline{\{v\}}$, $\overline{[v]}$ the L^2-projection of $\{v\}$ resp. $[v]$ onto W_h^0 resp. $[W_h^0]^d$, i.e.

$$(\overline{\{v\}}, w_h)_{\mathscr{F}} = (\{v\}, w_h)_{\mathscr{F}}, \qquad \forall w_h \in W_h^0,$$
$$(\overline{[v]}, w_h)_{\mathscr{F}} = ([v], w_h)_{\mathscr{F}}, \qquad \forall w_h \in [W_h^0]^d.$$

2.2 *Technical lemmas*

In this section we recall some well known results. For the proofs we refer to the book of Ciarlet (2002).

LEMMA 2.2 (Inverse inequality) Let $v_h \in V_{bs}$, then there exists a constant $c_I > 0$ independent of h such that

$$c_I^{-1}\|\mathtt{h}^2\Delta v_h\|_{\mathscr{K}} \leqslant \|\mathtt{h}\nabla v_h\|_{\mathscr{K}} \leqslant c_I\|v_h\|_{\mathscr{K}}.$$

Next, we present the standard trace inequality for discrete and non-discrete functions.

LEMMA 2.3 (Trace inequality) Let $v \in [H^1(\mathscr{K})]^m$ and $v_h \in [V_{bs}]^m$ with $m \in \{1,d\}$, then there exists a constant $c_T > 0$ independent of h such that

$$\|\{v\}\|_{\mathscr{F}} + \|[v]\|_{\mathscr{F}} \leqslant c_T\left(\|\mathtt{h}^{-\frac{1}{2}}v\|_{\mathscr{K}} + \|\mathtt{h}^{\frac{1}{2}}\nabla v\|_{\mathscr{K}}\right),$$
$$\|\{v_h\}\|_{\mathscr{F}} + \|[v_h]\|_{\mathscr{F}} \leqslant c_T\|\mathtt{h}^{-\frac{1}{2}}v_h\|_{\mathscr{K}}.$$

Finally, we define the following norms by

$$|||v|||^2 = \|\varepsilon^{\frac{1}{2}}\nabla v\|_{\mathscr{K}}^2 + \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}[v]\|_{\mathscr{F}}^2, \tag{2.3}$$
$$|||v|||_{\mathtt{e}}^2 = |||v|||^2 + \|\tau^{\frac{1}{2}}v\|_{\mathscr{K}}^2 \tag{2.4}$$

for all $v \in H^1(\mathscr{K})$ and where

$$\varepsilon_{\mathtt{F}} = \begin{cases} \max(\varepsilon|_{\kappa_1}, \varepsilon|_{\kappa_2}) & \text{if } F = \partial\kappa_1 \cap \partial\kappa_2 \in \mathscr{F}_i, \\ \varepsilon|_{\kappa} & \text{if } F = \partial\kappa \cap \partial\Omega \in \mathscr{F}_e. \end{cases}$$

Observe that the norm defined by equation (2.3) is indeed a norm due to the Poincaré inequality proven by Brenner (2003).

2.3 *Projections*

We denote by $\pi_p : L^2(\Omega) \to V_h^p$ the L^2-projection onto V_h^p defined by

$$\int_\Omega \pi_p(v)w_h\,dx = \int_\Omega v w_h\,dx \qquad \forall w_h \in V_h^p.$$

Then π_p satisfies the following approximation result: Let $v \in H^{p+1}(\mathscr{K})$, then

$$\|v - \pi_p v\|_{\mathscr{K}} + \|\mathtt{h}\nabla(v - \pi_p v)\|_{\mathscr{K}} \leqslant c|\mathtt{h}^{p+1}v|_{p+1,\mathscr{K}}. \tag{2.5}$$

Additionally let us denote by $\mathtt{I}_{\mathrm{sz}} : H^1(\Omega) \to V_{h,c}^1$ the Scott-Zhang interpolant, (Ern and Guermond, 2004; Scott and Zhang, 1990), satisfying the following approximation result: if $v \in H^1(\Omega)$, then

$$\|\mathtt{h}^{-1}(v - \mathtt{I}_{\mathrm{sz}}v)\|_{\mathscr{K}} + \|\nabla(v - \mathtt{I}_{\mathrm{sz}}v)\|_{\mathscr{K}} \leqslant c\,\|\nabla v\|_{\mathscr{K}}. \tag{2.6}$$

In addition, the the Scott-Zhang interpolant conserves homogeneous boundary conditions, i.e. $\mathtt{I}_{\mathrm{sz}}v|_{\partial\Omega} = 0$ if $v|_{\partial\Omega} = 0$.

Finally, we present the following projection that will be useful for the a posteriori analysis.

LEMMA 2.4 Let $a_h \in V_h^0$ and $b_h, c_h \in W_h^0$ be fixed. Then, there exists a unique function $\phi_h \in V_{bs}$ such that

$$\left\{ \begin{array}{rcll} \pi_0 \phi_h &=& a_h, & \\ \{\varepsilon \nabla \phi_h\}|_F \cdot n_F &=& b_h|_F & \forall F \in \mathscr{F}, \\ \overline{\{\phi_h\}}|_F &=& c_h|_F & \forall F \in \mathscr{F}_i. \end{array} \right. \tag{2.7}$$

Moreover ϕ_h satisfies the following stability result

$$\|\varepsilon^{\frac{1}{2}} \mathtt{h}^{-1} \phi_h\|_{\mathscr{K}}^2 + \|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} [\varepsilon \nabla \phi_h]\|_{\mathscr{F}_i}^2 + \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} \overline{[\phi_h]}\|_{\mathscr{F}}^2 \leqslant c_\phi^2 \left(\|\varepsilon^{\frac{1}{2}} \mathtt{h}^{-1} a_h\|_{\mathscr{K}}^2 + \|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} b_h\|_{\mathscr{F}}^2 + \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} c_h\|_{\mathscr{F}_i}^2 \right). \tag{2.8}$$

Proof. We refer to Lemma 9 in (Burman and Stamm, 2008a) to get existence and uniqueness of the projection as well as the following stability estimate

$$\|\varepsilon^{\frac{1}{2}} \mathtt{h}^{-1} \phi_h\|_{\mathscr{K}}^2 + \|\varepsilon^{\frac{1}{2}} \nabla \phi_h\|_{\mathscr{K}}^2 + \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [\phi_h]\|_{\mathscr{F}}^2 \leqslant c \left(\|\varepsilon^{\frac{1}{2}} \mathtt{h}^{-1} a_h\|_{\mathscr{K}}^2 + \|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} b_h\|_{\mathscr{F}}^2 + \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} c_h\|_{\mathscr{F}_i}^2 \right)$$

in the case of $\varepsilon \equiv 1$. The extension to general diffusion coefficients ε is straightforward. Applying additionally the trace inequality, Lemma 2.3, and using the stability of the facewise L^2-projection yields the estimate (2.8). For reasons of completeness the proof is attached in the appendix. □

3. Bubble Stabilized Discontinuous Galerkin Method

The BSDG-method consists of the classical bilinear form for the Symmetric Interior Penalty Galerkin (SIPG) method without jump penalization operator and the bubble enriched discontinuous finite element space V_{bs}. The problems consists of: find $u_h \in V_{bs}$ such that

$$a(u_h, v_h) = F(v_h) \qquad \forall v_h \in V_{bs}, \tag{3.1}$$

with

$$\begin{aligned} a(v,w) &= (\varepsilon \nabla v, \nabla w)_{\mathscr{K}} - (\{\varepsilon \nabla v\}, [w])_{\mathscr{F}} - ([v], \{\varepsilon \nabla w\})_{\mathscr{F}} + (\tau v, w)_{\mathscr{K}}, \\ F(w) &= (f, v_h)_{\mathscr{K}} \end{aligned}$$

for all $v, w \in H^2(\mathscr{K})$.

REMARK 3.1 The discrete solution u_h of (3.1) satisfies the following local mass conservation property

$$\int_\kappa \tau u_h \, dx - \int_{\partial \kappa} \{\varepsilon \nabla u_h\} \cdot n_\kappa \, ds = \int_\kappa f \, dx \qquad \forall \kappa \in \mathscr{K}.$$

The corresponding *a priori* estimates are developed in (Burman and Stamm, 2008a) for a diffusion coefficient equal to one. However, the results can be extended to general piecewise constant diffusion coefficients in a straightforward manner.

