Techniques of Integration

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1 Integration Tools

In many ways, the operation of differentiation can be condensed into a black box: once all the rules are known, one can simply create an algorithm to analyze any expression and apply those rules as necessary to obtain the desired result. In most cases, no creativity or ingenuity is needed to perform differentiation. However, integration is quite the opposite. Every integral is different and requires a different combination of techniques and methods to derive the answer. In essence, integration is an art.

In all that follows, we shall omit including the constant of integration when giving expressions for the antiderivatives, but the reader should always be mindful of its implied presence.

1.1 The Basics

For completeness, we shall include the basic integration formulas including the inverse power rule, and some basic trigonometric integrals.

$$\int x^n \,\mathrm{d}x = \frac{1}{n+1}x^{n+1} \qquad (\mathrm{n} \neq -1) \tag{1}$$

$$\int \sin x \,\mathrm{d}x = -\cos x \tag{2}$$

$$\int \cos x \,\mathrm{d}x = \sin x \tag{3}$$

$$\int \ln x \,\mathrm{d}x = \frac{1}{x} \tag{4}$$

1.2 u-Substitution

The method of u-substitution is a change of variables in an attempt to transform the integrand to something that is recognizably integrable. It is here where we first see that integration requires some adroitness. The method of u-substitution can be summarized as follows:

$$\int f(x)\,\mathrm{d}x = \int f(u)\frac{\mathrm{d}x(u)}{\mathrm{d}}\,\mathrm{d}u \tag{5}$$

1. Select of function of x (normally a portion of the integrand) to mold into a new variable u. I.e. make a definition of a new variable u in terms of x. This requires some foresight in that we hope to also find $u'(x)$ in the integrand so that we can complete u-substitution. (see step 4.)
2. Calculate $\mathrm{d}u = \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$
3. Replace the expression $u(x)$ in the integrand with the variable u wherever appropriate.
4. Identify $\frac{\mathrm{d}u}{\mathrm{d}x}$ in the integrand and replace $\frac{\mathrm{d}u}{\mathrm{d}x}dx$ with du. This is often the stumbling point in u-substitution.

Hopefully, at the end of these steps, we end up with an integral that we know how to compute. Just as a picture is worth a thousand words, so much more valuable an example is than the theory:

$$\int \frac{\ln x}{x}\,\mathrm{d}x \tag{6}$$

In order to compute this integral, we notice that it is not an integral of any basic form and so we do not have any rules that we can immediately apply to determine the answer. However, after staring at the page for a minute or two, we may finally realize that $\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$ so if we define $u(x)$ to be $\ln x$, then $u'(x)$ is in the integrand and we can complte the steps in u-substitution. Putting it all down on paper we have:

$$\begin{aligned} u(x) &= \ln x \\ \mathrm{d}u &= \frac{1}{x}\mathrm{d}x \end{aligned}$$

$$\begin{aligned}\int \frac{\ln x}{x}\,\mathrm{d}x &= \int u\left(\frac{1}{x}\,\mathrm{d}x\right)\\ &= \int u\,\mathrm{d}u\\ &= \frac{1}{2}u^2\\ &= \frac{1}{2}\left(\ln x\right)^2\end{aligned}$$

1.3 Properties and Identities of Mathematical Functions

There are many situations in which very nasty-looking integrals actually have relatively simple solutions because the integrand can be greatly simplified by using identities of functions. For example,

$$\int \frac{2\cos^2 x - \cos 2x - 1}{(1+\cos 2x)\sin^2 x}\,\mathrm{d}x$$

may look like a relatively complicated integral at first glance. However, by using some trigonometric identities, we can see that in fact the integrand reduces to 1, and so this integral is trivially easy to compute. While the above example may be an extreme case of simplification, many seemingly untractable problems become quite manageable after some applications of such identities. Below are a few examples of identities and properties to keep in mind while trying to integrate expressions.

