The Error Function

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The error function is defined as

$$\operatorname{erf} x \equiv \int_0^z f(x)\,\mathrm{d}x = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2}\,\mathrm{d}x \tag{1}$$

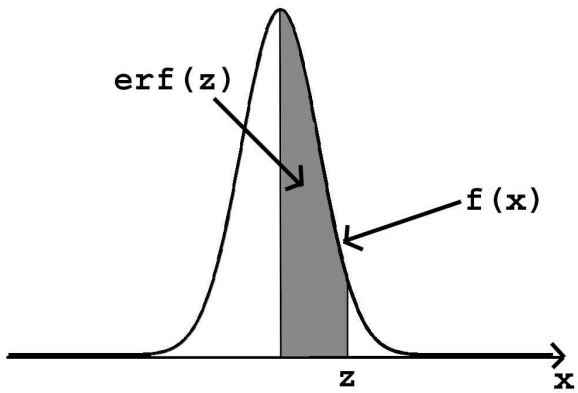

Figure 1: Visual interpretation of the error function

One way to interpret the definition of the erf function is that it represents the area under the curve of $f(x)$ from 0 to z. Of course, this is the same as the integral of $f(x)$, but the analytic form of the antiderivative of e^{-x^2} is unknown, so we simply assign the function $erf(z)$ to represent a scaled version of the that antiderivative. In addition to the error function, we can define the complementary error function

$$\operatorname{erfc} z = 1 - \operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-x^2}\,\mathrm{d}x \tag{2}$$

Common properties of the erf and erfc functions include:

$$\begin{aligned}
\operatorname{erf}(-z) &= -\operatorname{erf} z & \operatorname{erf} 0 &= 0 & \operatorname{erf} \infty &= 1 & \operatorname{erf}(-\infty) &= -1 \\
\operatorname{erfc}(-z) &= 2 - \operatorname{erfc} z & \operatorname{erfc} 0 &= 1 & \operatorname{erfc} \infty &= 0 & \operatorname{erfc}(-\infty) &= 2 \\
\operatorname{erf} z + \operatorname{erfc} z &= 1
\end{aligned}$$

As an example, we now consider solving the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{2}}{\pi} e^{-x^2} \qquad\qquad y(x_0) = y_0$$

We proceed in the straightforward manner by integrating both sides with respect to x and then rewriting the result:

$$\begin{aligned}
y(x) - y(x_0) &= \int_{x_0}^{x} \frac{\sqrt{2}}{\pi} e^{-t^2}\,\mathrm{d}t \\
y(x) &= \int_{0}^{x} \frac{\sqrt{2}}{\pi} e^{-t^2}\,\mathrm{d}t + \int_{x_0}^{0} \frac{\sqrt{2}}{\pi} e^{-t^2}\,\mathrm{d}t + y_0 \\
&= y_0 + \operatorname{erf} x - \operatorname{erf} x_0
\end{aligned}$$