SCHEDULING AND CONTROL OF MULTI-NODE MOBILE COMMUNICATIONS SYSTEMS WITH RANDOMLY-VARYING CHANNELS BY STABILITY METHODS

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Abstract. We consider a communications network consisting of many mobiles. There are random external data processes arriving at some of the mobiles, each destined for a unique destination or set of destinations. Each mobile can serve as a node in the possibly multi-hop (and not necessarily unique) path from source to destination. At each mobile the data is queued according to the source-destination pair.. Time is divided into small scheduling intervals. The capacity of the connecting channels are randomly varying. The system resources such as transmission power and/or time, bandwidth, and perhaps antennas, must be allocated to the various queues in a queue and channel-state dependent way to assure stability and good operation. Lost packets might or might not have to be retransmitted. At the beginning of the intervals, the channels are estimated via pilot signals and this information is used for the scheduling decisions, which are made at the beginning of the intervals. Stochastic stability methods are used to develop scheduling policies. The resulting controls are readily implementable and allow a range of tradeoffs between current rates and queue lengths, under very weak conditions. The basic methods are an extension of recent works for a system with one transmitter that communicates with many mobiles. The choice of Liapunov function allows a choice of the effective performance criteria. All essential factors are incorporated into a "mean rate" function, so that the results cover many different systems. Because of the non-Markovian nature of the problem, we use the perturbed Stochastic Liapunov function method, which is designed for such problems. Various extensions (such as the requirement of acknowledgments) are given, as well as a useful method for getting the a priori routes.

Key words. Scheduling in stochastic networks, randomly varying link capacities, mobile networks, stochastic stability, stability of networks with randomly varying links, routing in ad-hoc networks, perturbed stochastic Liapunov functions,

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1. Introduction. The paper considers the problem of scheduling in a network of M mobiles (to be referred to as nodes) with time varying link capacities. There are many (S) external sources with bursty data processes, each sending its data to its unique origin node, to be sent through the network to a unique (except for the multicasting case) destination node. At each mobile, the data is queued until transmitted, in an infinite buffer depending on the source-destination pair. Some mobiles serve as intermediaries in the possibly multi-hop connections between sources and destinations. The routes between source and destination need not be unique. We are concerned with the efficient and stabilizing allocation of the sys-

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tem resources, say, transmission power, time and bandwidth, to the various queues at each mobile in a queue and channel-state dependent way. Time is divided into small scheduling intervals. The capacities of the connecting channels in each interval form a correlated random process. At the beginning of the intervals, the capacities (or surrogates such as the S/N ratios) are estimated where possible via pilot signals and this information is used for the scheduling during that interval. The resource allocation decisions are made at the beginning of the intervals. Owing to the random nature of the arrival and channel processes, the computation or even the existence of stabilizing policies is not at all obvious. The approach is a network extension of the development for the one-node case in [4].

The channel processes are usually non-Markovian.[1] Even if it and the arrival processes were Markovian, it would be extremely difficult to use classical stability methods, but the versatile perturbed Liapunov function method [4, 7] can be used to obtain stabilizing scheduling policies. Let X denote the vector of all queue values at all of the nodes (all data quantities are measured in packets). With the perturbed Liapunov function method one starts with a basic Liapunov function $V(X)$ that works for an approximating "mean flow" system where the randomness has been averaged out in a particular "controlled" way. Then one gets a perturbation $\delta V(n)$ so that $V(X(n)) + \delta V(n)$ can be used as a Liapunov function for the true non-Markov physical system. Analogously to the "stability" method for selecting controls, the controls are determined by "approximately" minimizing a conditional expectation of the rate of change of the basic Liapunov function along the random path. The formulas are simple and the algorithm is readily implemented. For simplicity, we use a basic Liapunov function that is a polynomial which is the sum of terms, each depending on a single component of the state of the queue. This seems to be adequate for current needs, but a large family of strictly convex separable functions can also be used. The end result is that, if a certain "mean flow" is stabilizable, then so is the physical system under our scheduling rule. This stabilizability condition can often be readily verified, and appears to be very close to a necessary condition.