PROPOSITION 3.1 (Inf-sup condition, Proposition 1 in (Burman and Stamm, 2008a)) There exists a constant $c > 0$ independent of h such that for $\tau = 0$ or, $\tau > 0$ with $\mathtt{h}^2 < c_s \varepsilon / \tau$ on each element for some constant $c_s > 0$ independent of h, ε and τ, there holds

$$\forall v_h \in V_{bs}, \qquad c\, |||v_h|||_{\mathtt{e}} \leqslant \sup_{0 \neq w_h \in V_{bs}} \frac{a(v_h, w_h)}{|||w_h|||_{\mathtt{e}}}.$$

COROLLARY 3.1 (Stability) Under the assumption of Proposition 3.1, the discrete problem (3.1) has a unique solution. Furthermore the following estimation holds

$$|||u_h|||_{\mathrm{e}} \leqslant c\,\|\varepsilon^{-\frac{1}{2}} f\|_{\mathscr{K}}.$$

THEOREM 3.1 (Convergence in energy and L^2-norm, Theorem 1 and 2 in (Burman and Stamm, 2008a)) Let $u \in H^2(\Omega)$ be the solution of (1.1) and u_h be the discrete solution of (3.1). Under the assumption of Proposition 3.1, there holds

$$\|u - u_h\|_{\mathscr{K}} + h|||u - u_h|||_{\mathrm{e}} \leqslant ch^2 |u|_{2,\mathscr{K}}.$$

PROPOSITION 3.2 (Superconvergence of residual quantities, Proposition 2 in (Burman and Stamm, 2008a)) Let u_h be the solution of (3.1). Then, under the assumption of Proposition 3.1 the following estimation holds

$$\begin{aligned} \|\varepsilon_{\mathrm{F}}^{-\frac{1}{2}} \mathrm{h}^{\frac{1}{2}} [\varepsilon \nabla u_h]\|_{\mathscr{F}_i} + \|\varepsilon_{\mathrm{F}}^{\frac{1}{2}} \mathrm{h}^{-\frac{1}{2}} \overline{[u_h]}\|_{\mathscr{F}} + h\|\varepsilon^{-\frac{1}{2}}(f + \nabla\cdot\varepsilon\nabla u_h - \tau u_h)\|_{\mathscr{K}} \\ \leqslant ch \left(\|\varepsilon^{-\frac{1}{2}}(f - \pi_0 f)\|_{\mathscr{K}} + \tau h \|\varepsilon^{-1} f\|_{\mathscr{K}} \right), \end{aligned}$$

and if $f \in H^1(\mathscr{K})$ there holds

$$\begin{aligned} \|\varepsilon_{\mathrm{F}}^{-\frac{1}{2}} \mathrm{h}^{\frac{1}{2}} [\nabla u_h]\|_{\mathscr{F}_i} + \|\varepsilon_{\mathrm{F}}^{\frac{1}{2}} \mathrm{h}^{-\frac{1}{2}} \overline{[u_h]}\|_{\mathscr{F}} + h\|\varepsilon^{-\frac{1}{2}}(f + \nabla\cdot\varepsilon\nabla u_h - \tau u_h)\|_{\mathscr{K}} \\ \leqslant ch^2 \left(\|\varepsilon^{-\frac{1}{2}} \nabla f\|_{\mathscr{K}} + \tau \|\varepsilon^{-1} f\|_{\mathscr{K}} \right) \end{aligned}$$

independent of the regularity of the solution.

Observe that in Proposition 3.2 all the quantities are residual-based and that the result also holds for non-convex domains.

4. Splitting of Finite Element Space

In this section we define a continuous interpolation operator to split the discontinuous finite element space V_{bs} in a conforming and a non-conforming part. We further derive a norm equivalence result between the non-conforming part of the energy norm and the whole energy norm for non-conforming functions in the spirit of Houston et al. (2008).

Let us first as preliminary result introduce a continuous interpolant. Fix $\kappa \in \mathscr{K}$ and for any vertex ν in κ, set $\mathscr{K}_\nu = \left\{ \kappa' \in \mathscr{K} \;\middle|\; \nu \in \overline{\kappa}' \right\}$. Then, for $w_h \in V_h^1$, define $\mathtt{I}_{\mathrm{c}} w_h$ locally in κ by the value it takes at all the vertices of κ by setting

$$\mathtt{I}_{\mathrm{c}} w_h(\nu) = \begin{cases} \frac{1}{\mathrm{card}(\mathscr{K}_\nu)} \sum_{\kappa \in \mathscr{K}_\nu} w_h|_\kappa(\nu) & \text{if } \nu \in N_{\mathrm{int}}, \\ 0 & \text{if } \nu \in N_{\mathrm{ext}}, \end{cases} \tag{4.1}$$

where N_{int} and N_{ext} denotes the set of interior resp. exterior nodes. Clearly, $\mathtt{I}_{\mathrm{c}} w_h \in V_{h,c}^1$. There exists $c > 0$, such that the following estimate holds (Karakashian and Pascal, 2003, Thm. 2.2):

$$\|\nabla(v_h - \mathtt{I}_{\mathrm{c}} v_h)\|_{\mathscr{K}} \leqslant c\,\|\mathrm{h}^{-\frac{1}{2}} [v_h]\|_{\mathscr{F}},$$

for all $v_h \in V_h^1$. This result can be extended in order to take into account the diffusion coefficient which yields

$$\|\varepsilon^{\frac{1}{2}} \nabla (v_h - \mathtt{I}_\mathtt{c} v_h)\|_{\mathscr{K}} \leqslant c_c \, \|\varepsilon_\mathtt{F}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h]\|_{\mathscr{F}} \tag{4.2}$$

as long as the diffusion coefficient is of bounded variation across faces, which we assume here, and the constant c_c gets dependent on ρ.

Further a L^2-estimate can be shown. Indeed, using the norm equivalence of discrete spaces we get that there exists a constant $c > 0$ such that

$$\|\varepsilon^{\frac{1}{2}} \mathtt{h}^{-1} (v_h - \mathtt{I}_\mathtt{c} v_h)\|_{\mathscr{K}} \leqslant c \left(\|\varepsilon^{\frac{1}{2}} \nabla (v_h - \mathtt{I}_\mathtt{c} v_h)\|_{\mathscr{K}} + \|\varepsilon_\mathtt{F}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h]\|_{\mathscr{F}}^2 \right)^{\frac{1}{2}} \leqslant c \, \|\varepsilon_\mathtt{F}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h]\|_{\mathscr{F}} \tag{4.3}$$

where the second inequality is deduced from (4.2) We will now use the technique of splitting the finite element space into a continuous and discontinuous part, cf. (Houston et al., 2008). In our case, only the part of piecewise linear functions need to be split since the space of additional bubbles V_h^b does not contain any continuous function satisfying the homogeneous boundary conditions. Thus, we focus first on V_h^1. Define $V_h^{1,\shortparallel} = V_h^1 \cap H_0^1(\Omega)$ and denote its orthogonal component with respect to the bilinear form

$$(\varepsilon \nabla v, \nabla w)_{\mathscr{K}} + (\tau v, w)_{\mathscr{K}} + (\varepsilon_\mathtt{F} \mathtt{h}^{-1} [v], [w])_{\mathscr{F}}, \qquad \forall v, w \in H^1(\mathscr{K})$$

by $V_h^{1,\perp}$. Indeed this splitting is a direct sum. In the particular case of $\tau = 0$ this property is conserved by the Poincaré inequality, i.e. by the fact that

$$\|\varepsilon^{\frac{1}{2}} \nabla v_h\|_{\mathscr{K}}^2 + \|\varepsilon_\mathtt{F}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h]\|_{\mathscr{F}}^2$$

is a norm.

We finally split V_{bs} in a continuous part $V_{bs}^{\shortparallel} = V_h^{1,\shortparallel}$ and define the complementary part by $V_{bs}^{\perp} = V_h^{1,\perp} \oplus V_h^b$. Then, we are ready to stage some norm equivalence results concerning $V_{bs}^{\perp}$.

LEMMA 4.1 (Norm equivalence for $V_h^{1,\perp}$) For $\tau = 0$ or, $\tau > 0$ with $\mathtt{h}^2 < c_s \varepsilon / \tau$ on each element for some constant $c_s > 0$ independent of h, ε and τ, there exists a constant $c > 0$ only dependent on ρ, c_s and c_c such that for each $v_h^{\perp} \in V_h^{1,\perp}$ there holds

$$|||v_h^{\perp}|||_\mathtt{e} \leqslant c \, \|\varepsilon_\mathtt{F}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h^{\perp}]\|_{\mathscr{F}}.$$

Proof. Let $v_h^{\perp} \in V_h^{1,\perp}$ and observe that for any $w_h^{\shortparallel} \in V_h^{1,\shortparallel}$ we may write

$$\begin{aligned} |||v_h^{\perp}|||_\mathtt{e}^2 &= (\varepsilon \nabla v_h^{\perp}, \nabla (v_h^{\perp} - w_h^{\shortparallel}))_{\mathscr{K}} + (\tau v_h^{\perp}, v_h^{\perp} - w_h^{\shortparallel})_{\mathscr{K}} + (\varepsilon_\mathtt{F} \mathtt{h}^{-1} [v_h^{\perp}], [v_h^{\perp} - w_h^{\shortparallel}])_{\mathscr{F}} \\ &\leqslant |||v_h^{\perp}|||_\mathtt{e} \, |||v_h^{\perp} - w_h^{\shortparallel}|||_\mathtt{e} \end{aligned}$$

by the orthogonality relation between $V_h^{1,\perp}$ and $V_h^{1,\shortparallel}$. Therefore

$$|||v_h^{\perp}|||_\mathtt{e}^2 \leqslant \|\varepsilon^{\frac{1}{2}} \nabla (v_h^{\perp} - w_h^{\shortparallel})\|_{\mathscr{K}}^2 + \|\tau^{\frac{1}{2}} (v_h^{\perp} - w_h^{\shortparallel})\|_{\mathscr{K}}^2 + \|\varepsilon_\mathtt{F}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h^{\perp}]\|_{\mathscr{F}}^2.$$

