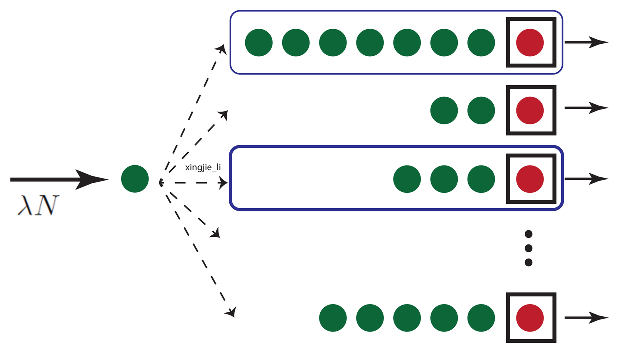
PDE Method for Randomized Load Balancing

Reza Aghajani, Kavita Ramanan, Xingjie Li, Division of Applied Mathematics, Brown University.

Model

Network with N servers



Load Balancing Algorithm:

- How to assign incoming jobs to servers to achieve good performance with low computational cost?

A performance parameter:

Steady-State Queue Length Probabilities:

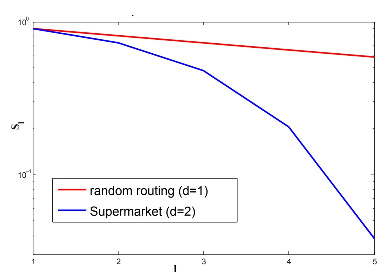
$$S_\ell = \lim_{N\to\infty} \mathbb{P}_{ss}\{\text{a typical queue length} \geq \ell\}$$

Common Load Balancing Algorithms

- ~~Joins the Shortest Queue~~ not feasible for large N
- $SQ(d)$ (supermarket) algorithm:
  - chooses d queues out of N, u.a.r.
  - joins the shortest among d

Exponential Service Dist.

SQ(d) for Exponential Service Distribution [VDK'96]:



- Random Routing (d=1): exponential decay.
- one additional choice (d=2): double-exponential decay.

Power of Two Choices

General Service Dist.

Statistical Observation:

- real-world service time distributions are non-exponential

Questions:

- Does the "power of two choices" also hold for general distributions? Partial answer by [Bramson-Lu-Prabhakar'13]: $SQ(2)$ for Pareto service distribution $\bar{G}(x) \sim x^{-\beta}$:
  - $\beta > 2$ (finite variance): double-exponential decay
  - $\beta = 2$: exponential decay
  - $\beta < 2$ (infinite variance): power-law decay

  How about other distributions (e.g. Log-Normal, Weibull?)
- How long does it take to reach "stationarity"?
- How about the transient behavior?
- How about time-inhomogeneous (e.g. periodic) arrivals ?

Challenges? We got an Idea!

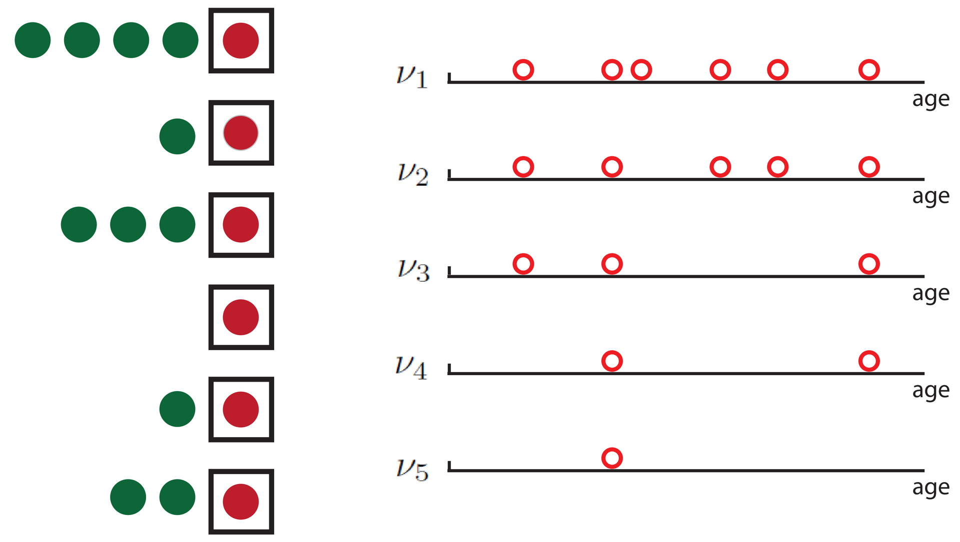
Challenges of General Service Dist.

