The Definite and Indefinite Integrals

September 23, 2004

1 The Integral

1.1 Indefinite Integrals

The Indefinite integral, often written as

$$\int f(x)dx \tag{1}$$

should be interpreted as the inverse of differentiation. Equation (1) tells you to find the function of x which, upon differentiation, yields $f(x)$ (i.e. to find the "antiderivative" of $f(x)$). The dx notation is there to remind you that you are "antidifferentiating" with respect to the variable x, and should ignore any other variables. For example,

$$\int (x^2 + y^2)dy = yx^2 + \frac{y^3}{3} + C \tag{2}$$

where C is any "constant". Because we are integrating with respect to the variable y, the variable x is held constant, so C may very well be a function of x. You can check that by taking the partial derivative with respect to y of the right hand side of (2), we obtain the integrand on the left hand side.

Integration is often much more difficult than differentiation becuase while differentiation has concrete rules describing steps to take, integration is more of an art form and there exist many examples of functions whose analytic antiderivatives do not have a closed form expression.

1.2 Definite Integrals

The Definite integral is often written

$$\int_a^b f(x)dx \tag{3}$$

This should be interpreted as the area between the curve of $f(x)$ and the x-axis from $x = a$ to $x = b$. With this definition, take special note that while the notation in (3) and (1) is almost identical, **they explicitly have no relation whatsoever**.

The definite integral is the limit of a Riemannian sum (the Riemann sum is used to approximate the area under the curve). Computing the value of (3) can be extremely cumbersome and difficult, even for a very simple function $f(x)$.

2 The Fundamental Theorem of Calculus

The definition of the definite and indefinite integrals do not admit any immediate relationship between the two entities. The indefinite integral in (1) is a function of x, wheras the definite integral in (3) is a number. The two quantities are stitched together only by the Fundamental Theorem of Calculus, a special version of which states that if the function $f(x)$ has an antiderivative $F(x)$, then

$$\int_a^b f(x)dx = \left(\int f(x)dx\right)\Bigg|_a^b = F(b) - F(a) \tag{4}$$

Another statement of the Fundamental Theorem of Calculus is that

$$G(x) = \int_a^x f(t)dt \tag{5}$$

is an antiderivative of $f(x)$. Note that here is the first relation between an indefinite integral and a definite one. It is not too surprising that definite integrals on e.g. homework assignments are frequently treated as if they are simply indefinite integrals evaluated at the limits. While this is technically true, one should realize that the definite integral is a statement about areas under curves, and that the Fundamental Theorem of Calculus is what allows us to associate that area under the curve with an antiderivative, or an indefinite integral.

Of course this view is quite cumbersome to carry along for every definite integral we have to compute, and so the above rumination is frequently discarded immediately as superflouous. However, the examples in the next section will illustrate the value of understanding the difference between a definite and an indefinite integral.

Other, more general statements of the Fundamental Theorem of Calculus include:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(t)\,\mathrm{d}t = f(b(x))b'(x) - f(a(x))a'(x) \tag{6}$$

$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(x,t)\,\mathrm{d}t = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,t)\,\mathrm{d}t \tag{7}$$

One last note is that there is often some notational confusion with integrals. For example, the expression

$$F(x) = \int_a^b f(x)dx \tag{8}$$

seems to be a perfectly logical statement since the variable x appears on both sides of the equation. However, remember that the expression on the right hand side is actually just the area under a curve. The variable x in the integrand is just a "dummy" variable, and in fact, we could just as easily state that

$$F(x) = \int_a^b f(t)dt = \int_a^b f(\clubsuit)d\clubsuit \tag{9}$$

The bottom line is that it doesn't matter what letter or symbol one assigns to be the dummy variable. However, one caveat is illustrated by the following expression:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b f(x)\,\mathrm{d}x \tag{10}$$

As we've learned, we can rename the dummy variable of integration however, we wish. If we choose to do so, then we run into the following problem: which of the following expressions does (10) refer to?

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b f(t)\,\mathrm{d}t \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}x}\int_a^b f(x)\,\mathrm{d}t$$

In other words, in (10), is the function f a function of the dummy variable of integration, or of the "external" variable x which we're taking the derivative with respect to? It does indeed matter since both interpretations give different results. However, the expression on the right is rarely the correct interpretation. In fact, if one runs into an expression like (10), 99 times out of 100, it will refer to the expression on the left (note that (8) adopts this convention as well). The accepted convention is that if the variable of integration is x, then every x that appears in the integrand is also the variable of integration.

3 Exercises

In the following exercises, let f be a function, perhaps of multiple variables, and let F denote its antiderivative (with respect to whichever variable is appropriate). Let f be sufficiently smooth so that all its derivatives exist. Compute the following: (answers on the following page)

1. $\frac{d}{dx}\int_a^b f(x)\,\mathrm{d}x$
2. $\int_a^b \frac{d}{dx}f(x)\,\mathrm{d}x$
3. $\frac{d}{dr}\int_a^r f(t)\,\mathrm{d}t$
4. $\frac{d}{du}\int_u^{u^2} f(z)\,\mathrm{d}z$
5. $\frac{d}{da}\int_a^b f(x)\,\mathrm{d}x$
6. $\int_a^x \frac{d}{dx}f(t)\,\mathrm{d}t$
7. $\frac{d}{dp}\int_s^p \frac{d}{dx}f(x)\,\mathrm{d}x$
8. $\frac{d}{dx}\int_a^x (f(c))^2\,\mathrm{d}c$
9. $\frac{\partial}{\partial t}\int_a^b f(x,t)\,\mathrm{d}x$
10. $\frac{\partial}{\partial s}\int_{s^2}^{s^3} f(s,x)\,\mathrm{d}x$

1. 0
2. $f(b) - f(a)$
3. $f(r)$
4. $2uf(u^2) - f(u)$
5. $-f(a)$
6. 0
7. $f'(p)$
8. $(f(x))^2$
9. $\int_a^b \frac{\partial f}{\partial t}(x,t)\,\mathrm{d}x$
10. $3s^2 f(s,s^3) - 2sf(s,s^2) + \int_{s^2}^{s^3} \frac{\partial f}{\partial s}(s,x)\,\mathrm{d}x$