Denote by $\mathtt{I}_\mathtt{c} : V_h^1 \to V_h^{1,\shortparallel}$ the continuous interpolant defined by (4.1). For $\tau = 0$, choosing $w_h^{\shortparallel} = \mathtt{I}_\mathtt{c} v_h^{\perp}$ yields immediately

$$|||v_h^{\perp}||| \leqslant c \, \|\varepsilon_\mathtt{F}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h^{\perp}]\|_{\mathscr{F}}$$

by (4.2). For $\tau > 0$, the condition on τ and (4.3) implies that

$$\|\tau^{\frac{1}{2}}(v_h^\perp - \mathtt{I_c} v_h^\perp)\|_{\mathscr{K}} \leqslant c_s \|\varepsilon^{\frac{1}{2}} \mathtt{h}^{-1}(v_h^\perp - \mathtt{I_c} v_h^\perp)\|_{\mathscr{K}} \leqslant c \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h^\perp]\|_{\mathscr{F}}$$

Note that only here enters the dependence on the constant on c_s. □

LEMMA 4.2 (Norm equivalence for V_h^b) There exists a constant $c > 0$ such that for each $v_h^b \in V_h^b$ there holds

$$|||v_h^b|||_{\mathtt{e}} \leqslant c \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h^b]\|_{\mathscr{F}}.$$

Proof. We will first prove that $\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [\cdot]\|_{\mathscr{F}}$ is a norm on V_h^b and then conclude by norm equivalence on discrete spaces. Thus assume that $\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h^b]\|_{\mathscr{F}} = 0$ for some $v_h^b \in V_h^b$. In consequence v_h^b is a continuous function of the form $v_h^b = \alpha x \cdot x$ with $\alpha \in \mathbb{R}$ and in addition

$$0 = v_h^b|_{\partial\Omega} = \alpha x \cdot x|_{\partial\Omega}$$

and therefore $\alpha = 0$. In consequence $v_h^b \equiv 0$ and the inequality follows by norm equivalence on discrete spaces. □

LEMMA 4.3 Let $v_h^\perp \in V_h^{1,\perp}$ and $v_h^b \in V_h^b$. Then there exists a constant $c > 0$ such that there holds

$$\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h^\perp]\|_{\mathscr{F}} + \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h^b]\|_{\mathscr{F}} \leqslant c \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h^\perp + v_h^b]\|_{\mathscr{F}}$$

Proof. This inequality follow by the fact that the decomposition $V_h^{1,\perp} + V_h^b$ is direct. □

COROLLARY 4.1 For $\tau = 0$ or, $\tau > 0$ with $\mathtt{h}^2 < c_s \varepsilon / \tau$ for some constant $c_s > 0$ independent of h, ε and τ, there exists a constant $c > 0$ such that for $v_h^\perp \in V_{bs}^\perp = V_h^{1,\perp} \oplus V_h^b$ there holds

$$|||v_h^\perp|||_{\mathtt{e}} \leqslant c \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h^\perp]\|_{\mathscr{F}}.$$

Proof. Combining the triangle inequality and Lemma 4.1, 4.2 and 4.3 yields

$$|||v_h^\perp + v_h^b|||_{\mathtt{e}} \leqslant |||v_h^\perp|||_{\mathtt{e}} + |||v_h^b|||_{\mathtt{e}} \leqslant \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h^\perp]\|_{\mathscr{F}} + \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h^b]\|_{\mathscr{F}} \leqslant c \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [v_h^\perp + v_h^b]\|_{\mathscr{F}}$$

where $v_h^\perp = v_h^\perp + v_h^b \in V_{bs}^\perp = V_h^{1,\perp} \oplus V_h^b$. □

5. A posteriori estimates for the BSDG method for the diffusion problem

Let us first discuss the pure diffusion equation, i.e. equation (1.1) with $\tau = 0$. The extension to the diffusion-reaction equation is presented in Section 6.

Let us define the following local error indicators

$$\eta_{R,\kappa}^2 = \|\varepsilon^{-\frac{1}{2}} \mathtt{h}(f + \nabla \cdot \varepsilon \nabla u_h)\|_\kappa^2, \qquad \eta_{J,\kappa}^2 = \tfrac{1}{2} \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [u_h]\|_{\partial\kappa}^2, \qquad \eta_{F,\kappa}^2 = \tfrac{1}{2} \|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} [\nabla u_h]\|_{\partial\kappa \setminus \partial\Omega}^2$$

and denote their sum by $\eta_\kappa^2 = \eta_{R,\kappa}^2 + \eta_{J,\kappa}^2 + \eta_{F,\kappa}^2$.

THEOREM 5.1 (Upper bound) Let $u \in H^1(\Omega)$ be the exact solution of (1.1) and let $u_h \in V_{bs}$ be its BSDG-approximation. Then, there exists a constant $c > 0$ such that there holds

$$|||u - u_h|||^2 \leqslant c \sum_{\kappa \in \mathscr{K}} \eta_\kappa^2.$$

Proof. Observe that for $e = u - u_h$ there holds

$$|||e|||^2 = \|\varepsilon^{\frac{1}{2}}\nabla e\|^2_{\mathscr{K}} + \|\varepsilon_{\mathrm{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}[u_h]\|^2_{\mathscr{F}} \leqslant \|\varepsilon^{\frac{1}{2}}\nabla e\|^2_{\mathscr{K}} + \sqrt{2}\,\|\varepsilon_{\mathrm{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}[e]\|_{\mathscr{F}} \left(\sum_{\kappa\in\mathscr{K}} \eta^2_{J,\kappa}\right)^{\frac{1}{2}}. \tag{5.1}$$

Further split $u_h = u_h^{\shortparallel} + u_h^{\perp}$ with $u_h^{\shortparallel} \in V_{bs}^{\shortparallel}$ and $u_h^{\perp} \in V_{bs}^{\perp}$. Define the error by $e = u - u_h = u - u_h^{\shortparallel} - u_h^{\perp}$ and denote $e^{\shortparallel} = u - u_h^{\shortparallel} \in H^1_0(\Omega)$. Thus we may write

$$\|\varepsilon^{\frac{1}{2}}\nabla e\|^2_{\mathscr{K}} = (\varepsilon\nabla e, \nabla e^{\shortparallel})_{\mathscr{K}} - (\varepsilon\nabla e, \nabla u_h^{\perp})_{\mathscr{K}} = I_1 + I_2. \tag{5.2}$$

Let $\phi_h \in H^1_0(\Omega)$ and develop the first term

$$I_1 = (\varepsilon\nabla u, \nabla e^{\shortparallel})_{\mathscr{K}} - (\varepsilon\nabla u_h, \nabla(e^{\shortparallel} - \phi_h))_{\mathscr{K}} - (\varepsilon\nabla u_h, \nabla\phi_h)_{\mathscr{K}}.$$

Observe firstly that $(\varepsilon\nabla u, \nabla e^{\shortparallel})_{\mathscr{K}} = (f, e^{\shortparallel})_{\mathscr{K}}$ since $u \in H^1_0(\Omega)$ is the exact solution and $e^{\shortparallel} \in H^1_0(\Omega)$, secondly that by integration by parts

$$-(\varepsilon\nabla u_h, \nabla(e^{\shortparallel} - \phi_h))_{\mathscr{K}} = (\nabla\cdot\varepsilon\nabla u_h, e^{\shortparallel} - \phi_h)_{\mathscr{K}} - ([\varepsilon\nabla u_h], e^{\shortparallel} - \phi_h)_{\mathscr{F}_i},$$

since $e^{\shortparallel} - \phi_h \in H^1_0(\Omega)$. Thirdly since $a(u_h, \phi_h) = F(\phi_h)$ implies that

$$-(\varepsilon\nabla u_h, \nabla\phi_h)_{\mathscr{K}} = -(f, \phi_h)_{\mathscr{K}} - ([u_h], \{\varepsilon\nabla\phi_h\})_{\mathscr{F}} = -(f, \phi_h)_{\mathscr{K}} - ([u_h], \{\varepsilon\nabla\phi_h\})_{\mathscr{F}}.$$

Respecting all three arguments yields

$$I_1 = (f + \nabla\cdot\varepsilon\nabla u_h, e^{\shortparallel} - \phi_h)_{\mathscr{K}} - ([\varepsilon\nabla u_h], e^{\shortparallel} - \phi_h)_{\mathscr{F}_i} - ([u_h], \{\varepsilon\nabla\phi_h\})_{\mathscr{F}}.$$