Some useful extensions are discussed in Section 4. There we give the modest changes that are required when a packet can be lost and the receipts on each individual link must be acknowledged. The multicasting case is briefly outlined and there is a discussion of a simple model where the number of sources can vary in time. Various extensions are implicitly included in the basic formulation. For example, channel breakdowns, priority users, and random connectivity,

The $(n+1)$st scheduling interval will be called the nth slot. The argument (n) denotes the beginning of the nth slot, and is referred to as "time n." Let $X_{i,k}(n)$ denote the queue size at time n at node k of

[1] E.g., Rayleigh fading.

data coming from source i (defined to be zero if node k is not on the path for source i). Define the vectors $X_k(n) = \{X_{i,k}, i \le S\}$ and $X(n) = \{X_k(n), k \le M\}$, with canonical values X_k and X, resp. With given weights $w_{i,k}$, the basic Liapunov function will be[2]

$$V(X) = \sum_{i,k} w_{i,k} X_{i,k}^p, \quad p \ge 2. \tag{1.1}$$

A stability analysis should assure robustness of behavior to small changes in the process dynamics; hence it is preferable to use methods that do not require the Markov property. The perturbed Liapunov function method does not require Markovianness. In applications, there are often many criteria that are of interest, e.g., mean delay and variance of delay. One should experiment with the form of the Liapunov function and examine the effects of the associated scheduling rules in order to get insight into the tradeoffs between competing criteria. Such an experimental procedure would give more insight and better rules than those obtained with a single fixed rule. The wide choice of functions $V(X)$ facilitates such experimentation.

There is much work on scheduling in the presence of various types of channel and data process randomness. But very little is available on scheduling for the general network case when the channels are randomly varying in a non-trivial way. For the one-node case where the rate of transmission is proportional to power, [1, 9] gets rules for power allocation whose form is similar to ours when $p = 2$ (such rules are called "max weight" there), and which are based on stability considerations. The method uses large deviations estimates and the setup is Markovian. See also [4]. The reference [11], perhaps the first to deal with random channels in a network, allocates power. Since their channel-rate and data-arrival processes are all i.i.d. sequences (this assumption is required by their method), the possible applications are very limited.

The papers [2, 3] deal with related problems, again essentially for one-node systems. There is a set of parallel processors, and the connectivities between the sources and the processors (but not the outgoing channels) vary randomly. They prove results concerning the limit (as $t \to \infty$) of (queue length at t)/t, and give conditions under which this limit is zero. This is used to show that the integral of the "rates" of transmission per unit time converges. They allocate a single resource (e.g., bandwidth) and the

[2]We could use $\sum_{i,k} w_{i,k}[X_{i,k} + h_{i,k}]^{p_{i,k}}$, where $p_{i,k} \ge 2, h_{i,k} \ge 0$ or $V(X) = \sum_{i,k} V_{i,k}(X_{i,k})$, where the $V_{i,k}(\cdot)$ are strictly convex non-negative functions, whose first derivative $DV_{i,k}(X_{i,k})$ is $o(V_{i,k}(X_{i,k}))$ and second derivative is $o(DV_{i,k}(X_{i,k})$. One can choose the function, for example, to model upper bound constraints on some queues. The choice of the functions and powers allows a variety of tradeoffs between queue size and throughput. We use (1.1) since the notation is simpler. But the development is parallel for the other cases, and the same conditions would be used.

rate is proportional to the allocation. Our proof is easily adapted to that problem, with the definition of stability to be used here. The work [10], for a one node model, has a Markovian channel-state process, the data input sequence is i.i.d., and a "complete resource pooling" condition is required. The decision rule is the same as ours for a quadratic Liapunov function. The emphasis is on stability and simplification of the model in the heavy traffic limit. The paper [6] treats the same subject as this paper, but the routes are restricted to be unique, and the set of extensions is different. When acknowledgments are required (as in Section 4), they are sent to the origin node. Here transmission on each link must be acknowledged. The developments differ in the type of Liapunov function perturbations that are used. See also [5] for a stability analysis as the heavy traffic regime is approached.

2. The Problem Formulation. Definitions. Let k denote a canonical node and let (i,k) denote the queue of source i data at node k. Since the routing is not necessarily unique, queue (i,k) might have possible forward links to any number of other nodes. Let $\{f(i,k,\alpha),\alpha\}$ denote the possible next nodes for queue (i,k). These are indexed by the parameter α, whose value ranges over a set that depends on i,k. This set will not be specified, but all summations over α, for fixed i,k, are assumed to be over this set. Similarly, queue (i,k) might receive data from any number of other nodes. Let $\{b(i,k,\beta),\beta\}$ denote the possible nodes from which (i,k) can receive data, indexed by the parameter β, whose value ranges over a set that depends on i,k. This set will not be specified, but all summations over β, for fixed i,k, are assumed to be over this set. If no route for source i uses node k, then queue (i,k) does not exist, and we ignore $X_{i,k}$, $f(i,k,\alpha)$ and $b(i,k,\beta)$. If the routing from (i,k) is unique, then α takes only one value. If k is the origin node for source i, then terms involving $b(i,k,\beta)$ are ignored, as are terms involving $f(i,k,\alpha)$ if node k is the terminal node for source i.

