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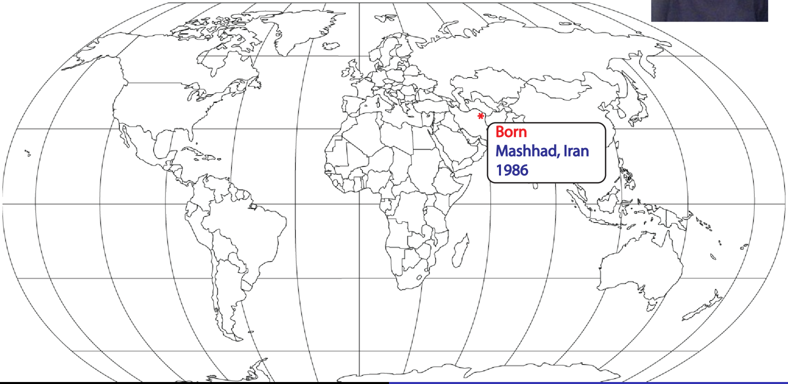

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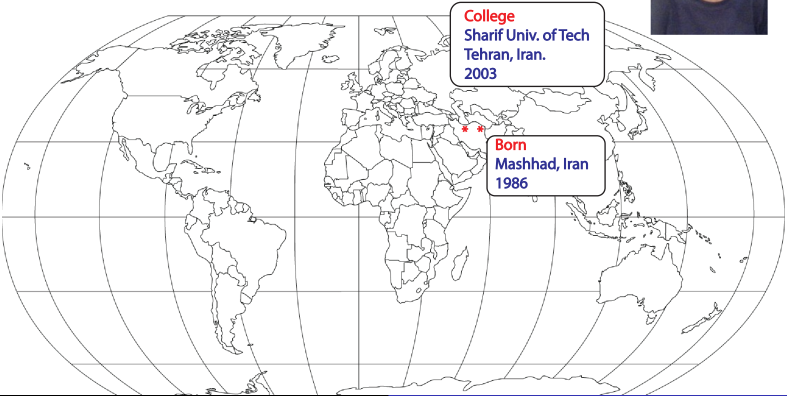

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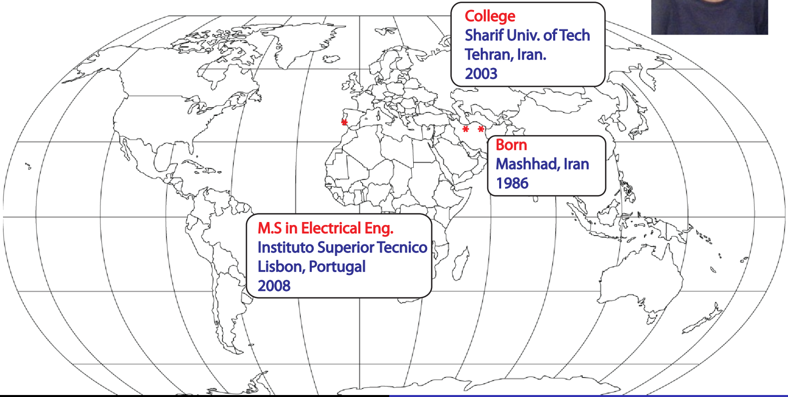

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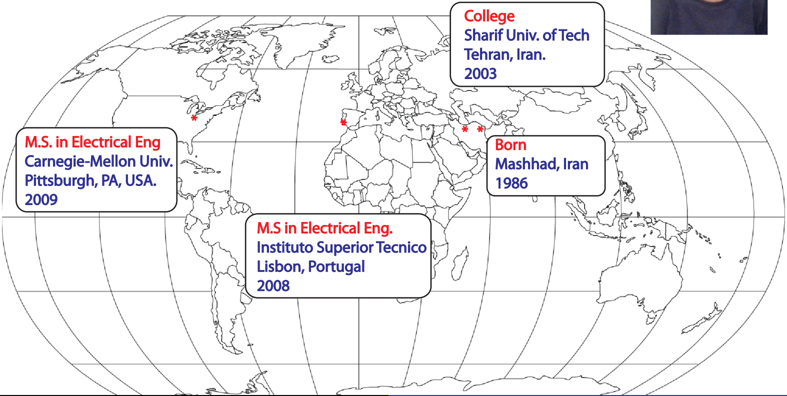