Now, choose $\phi_h = \mathtt{I}_{\mathrm{sz}} e^{\shortparallel}$, $\mathtt{I}_{\mathrm{sz}}$ being the Scott-Zhang interpolant defined by (2.6). Using the Cauchy-Schwarz and trace inequality, the approximability and H^1-stability of $\mathtt{I}_{\mathrm{sz}}$ yields

$$\begin{aligned} I_1 &\leqslant \|\varepsilon^{-\frac{1}{2}}\mathtt{h}(f + \nabla\cdot\varepsilon\nabla u_h)\|_{\mathscr{K}} \|\varepsilon^{\frac{1}{2}}\mathtt{h}^{-1}(e^{\shortparallel} - \mathtt{I}_{\mathrm{sz}}e^{\shortparallel})\|_{\mathscr{K}} + \|\varepsilon_{\mathrm{F}}^{-\frac{1}{2}}\mathtt{h}^{\frac{1}{2}}[\varepsilon\nabla u_h]\|_{\mathscr{F}_i} \|\varepsilon_{\mathrm{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}(e^{\shortparallel} - \mathtt{I}_{\mathrm{sz}}e^{\shortparallel})\|_{\mathscr{F}_i} \\ &\quad + \|\varepsilon_{\mathrm{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}[u_h]\|_{\mathscr{F}} \|\varepsilon_{\mathrm{F}}^{-\frac{1}{2}}\mathtt{h}^{\frac{1}{2}}\{\varepsilon\nabla\mathtt{I}_{\mathrm{sz}}e^{\shortparallel}\}\|_{\mathscr{F}} \\ &\leqslant c\left(\sum_{\kappa\in\mathscr{K}} \eta^2_{R,\kappa} + \eta^2_{J,\kappa} + \eta^2_{F,\kappa}\right)^{\frac{1}{2}} \|\varepsilon^{\frac{1}{2}}\nabla e^{\shortparallel}\|_{\mathscr{K}} \leqslant c\left(\sum_{\kappa\in\mathscr{K}} \eta^2_{R,\kappa} + \eta^2_{J,\kappa} + \eta^2_{F,\kappa}\right)^{\frac{1}{2}} |||e||| \end{aligned} \tag{5.3}$$

since

$$\|\varepsilon^{\frac{1}{2}}\nabla e^{\shortparallel}\|_{\mathscr{K}} = |||e^{\shortparallel}||| \leqslant |||e||| + |||u_h^{\perp}||| \leqslant |||e||| + c\|\varepsilon_{\mathrm{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}[u_h^{\perp}]\|_{\mathscr{F}} \leqslant |||e||| + c\|\varepsilon_{\mathrm{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}[e]\|_{\mathscr{F}} \leqslant c\,|||e|||$$

using the norm equivalence of Corollary 4.1.

For the term I_2 observe that

$$I_2 \leqslant \|\varepsilon^{\frac{1}{2}}\nabla e\|_{\mathscr{K}} \|\varepsilon^{\frac{1}{2}}\nabla u_h^{\perp}\|_{\mathscr{K}} \leqslant |||e|||\,|||u_h^{\perp}||| \leqslant c\,|||e|||\,\|\varepsilon_{\mathrm{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}[u_h]\|_{\mathscr{F}} \leqslant c\,|||e||| \left(\sum_{\kappa\in\mathscr{K}} \eta^2_{J,\kappa}\right)^{\frac{1}{2}} \tag{5.4}$$

applying again the Cauchy-Schwarz inequality and the norm equivalence of Corollary 4.1 combined with the fact that $\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}[e^{\shortparallel}]\|_{\mathscr{F}}=0$. Respecting (5.3) and (5.4) in (5.2) resp. (5.1) finally yields

$$|||e||| \leqslant c\left(\sum_{\kappa\in\mathscr{K}}\eta_{R,\kappa}^2+\eta_{J,\kappa}^2+\eta_{F,\kappa}^2\right)^{\frac{1}{2}}.$$

□

The conclusion of Proposition 3.2 is that all the information about the error of the approximation contained in

$$\|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}}\mathtt{h}^{\frac{1}{2}}[\nabla u_h]\|_{\mathscr{F}_i}+\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}\overline{[u_h]}\|_{\mathscr{F}}+h\|\varepsilon^{-\frac{1}{2}}(f+\nabla\cdot\varepsilon\nabla u_h)\|_{\mathscr{K}}$$

is bounded by the term $h\|\varepsilon^{-\frac{1}{2}}(f-\pi_0 f)\|_{\mathscr{K}}$, at least on a global level. Secondly, in the case of resolved data and oscillation, i.e. when $h\|\varepsilon^{-\frac{1}{2}}(f-\pi_0 f)\|_{\mathscr{K}}$ is scaling as h^2, the leading term of the error estimation is $\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}([u_h]-\overline{[u_h]})\|_{\mathscr{F}}^2$ and scales like h. This is the motivation to introduce the fluctuation or oscillation of the data f on a given element $\mathrm{osc}_{R,\kappa}$ resp. of the jump of the solution u_h on the boundary of a given element $\mathrm{osc}_{J,\kappa}$ by

$$\mathrm{osc}_{R,\kappa}^2=\|\varepsilon^{-\frac{1}{2}}\mathtt{h}(f-\pi_0 f)\|_{\kappa}^2 \qquad \text{and} \qquad \mathrm{osc}_{J,\kappa}^2=\tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}([u_h]-\overline{[u_h]})\|_{\partial\kappa}^2$$

as possible error estimators. Further we define $\mathrm{osc}_{\kappa}^2=\mathrm{osc}_{R,\kappa}^2+\mathrm{osc}_{J,\kappa}^2$. Remark that $\mathrm{osc}_{R,\kappa}$ can alternatively be interpreted as the oscillation of the residual since $\nabla\cdot\varepsilon\nabla u_h\in V_h^0$ for $u_h\in V_{bs}$ and therefore $\mathrm{osc}_{R,\kappa}^2=\|\mathtt{h}(\mathtt{I}-\pi_0)(f+\nabla\cdot\varepsilon\nabla u_h)\|_{\kappa}^2$ which justifies the notation, i.e. the R in the index. Note that $\mathtt{I}$ denotes the identity operator. Then, the estimator η_κ is locally equivalent to the oscillatory term osc_κ.

LEMMA 5.1 There exists a constant $c>0$ such that there holds

$$\mathrm{osc}_\kappa \leqslant \eta_\kappa \leqslant c\,\mathrm{osc}_\kappa, \qquad \forall\kappa\in\mathscr{K}. \tag{5.5}$$

Proof. For the first inequality of (5.5) observe by the stability of the L^2-projection and since $\nabla\cdot\varepsilon\nabla u_h\in V_h^0$ that

$$\mathrm{osc}_{R,\kappa}^2=\|\varepsilon^{-\frac{1}{2}}\mathtt{h}(f-\pi_0 f)\|_{\kappa}^2=\|\varepsilon^{-\frac{1}{2}}\mathtt{h}(\mathtt{I}-\pi_0)(f+\nabla\cdot\varepsilon\nabla u_h)\|_{\kappa}^2\leqslant\|\varepsilon^{-\frac{1}{2}}\mathtt{h}(f+\nabla\cdot\varepsilon\nabla u_h)\|_{\kappa}^2=\eta_{R,\kappa}^2,$$
$$\mathrm{osc}_{J,\kappa}^2=\tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}([u_h]-\overline{[u_h]})\|_{\partial\kappa}^2\leqslant\tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}[u_h]\|_{\partial\kappa}^2=\eta_{J,\kappa}^2.$$

For the second inequality of (5.5) we use a localized variant of Proposition 3.2. Fix $\kappa\in\mathscr{K}$ and let $\phi_{h,\kappa}\in V_{bs}$ be the projection defined in Lemma 2.4 with

$$a_h|_{\kappa'}=\begin{cases}-\delta\varepsilon^{-1}\mathtt{h}^2(\pi_0 f+\nabla\cdot\varepsilon\nabla u_h) & \text{if } \kappa'=\kappa,\\ 0 & \text{otherwise,}\end{cases}$$
$$b_h|_F=\begin{cases}-\delta\varepsilon_{\mathtt{F}}\mathtt{h}^{-1}\overline{[u_h]}|_F\cdot n_F & \text{if } F\subset\partial\kappa,\\ 0 & \text{otherwise,}\end{cases}$$
$$c_h|_F=\begin{cases}\delta\varepsilon_{\mathtt{F}}^{-1}\mathtt{h}[\varepsilon\nabla u_h]|_F & \text{if } F\subset\partial\kappa,\\ 0 & \text{otherwise,}\end{cases}$$

for all $\kappa' \in \mathscr{K}$ and $F \in \mathscr{F}$ and with $\delta > 0$ an arbitrary constant. Let us denote by χ_κ the characteristic function such that

$$\chi_\kappa|_{\kappa'} = \begin{cases} 1 & \text{if } \kappa' = \kappa, \\ 0 & \text{otherwise.} \end{cases}$$