- A common Markovian Representation for all N
  - sequence of queue lengths are not Markovian
  - Need to keep track of *ages*
  - dimension of the Markovian state space grows with N
- Would like a more robust framework.

Our Idea: Sequence of Measure-valued Processes

- Use a common infinite-dimensional state space

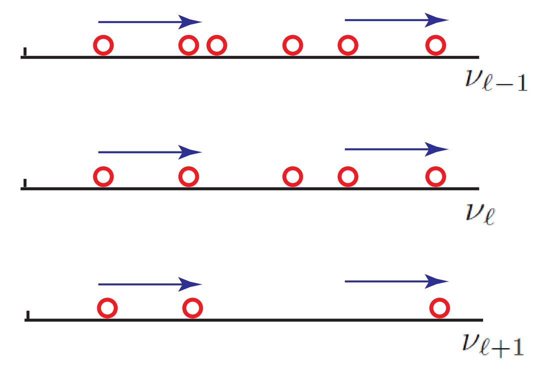
ν_ℓ: unit mass at the ages of jobs in servers with queues of length at least ℓ .



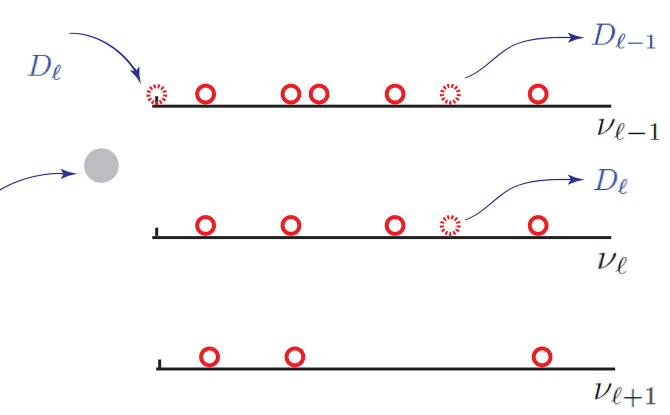
Inspired by a simpler framework in [Kaspi-Ramanan'11]

Dynamics of Measure-valued Processes

I. no arrivals/departures, masses move to the right with unit speed.

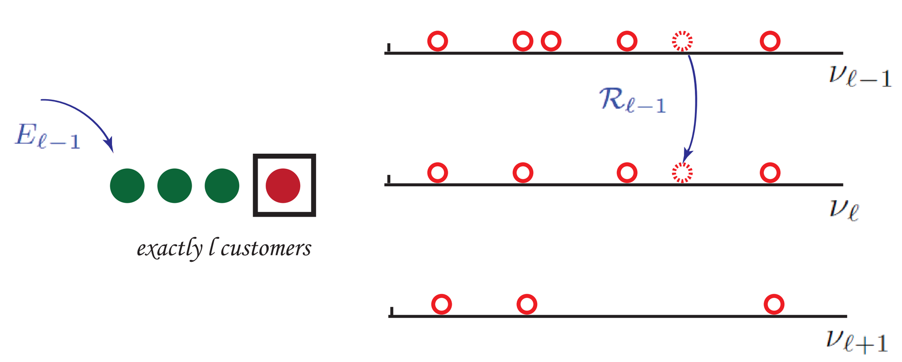


II. Upon departure from a queue with ℓ jobs,



- D_ℓ: cumulative departure process from servers with at least ℓ jobs

III. Upon arrival a queue with $\ell - 1$ jobs right before arrival,



- $\mathcal{R}_\ell$: *routing measure process*

Main Result: Hydrodynamic Limit

Definition A process $\nu = (\nu_\ell; \ell \geq 1)$ solves the *measure-valued hydrodynamic equations* if for all $f \in \mathbb{C}^1_b[0\times\infty)$,