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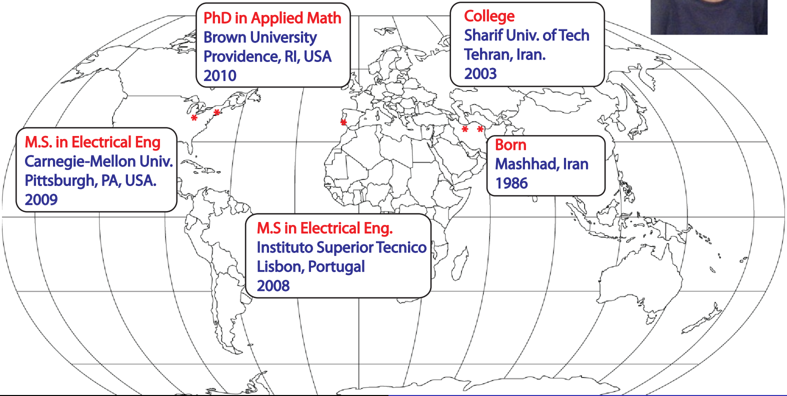

Asymptotic Coupling with Application in Queuing Systems

Mohammadreza Aghajani

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September 6 2012

1. Ergodicity Theorems for Markov Chains: Classical Results

Survey

1. Ergodicity Theorems for Markov Chains: Classical Results

2. Markov Chains in Infinite-Dimensions: Asymptotic Coupling

3. Application: Many-Server Queuing Systems

Stability of Markov Chains

Markov Chain on general space $(E, \mathcal{E})$

Given

- Initial distribution λ
- Transition Kernel $P(x, \cdot)$

We have

- $X \sim \mathbb{P}_\lambda$ on E^∞.
- $X(n) \sim \lambda P^n$.

Notions of Stability

- Invariant Distribution: $\pi = \pi P$.
- Ergodicity $\|\lambda P^N - \pi\| \to 0$.

Coupling

X, Y: Two random variables on $(E, \mathcal{E})$

Definition (Coupling)

$Z = (\tilde{X}, \tilde{Y})$ on $E \times E$ is a coupling of X and Y if

$$\tilde{X} \stackrel{d}{=} X, \qquad \tilde{Y} \stackrel{d}{=} Y.$$

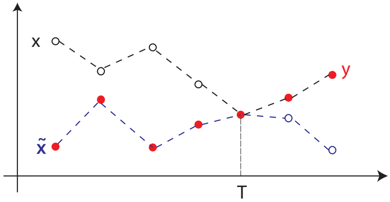
Coupling Inequality

$$\|\mathcal{L}\{X\} - \mathcal{L}\{Y\}\| \leq 2\mathbb{P}(\tilde{X} \neq \tilde{Y})$$

Coupling of Markov Chains

Two independent copies of a the chain $P(x,\cdot)$ on $E \subset \mathbb{Z}$:

- T: Coupling Time
- $Y_n \doteq \begin{cases} \tilde{X}_n & \text{if } n \leq T \\ X_n & \text{if } n > T \end{cases}$
- $Y \sim \tilde{X}$



By Coupling Inequality:

$$\boxed{\|\mathbb{P}_\lambda(X_n \in \cdot) - \mathbb{P}_{\tilde{\lambda}}(\tilde{X}_n \in \cdot)\| \leq 2\mathbb{P}_{\lambda\tilde{\lambda}}(T > n)}$$

When coupling is ‘successful’, ergodicity holds.

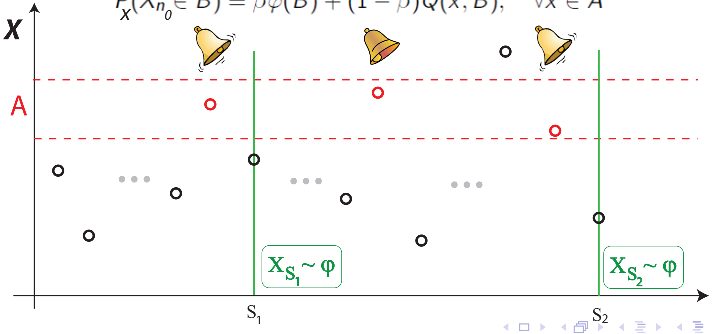
Ergodicity for Harris Chains

Definition (Harris Chain)

(i) $\mathbb{P}_x\left(X_n \in A;\ \text{for some } n\right) = 1, \quad \forall x \in E$ (recurrence)

(ii) $\mathbb{P}_x\left(X_{n_0} \in B\right) \geq \beta\varphi(B), \quad \forall x \in A, \forall B \in \mathcal{E}$ (small set)

$$P_x(X_{n_0} \in B) = \beta\varphi(B) + (1-\beta)Q(x,B), \quad \forall x \in A$$



Ergodicity for Harris Chains

Assume an invariant distribution π exists

Two independent copies of the chain:

- X is initialized at arbitrary $\lambda \rightarrow$ Corresponding $\{S_j\}$
- $\tilde{X}$ is initialized at $\pi \rightarrow$ Corresponding $\{\tilde{S}_j\}$

A 'successful' coupling:

- Coupling time $T = S_n = \tilde{S}_m$
- Renewal Theory $\Rightarrow$ T is almost surely finite.