Then, integration by parts yields

$$\begin{aligned} a(u_h, \chi_\kappa \phi_{h,\kappa}) &= -(\nabla \cdot \varepsilon \nabla u_h, \chi_\kappa \pi_0 \phi_{h,\kappa})_{\mathscr{K}} + ([\varepsilon \nabla u_h], \overline{\{\chi_\kappa \phi_{h,\kappa}\}})_{\mathscr{F}_i} - (\overline{[u_h]}, \{\chi_\kappa \varepsilon \nabla \phi_{h,\kappa}\})_{\mathscr{F}} \\ &= I_1 + I_2 + I_3, \end{aligned}$$

since $\varepsilon \nabla u_h \cdot n_\kappa$ is constant along faces. For the first term we can write

$$I_1 = -(\nabla \cdot \varepsilon \nabla u_h, a_h)_\kappa = -(\pi_0 f + \nabla \cdot \varepsilon \nabla u_h, a_h)_\kappa + (\pi_0 f, a_h)_\kappa = \delta \|\varepsilon^{\frac{1}{2}} \mathtt{h}(\pi_0 f + \nabla \cdot \varepsilon \nabla u_h)\|_\kappa^2 + (\pi_0 f, \phi_{h,\kappa})_\kappa$$

by the property of the projection $\phi_{h,\kappa}$. Observe for the second term that

$$\overline{\{\chi_\kappa \phi_{h,\kappa}\}}|_F = \{\chi_\kappa\} \overline{\{\phi_{h,\kappa}\}}|_F + \tfrac{1}{4}[\chi_\kappa] \cdot \overline{[\phi_{h,\kappa}]}|_F$$

for all $F \in \mathscr{F}_i$ and therefore we develop, using the property of the projection $\phi_{h,\kappa}$, the Cauchy-Schwarz and Young's inequality,

$$\begin{aligned} I_2 = ([\varepsilon \nabla u_h], \overline{\{\chi_\kappa \phi_{h,\kappa}\}})_{\mathscr{F}_i} &= \tfrac{1}{2}([\varepsilon \nabla u_h], \overline{\{\phi_{h,\kappa}\}})_{\partial\kappa \setminus \partial\Omega} + \tfrac{1}{4}([\varepsilon \nabla u_h], \overline{[\phi_{h,\kappa}]} \cdot n_\kappa)_{\partial\kappa \setminus \partial\Omega} \\ &\geqslant \delta \tfrac{1}{2} \|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} [\varepsilon \nabla u_h]\|_{\partial\kappa \setminus \partial\Omega}^2 - \tfrac{1}{4} \|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} [\varepsilon \nabla u_h]\|_{\partial\kappa \setminus \partial\Omega} \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} \overline{[\phi_{h,\kappa}]}\|_{\partial\kappa \setminus \partial\Omega} \\ &\geqslant \tfrac{1}{2}(\delta - \tfrac{1}{4}) \|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} [\varepsilon \nabla u_h]\|_{\partial\kappa \setminus \partial\Omega}^2 - \tfrac{1}{8} \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} \overline{[\phi_{h,\kappa}]}\|_{\partial\kappa}^2. \end{aligned}$$

Similarly we observe that

$$\begin{aligned} \overline{[u_h]} \cdot \{\chi_\kappa \varepsilon \nabla \phi_{h,\kappa}\}|_F &= \overline{[u_h]} \cdot \{\varepsilon \nabla \phi_{h,\kappa}\}\{\chi_\kappa\}|_F + \tfrac{1}{4}\overline{[u_h]} \cdot [\chi_\kappa][\varepsilon \nabla \phi_{h,\kappa}]|_{F \setminus \partial\Omega} \\ &= \overline{[u_h]} \cdot n_F \{\varepsilon \nabla \phi_{h,\kappa}\} \cdot n_F \{\chi_\kappa\}|_F + \tfrac{1}{4}\overline{[u_h]} \cdot [\chi_\kappa][\varepsilon \nabla \phi_{h,\kappa}]|_{F \setminus \partial\Omega} \end{aligned}$$

for all $F \in \mathscr{F}$ and thus we may write

$$\begin{aligned} I_3 = -(\overline{[u_h]}, \{\chi_\kappa \varepsilon \nabla \phi_{h,\kappa}\})_{\mathscr{F}} &= -\tfrac{1}{2}(\overline{[u_h]} \cdot n_F, \{\varepsilon \nabla \phi_{h,\kappa}\} \cdot n_F)_{\partial\kappa} - \tfrac{1}{4}(\overline{[u_h]} \cdot n_\kappa, [\varepsilon \nabla \phi_{h,\kappa}])_{\partial\kappa \setminus \partial\Omega} \\ &\geqslant \delta \tfrac{1}{2} \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} \overline{[u_h]}\|_{\partial\kappa}^2 - \tfrac{1}{4} \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} \overline{[u_h]}\|_{\partial\kappa} \|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} [\varepsilon \nabla \phi_{h,\kappa}]\|_{\partial\kappa \setminus \partial\Omega} \\ &\geqslant \tfrac{1}{2}(\delta - \tfrac{1}{4}) \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} \overline{[u_h]}\|_{\partial\kappa}^2 - \tfrac{1}{8} \|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} [\varepsilon \nabla \phi_{h,\kappa}]\|_{\partial\kappa \setminus \partial\Omega}^2. \end{aligned}$$

Further observe by the stability estimate of $\phi_{h,\kappa}$ that

$$\begin{aligned} &\|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} [\varepsilon \nabla \phi_{h,\kappa}]\|_{\partial\kappa \setminus \partial\Omega}^2 + \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} \overline{[\phi_{h,\kappa}]}\|_{\partial\kappa}^2 \\ &\qquad \leqslant c_\phi^2 \left(\|\varepsilon^{-\frac{1}{2}} \mathtt{h}(\pi_0 f + \nabla \cdot \varepsilon \nabla u_h)\|_\kappa^2 + \tfrac{1}{2} \|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} [\varepsilon \nabla u_h]\|_{\partial\kappa \setminus \partial\Omega}^2 + \tfrac{1}{2} \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} \overline{[u_h]}\|_{\partial\kappa}^2 \right) \end{aligned}$$

where $c_\phi > 0$ denotes the constant from (2.8). Thus, using that u_h is the solution of the discrete problem we get

$$\begin{aligned} &(\delta - \tfrac{c_\phi^2}{8}) \|\varepsilon^{-\frac{1}{2}} \mathtt{h}(\pi_0 f + \nabla \cdot \varepsilon \nabla u_h)\|_\kappa^2 + \tfrac{1}{2}(\delta - \tfrac{1}{4} - \tfrac{c_\phi^2}{8}) \|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} [\varepsilon \nabla u_h]\|_{\partial\kappa \setminus \partial\Omega}^2 + \tfrac{1}{2}(\delta - \tfrac{1}{4} - \tfrac{c_\phi^2}{8}) \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} \overline{[u_h]}\|_{\partial\kappa}^2 \\ &\qquad \leqslant a(u_h, \chi_\kappa \phi_{h,\kappa}) - (\pi_0 f, \chi_\kappa \phi_{h,\kappa})_\kappa = (f - \pi_0 f, \phi_{h,\kappa})_\kappa. \end{aligned}$$

Now, choosing $\delta = \frac{5}{4} + \frac{c_\phi^2}{8}$ yields $\frac{1}{2}(\delta - \frac{1}{4} - \frac{c_\phi^2}{8}) = \frac{1}{2}$ and $(\delta - \frac{c_\phi^2}{8}) = \frac{5}{4} > 1$. Additionally taking again into account the stability estimate of $\phi_{h,\kappa}$ we may write

$$\begin{aligned}
&\|\varepsilon^{-\frac{1}{2}}\mathtt{h}(\pi_0 f + \nabla\cdot\varepsilon\nabla u_h)\|_\kappa^2 + \tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}}\mathtt{h}^{\frac{1}{2}}[\varepsilon\nabla u_h]\|_{\partial\kappa\setminus\partial\Omega}^2 + \tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}\overline{[u_h]}\|_{\partial\kappa}^2 \\
&\qquad \leqslant (f - \pi_0 f, \phi_{h,\kappa})_\kappa \leqslant \mathrm{osc}_{R,\kappa}\|\varepsilon^{\frac{1}{2}}\mathtt{h}^{-1}\phi_{h,\kappa}\|_\kappa \\
&\qquad \leqslant c_\phi\ \mathrm{osc}_{R,\kappa}\left(\|\varepsilon^{-\frac{1}{2}}\mathtt{h}(\pi_0 f + \nabla\cdot\varepsilon\nabla u_h)\|_\kappa^2 + \tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}}\mathtt{h}^{\frac{1}{2}}[\varepsilon\nabla u_h]\|_{\partial\kappa\setminus\partial\Omega}^2 + \tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}\overline{[u_h]}\|_{\partial\kappa}^2\right)^{\frac{1}{2}}
\end{aligned}$$

and therefore

$$\|\varepsilon^{-\frac{1}{2}}\mathtt{h}(\pi_0 f + \nabla\cdot\varepsilon\nabla u_h)\|_\kappa + \eta_{F,\kappa} + \tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}\overline{[u_h]}\|_{\partial\kappa} \leqslant c_\phi\ \mathrm{osc}_{R,\kappa}. \tag{5.6}$$

Further observe that

$$\begin{aligned}
\eta_{R,\kappa} &= \|\varepsilon^{-\frac{1}{2}}\mathtt{h}(f + \nabla\cdot\varepsilon\nabla u_h)\|_\kappa \leqslant \|\varepsilon^{-\frac{1}{2}}\mathtt{h}(f - \pi_0 f)\|_\kappa + \|\varepsilon^{-\frac{1}{2}}\mathtt{h}(\pi_0 f + \Delta u_h)\|_\kappa \leqslant (1 + c_\phi)\ \mathrm{osc}_{R,\kappa}, \\
\eta_{J,\kappa} &= \tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}[u_h]\|_{\partial\kappa} \leqslant \tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}([u_h] - \overline{[u_h]})\|_{\partial\kappa} + \tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}\overline{[u_h]}\|_{\partial\kappa} \leqslant \mathrm{osc}_{J,\kappa} + c_\phi\ \mathrm{osc}_{R,\kappa}.
\end{aligned} \tag{5.7}$$

□

REMARK 5.1 Observe that if the constant c_ϕ is sufficiently small, then the constant in (5.5) is close to one.

