Asymptotic Analysis of Large Scale Systems

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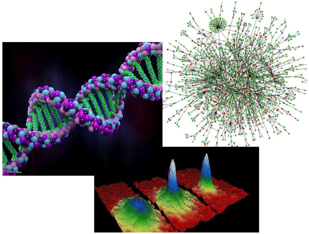
Large Scale Systems

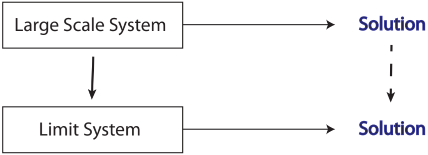
Large Scaling Systems Appear in

- Computer Science
- Biology
- Statistical Physics
- etc.



Common Characteristic: Hard to Analyze and Simulate

Asymptotic Analysis



History of Queueing Systems

- Johannsen, *Waiting Times and Number of Calls*, published in 1907 and reprinted in Post Office Electrical Engineers Journal, London, October, 1910.
- Erlang, A. K. , *The Theory of Probabilities and Telephone Conversations,* Nyt tidsskrift for Matematik, B, 20, 1909.

Many-Server Queues

Characterstics:

- Large Number of Servers
- General Service Distribution
- Heavy Traffic

Objectives

Quantities of Interest:

- Quality of Service parameters in Steady State
  - Probability that a customer has to wait upon arrival (α^N)
  - Average waiting time
  - etc.
- Service Costs

Input Parameters:

- Customer Arrival Rate
- Number of Servers (N, can be tuned)

Objective: Balance Between QoS and Costs

How should Number of Servers scale with Customer Arrival Rate to have $\alpha^N \in (0,1)$?

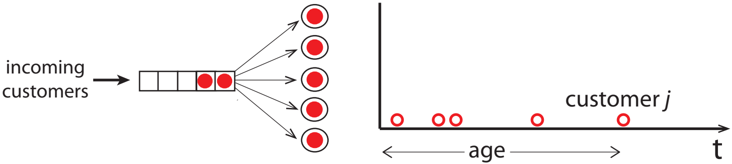
System State Representation

State Variable must contain

- Number of customers in system
- *Age* of each customer in service

Common Markovian State Space is Infinite Dimensional

A Measure Valued Representation



Problem Scheme

$$\hat{Y}^N(t)$$

Problem Scheme

$\hat{Y}^N(t)$

Dynamics of N-Server Queue

Problem Scheme

$$\hat{Y}^N(t) \longrightarrow \hat{\pi}^N$$

Problem Scheme

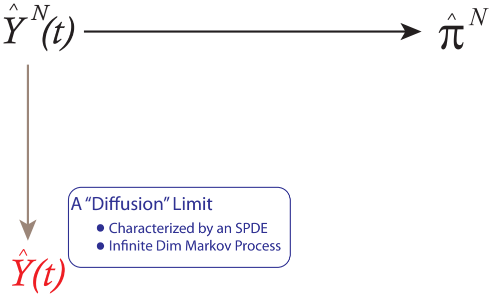
$$\hat{Y}^N(t) \longrightarrow \hat{\pi}^N$$