Coupling inequality gives ergodicity

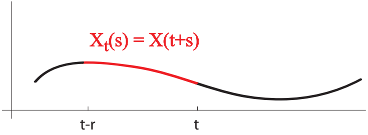
$$\|\mathbb{P}_\lambda(X_n \in \cdot) - \pi\| \leq \mathbb{P}_{\lambda\tilde{\lambda}}(T > n) \rightarrow 0$$

Survey

Infinite-Dimensional State Spaces: Example

Example: Stochastic Delay Differential Equation (SDDE)

$$dX(t) = -cX(t)dt + g\left(X(t-r)\right)dW_t$$



- $\{X_t; t \geq 0\}$ is a Markov Process on $\mathcal{C}([-r, 0])$
- Invariant Distribution Exists for large c.
- Given the solution X_t for any $t > 0$, X_0 can be recovered using Law of Iterated Logarithms

What Goes Wrong?

For SDDE and for typical inf-dim Markov chains:

$$\boxed{P(x, \cdot) \text{ and } P(y, \cdot) \text{ are mutually singular for } x \neq y}$$

Consequences:

- Only small sets are singletons
- Generally, singletons are not recurrent sets.

And therefore,

- Not Harris chains
- No successful coupling

Asymptotic Coupling

Definition (Asymptotic Coupling)

A measure Γ on $E^\infty \times E^\infty$ is an 'Asymptotic Coupling' for two initial distributions λ,μ on E, if

1. $\Gamma_1 \sim \mathbb{P}_\lambda$ and $\Gamma_2 \sim \mathbb{P}_\mu$.
2. $\Gamma\left(\{(x, y) \in E^\infty \times E^\infty; \lim_{n\to\infty} d(x_n, y_n) = 0\}\right) > 0$

Theorem (Hairer, Mattingly, Scheutzow)

If there exists a 'large enough' set $A \subset E$ such that for every $x, y \in A$ there exists an asymptotic coupling $\Gamma_{x,y}$ of δ_x and δ_y, then P has at most one invariant distribution.

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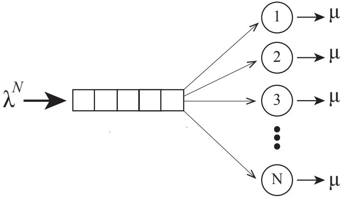
Many-Server Queues



Where do they arise?

- Call Centers
- Health Care
- Data Centers

A Markovian Representation

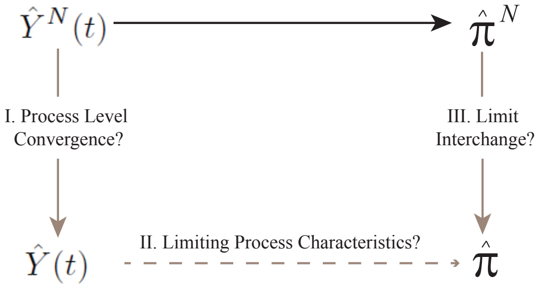
Analysis of GI/G/N systems:

- Usual representation is not Markovian
- A measure-valued (infinite-dimensional) Markovian representation[Kaspi, Ramanan]:

$$Y^N(t) = \left(X^N, \nu^N, Z^N\right) \in \mathbb{R} \times \mathbb{H}_{-2} \times \mathbb{W}^{1,1}$$

- Interested in invariant distribution π^N to assess Quality of Service.
- π^N is hard to characterize.

An Approximation Scheme



- π: invariant distribution of the limit process Y
- Hope: $\pi^N \Rightarrow \pi$
- A crucial question: Uniqueness of π

Asymptotic Coupling for Y

Theorem (Aghajani)

Y has a unique stationary distribution.

An asymptotic coupling scheme:

$$X(t) = X(0) + \sqrt{2}B(t) - \beta t - \int_0^t \langle h, \nu_s \rangle \, ds$$

Define

$$\tilde{X}(t) = \tilde{X}(0) + \sqrt{2}\tilde{B}(t) - \beta t - \int_0^t \langle h, \tilde{\nu}_s \rangle \, ds$$

where $\tilde{B}_t = B_t + \int_0^t \zeta(s) ds$. Choose ζ such that

- $\Delta X = X - \tilde{X}$ has a simpler from
- Girsanov Theorem holds