initial jobs; linear growth of ages; service entry; departure; Routing process

$$\langle f,\nu_\ell(t)\rangle = \langle f,\nu_\ell(0)\rangle + \int_0^t \langle f',\nu_\ell(s)\rangle ds + f(0)D_{\ell+1}(t) - \int_0^t \langle h\, f,\nu_\ell(s)\rangle ds + \int_0^t \langle f,\eta_\ell(s)\rangle ds.$$

$$\langle 1,\nu_\ell(t)\rangle - \langle 1,\nu_\ell(0)\rangle = D_{\ell+1}(t) + \int_0^t \langle 1,\eta_\ell(s)\rangle ds - D_\ell(t),$$ → mass balance

$$D_\ell(t) = \int_0^t \langle h,\nu_\ell(s)\rangle ds,$$ → departure rate

$$\eta_\ell(t) = \begin{cases} \lambda\left(1-\langle 1,\nu_1(t)\rangle^2\right)\delta_0 & \text{if } \ell = 1,\\ \lambda\langle 1,\nu_{\ell-1}(t)+\nu_\ell(t)\rangle(\nu_{\ell-1}(t)-\nu_\ell(t)) & \text{if } \ell \geq 2.\end{cases}$$

routing measure; routing probabilities

Theorem: : Existence/Uniqueness of Hydrodynamic Eqns.

For every initial condition $\nu(0) = (\nu_\ell(0); \ell \geq 1)$ with $\nu_\ell \geq \nu_{\ell+1}$, the hydrodynamic equations has a unique solution.

The hydrodynamic equations can be partially solved

$$\langle f,\nu_\ell(t)\rangle = \langle f(\cdot+t)\frac{\bar{G}(\cdot+t)}{\bar{G}(\cdot)},\nu_\ell(0)\rangle + \int_{[0,t]} f(t-s)\bar{G}(t-s)dD_{\ell+1}(s) + \int_0^t \langle f(\cdot+t-s)\frac{\bar{G}(\cdot+t-s)}{\bar{G}(\cdot)},\eta_\ell(s)\rangle ds \qquad (1)$$

Theorem: : Hydrodynamic Limit

Let $\{\nu^{(N)}(t) = (\nu^{(N)}_\ell(t))_\ell; t \geq 0\}$ be the representation for the N-server system with initial condition $\nu^{(N)}(0)$. If

1. arrival process $E^{(N)}$ is a renewal process with rate λ^N, and $\lambda^N/N \to \lambda$,
2. service distribution G has mean 1 and density g,
3. for every $\ell \geq 1$, $\nu^{(N)}_\ell(0)/N \to \nu_\ell(0)$,

for some $\nu_\ell(0)$, then

$$\frac{1}{N}\nu^{(N)}_\ell \to \nu_\ell,$$

where ν is the unique solution to the hydrodynamic equations with initial condition $\nu(0)$.

Corollary: : Propagation of Chaos

If the initial condition is *exchangeable*, then

$$\lim_{N\to\infty} \mathbb{P}\{X^{(N),1}(t) \geq \ell_1, ..., X^{(N),K}(t) \geq \ell_K\} = \prod_{k=1}^K \langle 1,\ \nu_{\ell_k}(t)\rangle.$$

Open question: propagation of chaos on the infinite interval

Remarks on Proof

- show tightness of the subsequence $\{\frac{1}{N}\nu^{(N)}\}$ identify compensators of certain processes,
- show that every sub-seq. limit solves the hydrodynamic equation.
- Reduce uniqueness of solutions to hydrodynamic equations to uniqueness of solutions to the system of non-linear PDEs.
- show the uniqueness of non-linear PDEs, (find the right metric)

Reduction to "PDEs"

If one is only interested in $S_\ell(t) = \langle 1,\nu_\ell(t)\rangle$,

$$\langle 1,\nu_\ell(t)\rangle = \langle\frac{\bar{G}(\cdot+t)}{\bar{G}(\cdot)},\nu_\ell(0)\rangle + \int_{[0,t]} \bar{G}(t-s)dD_{\ell+1}(s) + \int_0^t \langle\frac{\bar{G}(\cdot+t-s)}{\bar{G}(\cdot)},\eta_\ell(s)\rangle ds$$