1. $\sin^2 x + \cos^2 x = 1$
2. $\sin 2x = 2\sin x\cos x$
3. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
4. $\sin\left(\alpha \pm \beta\right) = \left(\sin\alpha\cos\beta\right) \pm \left(\cos\alpha\sin\beta\right)$
5. $\cos\left(\alpha \pm \beta\right) = \left(\cos\alpha\cos\beta\right) \mp \left(\sin\alpha\sin\beta\right)$
6. $e^{a+b} = e^a e^b$

7. $e^{ab} = (e^a)^b$
8. $\log ab = \log a + \log b$
9. $\log a^b = b \log a$
10. $\log_a b = \frac{\ln b}{\ln a}$
11. $a^x = e^{x \ln a}$

1.4 Integration by Parts

There are some integrals that cannot be dealt with by any of the means dictated so far. Actually, there are many such integrals. A very powerful tool is the integration by parts formula. The one dimensional case can be derived from the product rule:

$$(uv)' = u'v + uv'$$
$$\int (uv)' \, \mathrm{d}x = \int u'v \, \mathrm{d}x + \int uv' \, \mathrm{d}x$$

Rearranging terms, we obtain

$$\int uv' \, \mathrm{d}x = uv - \int u'v \, \mathrm{d}x \tag{7}$$

The power of this method can be seen in the following way: suppose we are trying to integrate a product of two functions, where one function is the derivative of something (e.g. v'). Then using this method, we essentially transfer the derivative onto the other function u and supplement such an action with a correction term, uv. This operation of "transferring derivatives" is indeed very useful in integrating a great many things. In order to illustrate the idea, we consider an example.

$$\int x \sin x \, \mathrm{d}x$$

No previous techniques are applicable to this example. However, let us consider using integration by parts. We want to "transfer" a derivative from one term onto another. We notice that $f(x) = x$ is the derivative of $\frac{1}{2}x^2$ and that $g(x) = \sin x$ is the derivative of $-\cos x$. If we transfer a derivative from $f(x)$ to $g(x)$, we will end up trying to integrate something proportional to $x^2 \cos x$, which is hardly better than we started off with. However, if instead we do the opposite, we will only have to integrate $\cos x$, which is quite within our capabilities. We then proceed as follows:

$$\begin{aligned} u(x) &= x & v'(x) &= \sin x \\ u'(x) &= 1 & v(x) &= -\cos x \end{aligned}$$

$$\begin{aligned} \int x \sin x \, \mathrm{d}x &= uv - \int uv' \, \mathrm{d}x \\ &= -x \cos x + \int \cos x \, \mathrm{d}x \\ &= -x \cos x + \sin x \end{aligned}$$

1.5 Trigonometric Substitution

When doing u-substitution, we were looking to assign a more complicated function of x a very compact name (namely u). In this form of substitution, we want to expand the variable x from itself into a trigonometric function. The motivation for this is as follows: suppose we wanted to integrate something like $\frac{1}{\sqrt{1-x^2}}$. None of the tricks we have learned so far enable us to do this. But what if the x^2 in the denominator were a $\sin^2 u$ instead? Then we could use our trigonometric identity $1 - \sin^2 x = \cos^2 x$ to get rid of the square root sign and integrate the result. We now present the mathematics of this substitution:

$$x = \sin u \qquad \Longrightarrow \qquad \mathrm{d}x = \cos u \, \mathrm{d}u$$

Then our integral transforms as follows

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} &= \int \frac{\cos u \,\mathrm{d}u}{\sqrt{1-\sin^2 u}} \\
&= \int \frac{\cos u \,\mathrm{d}u}{\sqrt{\cos^2 u}} \\
&= \int \mathrm{d}u \\
&= u \\
&= \arcsin x
\end{aligned}
$$

A trigonometric substitution allows us to deal with terms of the form $a - bx^2$, $bx^2 - a$, and $a + bx^2$ which are under a square root. The idea is that we wish to coalesce the two terms into one and then get rid of the square root. In certain cases we are even able to use trigonometric substitution to deal with higher powers of x that may be raised to a fractional power other than $\frac{1}{2}$.