THEOREM 5.2 (Lower bound) Let $u \in H^1(\Omega)$ be the exact solution of (1.1) and let $u_h \in V_{bs}$ be the BSDG-approximation. Then, there exists a constant $c > 0$ such that there holds locally

$$\eta_\kappa^2 \leqslant \tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}[u - u_h]\|_{\partial\kappa}^2 + c\ \mathrm{osc}_{R,\kappa}^2, \tag{5.8}$$

and globally

$$\sum_{\kappa\in\mathscr{K}}\eta_\kappa^2 \leqslant |||u - u_h|||^2 + c\sum_{\kappa\in\mathscr{K}}\mathrm{osc}_{R,\kappa}^2. \tag{5.9}$$

REMARK 5.2 Note that for regular right hand side, i.e. $f \in H^1(\mathscr{K})$, the quantity $\mathrm{osc}_{R,\kappa}$ converges to zero as h_κ^2 whereas $\mathrm{osc}_{J,\kappa}$ converges to zero as h_κ.

Proof. This is a direct consequence of the second inequality of the previous lemma, Lemma 5.1, with a sharper treatment of the constant. Indeed, note that

$$\eta_{J,\kappa}^2 = \tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}[u_h]\|_{\partial\kappa}^2 = \tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}}\mathtt{h}^{-\frac{1}{2}}[u - u_h]\|_{\partial\kappa}^2$$

and by (5.6) and (5.7) that

$$\eta_{R,\kappa}^2 + \eta_{F,\kappa}^2 \leqslant c\ \mathrm{osc}_{R,\kappa}^2.$$

Finally summing over all elements leads to the global estimate. □

Recall that by Lemma 5.1 the error estimator η_κ and the oscillation indicator osc_κ are equivalent as error estimations for $|||u - u_h|||$. Thus we propose the oscillatory terms osc_κ as error estimator and derive the following upper and lower bounds.

THEOREM 5.3 (Upper bound) Let $u \in H^1(\Omega)$ be the exact solution of (1.1) and let $u_h \in V_{bs}$ be the BSDG-approximation. Then, there exists a constant $c > 0$ such that there holds

$$|||u - u_h|||^2 \leqslant c \sum_{\kappa \in \mathscr{K}} \mathrm{osc}_\kappa^2.$$

Proof. This is a direct consequence of Theorem 5.1 and Lemma 5.1. □

THEOREM 5.4 (Lower bound) Let $u \in H^1(\Omega)$ be the exact solution of (1.1) and let $u_h \in V_{bs}$ be the BSDG-approximation. Then, there exists a constant $c > 0$ such that there holds locally

$$\mathrm{osc}_\kappa^2 \leqslant \tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} [u - u_h]\|_{\partial\kappa}^2 + \mathrm{osc}_{R,\kappa}^2$$

and globally

$$\sum_{\kappa \in \mathscr{K}} \mathrm{osc}_\kappa^2 \leqslant |||u - u_h|||^2 + \sum_{\kappa \in \mathscr{K}} \mathrm{osc}_{R,\kappa}^2.$$

Proof. The local result is a direct consequence of the definition of the estimator osc_κ. Indeed

$$\mathrm{osc}_{J,\kappa}^2 = \tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}}([u_h] - \overline{[u_h]})\|_{\partial\kappa}^2 \leqslant \tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}}[u_h]\|_{\partial\kappa}^2 = \tfrac{1}{2}\|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}}[u - u_h]\|_{\partial\kappa}^2.$$

Finally summing over all elements leads to the global estimate. □

REMARK 5.3 Observe by Proposition 3.2 that for $f \in V_h^0$ the error estimator η_κ and the oscillation indicator osc_κ coincide and that

$$\eta_\kappa = \eta_{J,\kappa} = \mathrm{osc}_\kappa = \mathrm{osc}_{J,\kappa}.$$

5.1 *Numerical examples*

Let us briefly present the test problems used for the numerical tests.

i) *Problem with smooth solution*
We consider problem (1.1) with $f(x,y) = 2(2 - x^2 - y^2)$ and $\varepsilon \equiv 1$ on the square $\Omega = (-1,1)^2$. The analytic exact solution is given by $u(x,y) = (x^2 - 1)(y^2 - 1) \in C^\infty(\overline{\Omega})$. A sequence of unstructured meshes is considered.

ii) *Problem with irregular solution*
Now choose the following L-shaped domain: $\Omega = \left([-1,1] \times [-1,0] \cup [0,1]^2\right)^\circ$. We consider problem (1.1) with $f \equiv 0$, $\varepsilon \equiv 1$ and non-homogeneous boundary conditions such that the solution is

$$u(x,y) = (x^2 + y^2)^{\frac{1}{3}} \sin\left(\frac{2}{3}\arctan_*\left(\frac{x}{y}\right)\right)$$

where $\arctan_*$ is chosen in the manner that it is a continuous function at points with $y = 0$. One can prove that $u \notin H^2(\Omega)$. A sequence of unstructured meshes is considered.

We analyse the effectivity of the error estimators derived in the previous section for the test problem i) and ii). To do that we define the effectivity index by

$$\mathrm{eff}_\eta = \frac{\left(\sum_{\kappa \in \mathscr{K}} \eta_\kappa^2\right)^{\frac{1}{2}}}{|||u - u_h|||} \qquad \text{and} \qquad \mathrm{eff}_{\mathrm{osc}} = \frac{\left(\sum_{\kappa \in \mathscr{K}} \mathrm{osc}_\kappa^2\right)^{\frac{1}{2}}}{|||u - u_h|||}. \tag{5.10}$$

h	$\|\|\|u-u_h\|\|\|$	eff_η	$\mathrm{eff}_{\mathrm{osc}}$	$c_{\eta,\mathrm{osc}}$
0.2	3.30E-01	0.72745	0.72425	1.00443
0.1	1.71E-01	0.72383	0.72237	1.00203
0.05	8.50E-02	0.72292	0.72261	1.00043
0.025	4.42E-02	0.72906	0.72898	1.00010

Table 1. Effectivity of a posteriori error estimators for test problem i) with smooth solution for different mesh sizes.

h	$\|\|\|u-u_h\|\|\|$	eff_η	$\mathrm{eff}_{\mathrm{osc}}$	$c_{\eta,\mathrm{osc}}$
0.2	1.34E-01	0.65123	0.65123	1
0.1	8.64E-02	0.65179	0.65179	1
0.05	5.62E-02	0.67006	0.67006	1
0.025	3.51E-02	0.65788	0.65788	1

Table 2. Effectivity of a posteriori error estimators for test problem ii) with non-smooth solution for different mesh sizes.

In order to compare the error estimators η_κ with the oscillation estimator osc_κ we define the following coefficient

$$c_{\eta,\mathrm{osc}} = \frac{\left(\sum_{\kappa\in\mathscr{K}} \eta_\kappa^2\right)^{\frac{1}{2}}}{\left(\sum_{\kappa\in\mathscr{K}} \mathrm{osc}_\kappa^2\right)^{\frac{1}{2}}}. \tag{5.11}$$

Note that we do not apply a refinement strategy by adaptivity and that a uniform refinement of the mesh is considered.

Table 1 shows the energy-error, the effectivity indices of the two estimators and the coefficient of the two estimators for different mesh sizes and for the test problem with smooth solution. Note that the two estimators are really equivalent for this test problem and that the effectivity index for both estimators is smaller than one, which in turn means that no overestimation of the error held, but a slight underestimation and can be explained by Theorem 5.2 and 5.4.

Table 3 illustrates the same quantities for the test problem with non-smooth solution, i.e. test problem ii), for different mesh sizes. Observe that according to Remark 5.3 the two estimators are identical and thus $c_{\eta,\mathrm{osc}} = 1$ since $f \equiv 0$.