An invariant family under equation (1)

$$\mathbb{F} = \left\{f_r(x) = \frac{\bar{G}(x+r)}{\bar{G}(x)}; r \geq 0\right\}$$

Theorem: : PDE representation

For every sequence of bounded, continuously differentiable functions ξ^0_ℓ, the sequence of partial integro-differential equations

$$\xi_\ell(t,r) = \xi^0_\ell(t+r) - \int_0^t \bar{G}(t+r-u)\xi'_{\ell+1}(u,0)du + \int_0^t \zeta_\ell(t,u,r)du$$

with

$$\zeta_\ell = \begin{cases} \bar{G}(t+r-u)(1-\xi_1(u,0)^2) & \ell = 1,\\ (\xi_{\ell-1}(u,0)+\xi_\ell(u,0))(\xi_{\ell-1}(u,t+r-u)-\xi_\ell(u,t+r-u)) & \ell \geq 2.\end{cases}$$

and with boundary condition

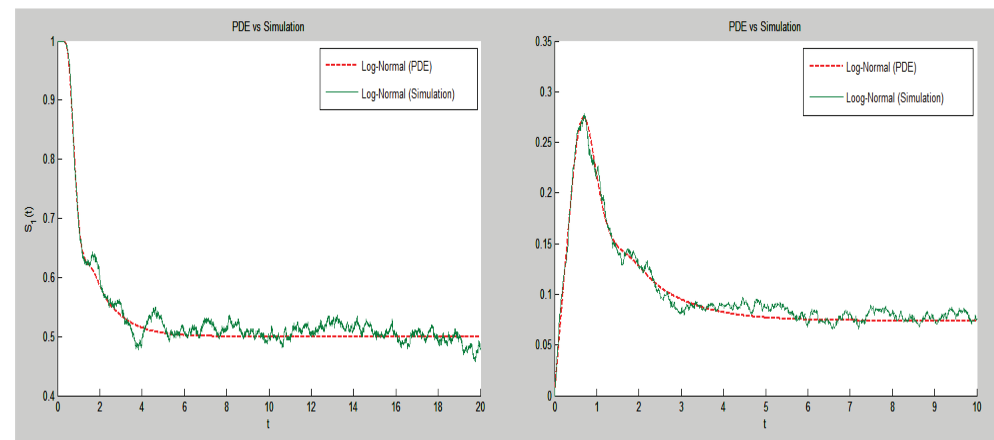
$$\xi_\ell(t,0) - \xi^0_\ell(0) = \int_0^t \lambda(u)\left(\xi_{\ell-1}(u,0)^2 - \xi_\ell(u,0)^2\right)du - \int_0^t \lambda(u)\left(\xi'_{\ell-1}(u,0) - \xi'_\ell(u,0)^2\right)du,$$

has a unique solution. Furthermore,

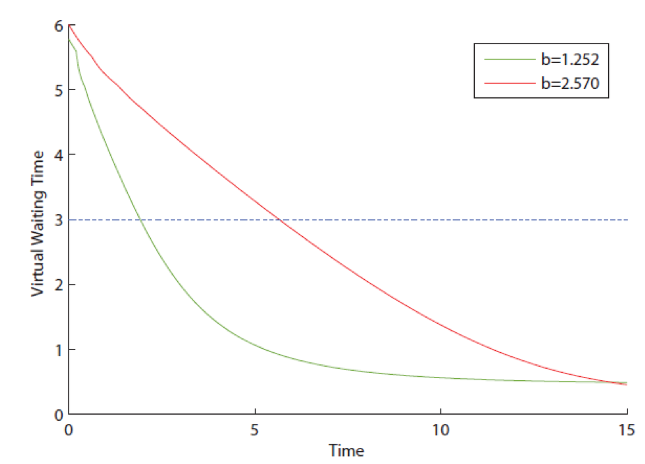
$$\xi_\ell(t,r) = \langle f^r,\nu_\ell(t)\rangle.$$

Numerical Results

Hydrodynamic PDEs can be numerically solve, and be used to approximate the transient behavior queue-length Probabilities:



or to gain non-trivial insight on the behavior of the N-server network, for example,



Service distribution with larger variance performs better on getting rid of initial backlog. A COUNTER-INTUITIVE RESULT!