1.6 Partial Fractions

Perhaps one of most difficult of the common techniques of integration to master is the method of partial fractions. This method is almost exclusively used in the cases of integrating rational polynomials, i.e. integrands of the form $\frac{P(x)}{Q(x)}$. Partial fractions is used to split up the integrand. If the degree of the denominator is greater than 1, we may have trouble integrating it depending on the degree of the numerator. Thus, we use partial fractions to split up a single rational polynomial with a non-unity degree denominator into multiple rational polynomials with more tractable denominators. Consider the following situations:

- Deg[Q(x)]=1, Deg[P(x)]=0: In this case we can simply use the fact that the integral of $\frac{1}{x}$ is $\ln x$

- Deg[Q(x)]=1, Deg[P(x)]>0: Here we can employ long division to break this up into a regular polynomial plus a rational polynomial of the previous form

- Deg[Q(x)]>1, Deg[Q(x)]>Deg[P(x)]: This case is the classic partial fractions situation. Here we can use partial fractions to split this rational polynomial up into multiple rational polynomials that have lower-degree denominators that we can deal with individually.

- Deg[Q(x)]>1, Deg[Q(x)]<=Deg[P(x)]: First we must use long division to obtain a regular polynomial and a rational polynomial that we perhaps need to use partial fractions to split up further.

Let us illustrate an example of the third type. We seek to calculate

$$\int \frac{1}{x^2+3x+2}\,\mathrm{d}x \tag{8}$$

We notice that we cannot really use any of the techniques we've learned before to integrate this. But we can use partial fractions to manage this rational polynomial. We begin by factoring the denominator and then making an educated *guess* as to what form this rational polynomial may split up into:

$$\frac{1}{x^2+3x^2} = \frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \tag{9}$$

We have introduced constants A and B, which we need to determine. Notice that *if* it is possible to find such constants A and B that satisfy the above equality, we can easily integrate the two terms on the right hand side. But how do we determine the constants A and B? Let us rearrange (9) by multiplying the entire expression by the denominator $(x+1)(x+2)$:

$$1 = A(x+2) + B(x+1) \tag{10}$$

We now have a simpler algebraic relation which we can use to determine the constants A and B. Equation (10) must hold for all values of x and because of this, we can match coefficients of like terms. That is, all the terms multiplying x on the left hand side must sum to match the sum of all the terms multiplying x on the right hand side, and all the constant terms on the left must sum to equal the sum of the constants terms on the right hand side. On the left hand side of (10) we have the polynomial $0x+1$ and on the right hand side we have $(A+B)x+(2A+B)$. From this we derive two linear equations for A and B:

$$0 = A + B \qquad\qquad\qquad 1 = 2A + B$$

From these equations we determine that $A = 1$ and $B = -1$. We have thus been able to find constants A and B so that relation (9) is valid. We can now rewrite (8) as follows:

$$\begin{aligned}\int \frac{1}{x^2 + 3x + 2}\,\mathrm{d}x &= \int \left(\frac{1}{x+1} - \frac{1}{x+2}\right)\mathrm{d}x \\ &= \int \frac{\mathrm{d}x}{x+1} - \int \frac{\mathrm{d}x}{x+2} \\ &= \ln(x+1) - \ln(x+2) \\ &= \ln\left(\frac{x+1}{x+2}\right)\end{aligned}$$

The modus operandi for partial fractions is pretty clear and straightforward once one is able to determine a reasonable guess for breaking up the fraction (like in (9)). This step is the crucial step in the method. The key is that you want to have as many unknowns as you have equations. If, after getting to an equation like (10), we find too many equations or too many unknowns, then we should back up to equation (9) to augment our guess. Consider the following example:

$$\int \frac{-2\,\mathrm{d}x}{x^3 + x}$$

As before, we seek a guess for how to break up the fraction:

$$\frac{-2}{x^3 + x} = \frac{A}{x} + \frac{B}{x^2 + 1}$$

$$\implies -2 = A(x^2 + 1) + Bx$$

We now notice from the above equation that we have a quadratic polynomial equation, and thus we will have three constraint equations (one for the x^2 term, one for the x term, and a final one from the constant terms).