6. A posteriori estimates for the BSDG method for the diffusion-reaction problem

Let us discuss the extension to the diffusion-reaction equation, i.e. equation (1.1) with $\tau > 0$. The above developed theory remains valid with some modifications and remarks. Of course the reaction term has to be taken into account for the quantities related to the residual and therefore we may write

$$\eta_{R,\kappa}^2 = \|\varepsilon^{-\frac{1}{2}}\mathtt{h}(f + \nabla\cdot\varepsilon\nabla u_h - \tau u_h)\|_\kappa^2 \qquad \text{and} \qquad \mathrm{osc}_{R,\kappa}^2 = \|\varepsilon^{-\frac{1}{2}}\mathtt{h}(\mathtt{I}-\pi_0)(f-\tau u_h)\|_\kappa^2$$

whereas the estimators $\eta_{J,\kappa}$, $\eta_{F,\kappa}$ and $\mathrm{osc}_{J,\kappa}$ remain unchanged. Using now the energy-norm $|||\cdot|||_{\mathtt{e}}$ defined by (2.4) and the above definition of $\eta_{R,\kappa}$ keeps Theorem 5.1 and Lemma 5.1 valid with only minor changes. Details are left to the reader. In principle Proposition 5.2 remains also valid, but since

h	$\|\|\|u-u_h\|\|\|$	eff_η	$\mathrm{eff}_{\mathrm{osc}}$	$c_{\eta,\mathrm{osc}}$
0.2	11.7226	0.956837	0.956837	1
0.1	11.286	0.956655	0.956655	1
0.05	10.7981	0.956694	0.956694	1
0.025	10.2732	0.956851	0.956851	1

Table 3. Effectivity of a posteriori error estimators for test problem iii) with heterogenous diffusion coefficient for different mesh sizes.

$\mathrm{osc}_{R,\kappa}$ depends on the solution itself and not solely on the data this lower bound is questionable. As remedy we propose the following solution: splitting the residual oscillation term into two parts yields

$$\sum_{\kappa\in\mathscr{K}} \mathrm{osc}_{R,\kappa}^2 = \|\varepsilon^{-\frac{1}{2}}\mathtt{h}(\mathtt{I}-\pi_0)(f-\tau u_h)\|_{\mathscr{K}}^2 \leqslant 2\,\|\varepsilon^{-\frac{1}{2}}\mathtt{h}(f-\pi_0 f)\|_{\mathscr{K}}^2 + 2\tau^2\|\varepsilon^{-\frac{1}{2}}\mathtt{h}(u_h-\pi_0 u_h)\|_{\mathscr{K}}^2$$

and bounding the second term by using the stability estimate of Corollary 3.1, i.e.

$$\|\varepsilon^{-\frac{1}{2}}\mathtt{h}(u_h-\pi_0 u_h)\|_{\mathscr{K}}^2 \leqslant c\,\|\varepsilon^{-\frac{1}{2}}\mathtt{h}^2\nabla u_h\|_{\mathscr{K}}^2 \leqslant c\,\|\varepsilon^{-1}\mathtt{h}^2 f\|_{\mathscr{K}}^2,$$

yields

$$\sum_{\kappa\in\mathscr{K}} \eta_\kappa^2 \leqslant |||u-u_h|||_{\mathtt{e}}^2 + 2\,\|\varepsilon^{-\frac{1}{2}}\mathtt{h}(f-\pi_0 f)\|_{\mathscr{K}}^2 + c\,\tau^2\|\varepsilon^{-1}\mathtt{h}^2 f\|_{\mathscr{K}}^2$$

as equivalent estimate to (5.9). The local bound (5.8) can not be established since the stability estimation of Corollary 3.1 holds only on a global level. However, observe that only the additional term related to τ is globally coupled and that this quantity is superconverging. Theorem 5.3 still holds with a reaction term whereas for Theorem 5.4 suffer from the same restriction as described above, i.e.

$$\sum_{\kappa\in\mathscr{K}} \mathrm{osc}_\kappa^2 \leqslant |||u-u_h|||_{\mathtt{e}}^2 + 2\,\|\varepsilon^{-\frac{1}{2}}\mathtt{h}(f-\pi_0 f)\|_{\mathscr{K}}^2 + c\,\tau^2\|\varepsilon^{-1}\mathtt{h}^2 f\|_{\mathscr{K}}^2$$

and without local lower bound for osc_κ. Further, let us denote by $\mathtt{data_f}$ the following expression

$$\mathtt{data_f} = \tau\,\|\varepsilon^{-1}\mathtt{h}^2 f\|_{\mathscr{K}}$$

Thus, for small enough mesh sizes h, i.e. when $\mathtt{data_f}$ is small compared to $|||u-u_h|||$, we expect the estimates η_κ and osc_κ to be efficient as well. All together we can now state the following Propositions.

THEOREM 6.1 (Upper bounds) Let $u \in H^1(\Omega)$ be the exact solution of (1.1) and let $u_h \in V_{bs}$ be its BSDG-approximation. Then, there exists a constant $c>0$ such that if $\mathtt{h}^2 < c_s\varepsilon/\tau$ on each element for some constant $c_s>0$ independent of h, ε and τ, there holds

$$|||u-u_h|||_{\mathtt{e}}^2 \leqslant c\sum_{\kappa\in\mathscr{K}}\eta_\kappa^2, \qquad \text{and} \qquad |||u-u_h|||_{\mathtt{e}}^2 \leqslant c\sum_{\kappa\in\mathscr{K}}\mathrm{osc}_\kappa^2.$$

THEOREM 6.2 (Lower bounds) Let $u \in H^1(\Omega)$ be the exact solution of (1.1) with $\tau>0$ and let $u_h \in V_{bs}$ be the BSDG-approximation. There exists a constant $c>0$ such that

$$\sum_{\kappa\in\mathscr{K}}\eta_\kappa^2 \leqslant |||u-u_h|||_{\mathtt{e}}^2 + c\sum_{\kappa\in\mathscr{K}}\mathrm{osc}_{R,\kappa}^2 + c\,\mathtt{data_f},$$
$$\sum_{\kappa\in\mathscr{K}}\mathrm{osc}_\kappa^2 \leqslant |||u-u_h|||_{\mathtt{e}}^2 + \sum_{\kappa\in\mathscr{K}}\mathrm{osc}_{R,\kappa}^2 + c\,\mathtt{data_f}.$$

6.1 *Numerical examples*

A variable reaction coefficient is used to study the influence of dominating reaction on the a posteriori error estimates. The following numerical test is studied in this section.

iii) *Problem with variable reaction term*
We consider problem (1.1) with $\tau > 0$, $\varepsilon \equiv 1$ and $f \equiv 0$ on $(0,1)^2$. The boundary condition is given by

$$g(x,y) = \begin{cases} \sin(\pi x) & \text{if } y = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding exact solution is then given by

$$u(x,y) = \frac{\sin(\pi x)\sinh(\sqrt{\tau+\pi^2}y)}{\sinh(\sqrt{\tau+\pi^2})} \in C^\infty(\overline{\Omega}).$$

A sequence of unstructured meshes is considered. A similar numerical test is used in (Romkes et al., 2003).

In the following tables we reuse the definition of the efficiency indexes eff_η and eff_{osc} of (5.10) and of the comparison coefficient $c_{\eta,\text{osc}}$ of (5.11). Due to the non-homogeneous boundary conditions we define

$$\texttt{data}_{\texttt{f},\texttt{g}} = \tau \left(\|\varepsilon^{-1}\text{h}^2 f\|^2_{\mathscr{K}} + \|\text{h}^{\frac{3}{2}} g\|^2_{\mathscr{F}_e} \right)^{\frac{1}{2}}.$$

Table 4 illustrates the efficiency of the error indicator η_κ and the oscillation indicator osc_κ for different values of $\tau = 1, 100, 1000$ for the test problem iii). For $\tau = 1$, the quantity $\texttt{data}_{\texttt{f},\texttt{g}}$ is in this case small when compared to the energy-error and thus does not affect the efficiency of the estimates. The results are similar to the ones of the pure diffusion case discussed in the previous section.

For $\tau = 100$ and $\tau = 1000$, due to the relative large value of the reaction coefficient, compared to a diffusion coefficient of one, the quantity $\texttt{data}_{\texttt{f},\texttt{g}}$ affects the efficiency of the estimates for coarse meshes. For finer meshes, as the quantity $\texttt{data}_{\texttt{f},\texttt{g}}$ becomes of comparable size as the energy error, the estimators converge to a efficiency of around $0.7 \sim 0.8$ which was the standard value in the previous examples. This behaviour matches with the theoretical results of Theorem 6.2.