Stationary behavior for Large N

- Quantity of Interest
- Hard to Characterize

Problem Scheme

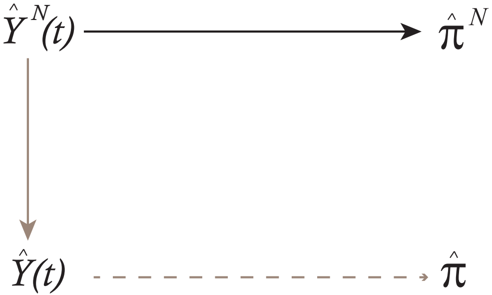
Problem Scheme



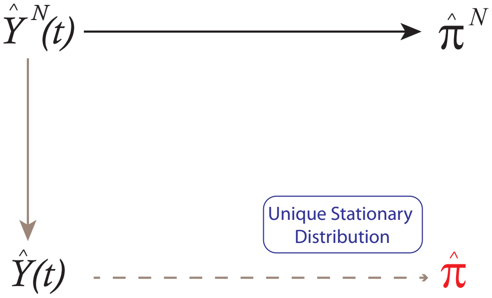
A "Diffusion" Limit

- Characterized by an SPDE
- Infinite Dim Markov Process

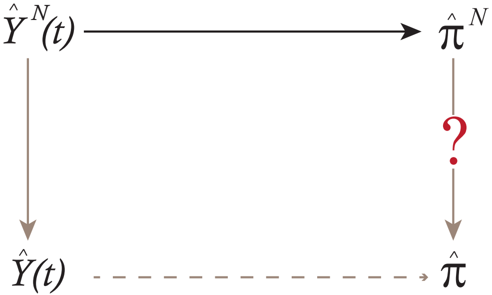
Problem Scheme



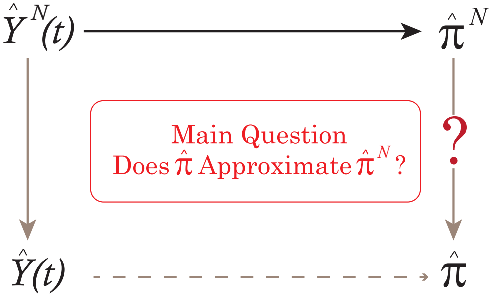
Problem Scheme



Problem Scheme



Problem Scheme



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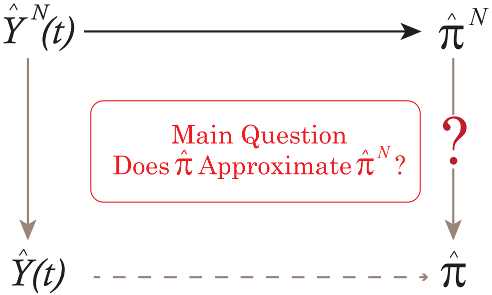
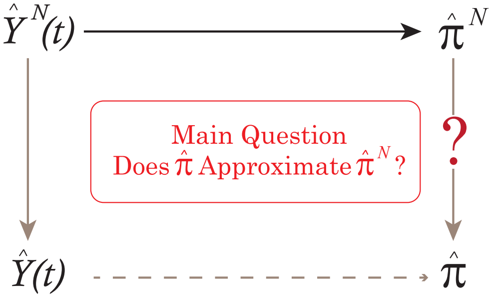


Exponential Case: Solved in 80's [Halfin-Whitt]

- $\alpha \in (0,1) \iff$ Arrival Rate $\sim N - \beta\sqrt{N}$
- β is given as a function of α

Problem Scheme



(More) General Case:

- Approximation scheme holds with the same scaling.
- A better understanding of $\hat{\pi}$ is needed yet.

Theoretical Components

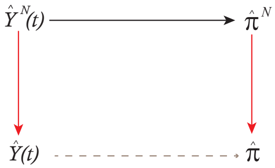
Classical Queueing Theory

- Ergodic Theorem for G/G/N Queues
- Harris Recurrent Chains

Theoretical Components

Functional LLN and CLT

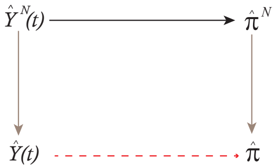
- Functional Analysis
  - Various Function Spaces
  - Convergence Criteria
- Probability Theory on General Spaces
  - Different notations of Convergence
  - Convergence of Probability Measures
  - Convergence of Measure-Valued processes



Theoretical Components

Stability of Solutions to SPDEs

- Basic SPDE Theory
  - Existence/Uniqueness Theorems
  - Stochastic Calculus
- Inf. Dim. Markov Processes
  - Asymptotic Coupling Method
- Renewal Theory



Where This Leads to

- Further on Many-Server Problem
  - More precise characterization of π
  - More realistic Assumptions: queues with Abandonment, network of queues, control.
  - Numerical Techniques for computing π
- Apply this set of techniques to other large scale problems
  - Biology,
  - etc.