However, we have only specified two unknowns, A and B. Therefore, unless we are extremely lucky, we shall not be able to break up the fraction as we have guessed. We therefore augment our guess as follows:

$$\frac{-2}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\Longrightarrow -2 = A(x^2+1) + (Bx+C)x$$

$$A = -2 \qquad B = 2 \qquad C = 0$$

And now we may successfully perform the integration we wish to accomplish.

1.7 Miscellaneous Techniques

There are many situations in which there simply are not any standard techniques or methods for integration. Sometimes one just has to be a bit clever. This section is devoted to such situations, and to other minor techniques which are deemed to be deserving of notice. We start with some integration results that are perhaps important enough to memorize:

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x \tag{11}$$

$$\int \frac{-\,\mathrm{d}x}{\sqrt{1-x^2}} = \arccos x \tag{12}$$

$$\int \frac{\mathrm{d}x}{1+x^2} = \arctan x \tag{13}$$

These formulas can be derived either by implicit differentiation on the functions $\arcsin x$, $\arccos x$, and $\arctan x$, or by trigonometric substitution. In any case, these formulas can perhaps save some time when one happens across an integral of the above forms.

We illustrate other techniques via examples. Using the identity $\sin(\alpha x \pm \beta x) = \sin(\alpha x)\cos(\beta x) \pm \sin(\beta x)\cos(\alpha x)$

$$\begin{aligned}\int \sin(\alpha x)\cos(\beta x)\,\mathrm{d}x &= \frac{1}{2}\int \left[\sin((\alpha+\beta)x) + \sin((\alpha-\beta)x)\right]\,\mathrm{d}x \\ &= -\frac{1}{2(\alpha+\beta)}\cos((\alpha+\beta)x)) - \frac{1}{2(\alpha-\beta)}\cos((\alpha-\beta)x)\end{aligned}$$

Another useful technique is completing the square. Consider the following integral and the solution method:

$$\begin{aligned}\int \frac{1}{x^2+x+1}\,\mathrm{d}x &= \int \frac{\mathrm{d}x}{(x^2+x+{}^1\!/_4)+{}^3\!/_4} \\ &= \int \frac{\mathrm{d}x}{(x+{}^1\!/_2)^2+{}^3\!/_4} \\ &= \int \frac{\frac{\sqrt{3}}{2}\,\mathrm{d}u}{{}^3\!/_4u^2+{}^3\!/_4} \qquad \left(u = \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) \\ &= \frac{2}{\sqrt{3}}\arctan u \\ &= \frac{2}{\sqrt{3}}\arctan\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]\end{aligned}$$

1.8 Exercises

Integrate the following expressions:

1. $\displaystyle\int \frac{\mathrm{d}x}{x\ln\left(\frac{1}{x}\right)}$

2. $\displaystyle\int \arctan x \,\mathrm{d}x$

3. $\displaystyle\int \ln x \,\mathrm{d}x$

4. $\displaystyle\int \frac{\mathrm{d}x}{x\sqrt{x^4-1}}$

5. $\displaystyle\int \frac{\mathrm{d}x}{\arcsin(x)\sqrt{1-x^2}}$

6. $\displaystyle\int \frac{3\,\mathrm{d}x}{x^2-5x+6}$

7. $\displaystyle\int \frac{x\,\mathrm{d}x}{x^2-5x+6}$

8. $\displaystyle\int \frac{4x^3\,\mathrm{d}x}{x^2-4}$

9. $\displaystyle\int \frac{\mathrm{d}x}{1-x^2}$

10. $\displaystyle\int xe^x \,\mathrm{d}x$

11. $\displaystyle\int \sqrt{1+x^{\frac{2}{3}}}\,\mathrm{d}x$

12. $\displaystyle\int \frac{\sin(2x)\,\mathrm{d}x}{1+\sin^2 x}$

13. $\displaystyle\int \frac{\mathrm{d}x}{x^2+4}$

14. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}$