7. Conclusion

In this work, we have proposed and analysed efficient and reliable a posteriori energy-norm error estimates for the Bubble Stabilized Discontinuous Galerkin (BSDG) method applied to the diffusion-reaction equation in two and three spatial dimensions. Two estimators are presented. The first one consists of the classical residual quantities as used for standard discontinuous Galerkin methods. The second one consists of the oscillation of the right hand side and the ”oscillation” of the jump of the solution across faces. For both estimators, upper and lower bounds are established and for resolved data, the lower bound turns out to hold without constant. Although no explicit control of the constants is given, both estimators behave similarly and are surprisingly stable with respect to variations of the mesh size, the problem and the variation of coefficients in the numerical tests. For high reaction coefficients however, the efficiency may be perturbed by underresolved data. This is illustrated on a theoretical and numerical level.

	h	$\|\|\|u-u_h\|\|\|_{\mathtt{e}}$	eff_η	$\mathrm{eff}_{\mathrm{osc}}$	$c_{\eta,\mathrm{osc}}$	$\mathtt{data}_{\mathtt{f},\mathtt{g}}$
$\tau=1$	0.1	2.41E-01	0.71335	0.71332	1.00004	2.24E-02
	0.05	1.24E-01	0.72118	0.72118	1.00001	7.91E-03
	0.025	6.14E-02	0.72389	0.72389	1.00000	2.80E-03
	0.0125	3.09E-02	0.72599	0.72599	1.00000	9.88E-04
$\tau=100$	0.1	5.71E-01	0.99257	0.96146	1.03235	2.236
	0.05	3.06E-01	0.79819	0.78346	1.01880	0.791
	0.025	1.51E-01	0.73899	0.73612	1.00389	0.280
	0.0125	7.69E-02	0.72691	0.72620	1.00098	0.099
$\tau=1000$	0.1	1.97E+00	2.87936	2.84151	1.01332	22.361
	0.05	1.25E+00	1.51674	1.43952	1.05364	7.906
	0.025	6.40E-01	0.92276	0.90437	1.02033	2.795
	0.0125	3.27E-01	0.78177	0.77584	1.00764	0.988

Table 4. Effectivity of a posteriori error estimators for test problem iii) with $\tau=1,100,1000$ for different mesh sizes as well as the quantity $\mathtt{data}_{\mathtt{f},\mathtt{g}}$.

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Appendix (Detailed proof of Lemma 2.4)

Proof. Let us first establish the a priori estimate. Observe that

$$\|\varepsilon^{\frac12}\mathtt{h}^{-1}\phi_h\|^2_{\mathscr{K}} \leqslant 2\,\|\varepsilon^{\frac12}\mathtt{h}^{-1}\pi_0\phi_h\|^2_{\mathscr{K}} + 2\,\|\varepsilon^{\frac12}\mathtt{h}^{-1}(\phi_h-\pi_0\phi_h)\|^2_{\mathscr{K}} \leqslant 2\|\varepsilon^{\frac12}\mathtt{h}^{-1}a_h\|^2_{\mathscr{K}} + c_\star\,\|\varepsilon^{\frac12}\nabla\phi_h\|^2_{\mathscr{K}} \quad (7.1)$$

and that

$$|||\phi_h|||^2 = \|\varepsilon_{\mathtt{F}}^{\frac12}\mathtt{h}^{-\frac12}[\phi_h]\|^2_{\mathscr{F}} + \|\varepsilon^{\frac12}\nabla\phi_h\|^2_{\mathscr{K}} \leqslant 2\,\|\varepsilon_{\mathtt{F}}^{\frac12}\mathtt{h}^{-\frac12}([\phi_h]-\overline{[\phi_h]})\|^2_{\mathscr{F}} + 2\,\|\varepsilon_{\mathtt{F}}^{\frac12}\mathtt{h}^{-\frac12}\overline{[\phi_h]}\|^2_{\mathscr{F}} + \|\varepsilon^{\frac12}\nabla\phi_h\|^2_{\mathscr{K}}.$$

Using further the Bremble-Hilbert lemma, the trace and inverse inequalities and the boundedness of ε over faces yields

$$\|\varepsilon_{\mathtt{F}}^{\frac12}\mathtt{h}^{-\frac12}([\phi_h]-\overline{[\phi_h]})\|_{\mathscr{F}} \leqslant c\,\|\varepsilon_{\mathtt{F}}^{\frac12}\mathtt{h}^{\frac12}[\nabla\phi_h]_t\|_{\mathscr{F}} \leqslant c\,\|\varepsilon^{\frac12}\nabla\phi_h\|_{\mathscr{K}}$$

where $[\cdot]_t$ stands for the tangential jump defined by $[\nabla\phi_h]_t = \nabla\phi_h|_{\kappa_1}\times n_1 + \nabla\phi_h|_{\kappa_2}\times n_2$. By the previous two equations, integration by parts and equation (2.1) it follows that

$$\begin{aligned} c\,|||\phi_h|||^2 &\leqslant \|\varepsilon_{\mathtt{F}}^{\frac12}\mathtt{h}^{-\frac12}\overline{[\phi_h]}\|^2_{\mathscr{F}} + \|\varepsilon^{\frac12}\nabla\phi_h\|^2_{\mathscr{K}} \\ &= \|\varepsilon_{\mathtt{F}}^{\frac12}\mathtt{h}^{-\frac12}b_h\|^2_{\mathscr{F}} - (\nabla\cdot\varepsilon\nabla\phi_h,\pi_0\phi_h)_{\mathscr{K}} + (\{\varepsilon\nabla\phi_h\}\cdot n_F,\overline{[\phi_h]}\cdot n_F)_{\mathscr{F}} + ([\varepsilon\nabla\phi_h],\overline{\{\phi_h\}})_{\mathscr{F}_i} \\ &= \|\varepsilon_{\mathtt{F}}^{\frac12}\mathtt{h}^{-\frac12}b_h\|^2_{\mathscr{F}} \underbrace{-(\nabla\cdot\varepsilon\nabla\phi_h,a_h)_{\mathscr{K}}}_{I} + \underbrace{(\overline{[\phi_h]}\cdot n_F,b_h)_{\mathscr{F}}}_{II} + \underbrace{([\varepsilon\nabla\phi_h],c_h)_{\mathscr{F}_i}}_{III} \end{aligned}$$

since $\nabla \cdot \varepsilon \nabla \phi_h \in V_h^0$ and $\varepsilon \nabla \phi_h \cdot n_\kappa$ is constant along faces. Applying the Cauchy-Schwarz, the inverse (I) or the trace (II, III) and then Young's inequality for each term yields respectively

$$
\begin{aligned}
I &\leqslant c_I \|\varepsilon^{\frac{1}{2}} \nabla \phi_h\|_{\mathscr{K}} \|\varepsilon^{\frac{1}{2}} \mathtt{h}^{-1} a_h\|_{\mathscr{K}} \leqslant \tfrac{1}{4} \|\varepsilon^{\frac{1}{2}} \nabla \phi_h\|_{\mathscr{K}}^2 + c_I^2 \|\varepsilon^{\frac{1}{2}} \mathtt{h}^{-1} a_h\|_{\mathscr{K}}^2 \\
II &\leqslant \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} \overline{[\phi_h]}\|_{\mathscr{F}} \|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} b_h\|_{\mathscr{F}} \leqslant c_T \|\varepsilon^{\frac{1}{2}} \mathtt{h}^{-1} \phi_h\|_{\mathscr{K}} \|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} b_h\|_{\mathscr{F}} \\
&\leqslant c_\star \delta \|\varepsilon^{\frac{1}{2}} \nabla \phi_h\|_{\mathscr{K}}^2 + 2\delta \|\varepsilon^{\frac{1}{2}} \mathtt{h}^{-1} a_h\|_{\mathscr{K}}^2 + c \|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} b_h\|_{\mathscr{F}}^2, \\
III &\leqslant c_T \|\varepsilon^{\frac{1}{2}} \nabla \phi_h\|_{\mathscr{K}} \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} c_h\|_{\mathscr{F}_i} \leqslant \tfrac{1}{4} \|\varepsilon^{\frac{1}{2}} \nabla \phi_h\|_{\mathscr{K}}^2 + c_T^2 \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} c_h\|_{\mathscr{F}_i}^2,
\end{aligned}
$$

where the constant δ can be chosen sufficiently small and using again the boundedness of ε over faces. Thus, combining with (7.1), yields

$$
\|\varepsilon^{\frac{1}{2}} \mathtt{h}^{-1} \phi_h\|_{\mathscr{K}}^2 + |||\phi_h|||^2 \leqslant c \left(\|\varepsilon^{\frac{1}{2}} \mathtt{h}^{-1} a_h\|_{\mathscr{K}}^2 + \|\varepsilon_{\mathtt{F}}^{-\frac{1}{2}} \mathtt{h}^{\frac{1}{2}} b_h\|_{\mathscr{F}}^2 + \|\varepsilon_{\mathtt{F}}^{\frac{1}{2}} \mathtt{h}^{-\frac{1}{2}} c_h\|_{\mathscr{F}_i}^2 \right).
$$

To conclude the proof, it now suffices to observe that (2.7) is nothing more than a square linear system of size $N_{\mathscr{K}} + N_{\mathscr{F}} + N_{\mathscr{F}_i} = (d+2)N_{\mathscr{K}}$, where $N_{\mathscr{K}}$, $N_{\mathscr{F}}$, $N_{\mathscr{F}_i}$ denotes respectively the number of elements, faces and interior faces. Hence, existence and uniqueness of a solution of the linear system are equivalent. Let us denote by $Aw = b$ the square linear system and assume that there is a vector w_1 and w_2 such that $Aw_i = b$, $i = 1, 2$. Further let us denote the difference between them by $e = w_1 - w_2$ and therefore $Ae = 0$. The a priori estimate (2.8) implies that $e = 0$ and thus the solution is unique and hence the matrix is regular. □

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