APMA 1650/1655 Homework 2

February 15, 2016

Due before class on Friday, Feb. 19th. It can be dropped off in the APMA 1650 homework box on the first floor of the APMA department, 182 George St OR at class (before it starts) on Friday.

APMA 1650: Complete all unstarred problems.
APMA 1655: Complete all unstarred and single starred problems.
Double starred problems are particularly challenging and I do not neccessarily expect you to answer them to completion. You should work on them and write up your (partial) solutions/attempts/computations, etc.

Show all work!

1. Two people play a fun game! They take turns flipping a fair coin, the first to flip H wins. A round constitutes just one flip (only one person gets to "go".)

   (a) What is the probability that the player who goes first wins?

   (b) What is the probability that the game lasts at least 4 rounds?

   (c) Given they have played 4 rounds, what is the probability that the game lasts at least 8 rounds?

   (d) * Suppose that the coin is biased with the probability of Heads equal to p. Suppose the game lasts exactly 4 rounds. What value of p maximizes the chances of the game lasting exactly 4 rounds? For k rounds?

   (e) ** Suppose now that player 1 wins the game if the sequence HHT appears before the sequence HTT and player 2 wins if HTT appears before HHT. What is the probability that player 1 wins the game?

2. Suppose a drawer contains x white socks and y black socks. We are going to pack socks as follows: (1) Pick out a sock equally likely at random from all socks in the drawer and put it in your suitcase. (2) Pick out a sock equally likely at random from the drawer. If its color is the same as the last sock chosen then put in the suitcase and repeat from step (2). If its color is not the same, put it back in the drawer and repeat from step (1).

   Find the probability that the last sock left in the drawer is white if:

   (a) $x = y = 1$ (there are only two socks in the drawer to begin with)

   (b) $x = 2$ and $y = 1$

   (c) * For any value of x and y

3. (A real-life scenario) There are 6 TAs in our class. Sitting around the kitchen table, they must grade 200 homeworks each Friday. Karen (one of the TAs) places 30 stickers on random homeworks using the uniform distribution.

   (a) You pick up 5 homework assignments (for yourself and four friends). What is the probability that there are at least 3 stickers among them?

   (b) What is the number of expected stickers among your 5 homework asssignments?

   (c) The next week, Clark is in charge of the 30 stickers. On the new week's homeworks, he sees that 20% of them are for 1655 students. He assigns 70% of the stickers to the 1655 homeworks (uniformly at random) and the remainder of the stickers onto the 1650 homeworks (uniformly at random). Given that your homework has a sticker, what is the chance that you are a 1655 student?

   (d) The following week, Will is in charge of the 30 stickers. (He secretly likes stickers and is loath to give them out.) He rolls a 10-sided fair die for every homework he grades. If the die's outcome is a "10", he then gives a sticker. How many homeworks does he expect to grade before he gives his first sticker?

   (e) Will eats all the stickers he doesn't place on homeworks. He only grades 50 homeworks. What is the expected number of stickers he eats? (Consider as negligible the event that he places all his 30 stickers before he finishes grading the 50 homeworks.)

4. Let A, B, C be events of a common probability space.

   (a) Prove the following equation holds:

$$P(A \cup B|C) = P(A|C) + P(B|C) - P(A \cap B|C)$$

   (b) Give an example that shows the following does not generally hold:

$$P(A|B \cup C) = P(A|B) + P(A|C) - P(A|B \cap C)$$

5. (a) Suppose that A and B are mutually exclusive events with $P(A) > 0$. Are A and B independent? State your reasons.

   (b) Suppose that $A \subset B$ and that $P(A) > 0$. Are A and B independent? If not, give a specific example where they are independent.