Let $L_k(n)$ denote the (vector) set of channel states at node k, at time n. It is a vector consisting of the states of all of the possible links $\{(i,f(i,k,\alpha));i,\alpha\}$ that are outgoing from node k. $L_k(n)$ could be just the set of S/N ratios at the receivers corresponding to unit transmitted power, or it might be some other indicator of the link capacities. It is notationally convenient to work with the vector $L_k(n)$, rather than with the individual links, since the decisions at each node k depend on the states of all of the possible outgoing links. $L_k(n)$ might denote other quantities in addition to the channel quality. For example, there might be power constraints that vary randomly due to interference from exogenous sources. These could be included in the $L_k(n)$. If some link at node k is unavailable at time n, then that fact could also be included in $L_k(n)$. For notational simplicity, we suppose that the channel state vector takes only finitely many values for each node k. The (vector-valued) symbol j is used for the canonical value of $L_k(n)$, for any k, n. The range of j will depend on the node k, but will be suppressed in the notation. Let $d_{i,k,\alpha}(n)$ denote the number of packets sent from queue i of node k to queue i of node α at time n. It will depend on the channel state and the allocated resources (e.g., power, frequency, bandwidth). It is always zero if node k is not on any path for source i. Let $a_{i,k}(n)$ denote the actual random number of arrivals in slot n from the exterior, if any, from source i at node k. These will be non-zero only for the unique node $k(i)$ at which source i enters the network.

Let $\mathcal{F}_n$ denote the minimal σ-algebra that measures all of the systems data up until time n as well as the channel states $\{L_n(k), k\}$ in slot n. These channel states are assumed to be available at time n. Let E_n denote the expectation conditioned on $\mathcal{F}_n$. We say that the packets sent in slot n are sent at time n, when the scheduling decisions are made.

Stability: Definition. An appropriate definition of stability is a "uniform mean recurrence time" property. Suppose that there are $0 < q_0 < \infty$ and a real-valued function $F(\cdot) \geq 0$ such that the following holds: For any n and the random time $\sigma_1 = \min\{k \geq n : |X(k)| \leq q_0\}$, we have[3]

$$E_n \left[\sigma_1 - n\right] \leq F(X(n)) I_{\{|X(n)| \geq q_0\}}. \tag{2.1}$$

Then the system is said to be stable. If $|X(n)|$ reaches a level $q_1 > q_0$, then the conditional expectation of the time required to return to a value q_0 or less is bounded by a function of q_1, uniformly in the past history and in n.[4] Note that the right side of (2.1) depends only on $X(n)$, and nothing else, even though there is a conditional expectation E_n on the left side, and the channel and arrival processes are random and correlated.

The decision rule. The number of packets, $d_{i,k,\alpha}(n)$, transmitted from queue (i, k) in slot n to node $f(i, k, \alpha)$ depends on the allocated resources, such as power, bandwidth, or time. Such resources are subject to constraints, either locally (at each node) or globally (for the entire network). The constraints might be just bounds on the total resources available at a node or on the number of packets than can be sent in a slot, in which case the determination of the $d_{i,k,\alpha}(n)$ for all i, α can be all made at node k. If the constraints involve more than one node, then making the assignments requires coordination among the nodes.

In classical control theory, stability ideas are often used to obtain controls that assure a stable system. Typically, one chooses a Liapunov function and then selects the control that minimizes its "rate of change" on the path. The idea is similar in our case. We will choose the $d_{i,k,\alpha}(n)$ that

[3] $\sigma_1 = \infty$, unless otherwise defined.

[4] This implies that the sequence $\{X(n)\}$ is tight or bounded in probability (see, for example, [7, Theorem 2, Chapter 6].

minimize an approximation to $E_n V(X(n+1)) - V(X(n))$. To motivate what will be done, let us start with the evaluation

$$\begin{aligned} & w_{i,k}\left[E_n X^p_{i,k}(n+1) - X^p_{i,k}(n)\right] \\ & \quad = w_{i,k} X^{p-1}_{i,k}(n)\left[-\sum_\alpha d_{i,k,\alpha}(n) + E_n a_{i,k}(n) + \sum_\beta d_{i,b(i,k,\beta),k}(n)\right] \\ & \qquad + \text{ terms of order } (p-2) \text{ in } X_{i,k}(n). \end{aligned}$$

Note that $d_{i,b(i,k,\beta),k}(n)$ = number of packets sent from queue $(i, b(i,k,\beta))$ to queue (i,k) at time n. Hence the last sum is the total number of packets arriving at node k at time n from all nodes. The sum over i,k of the terms in the second line that do not involve the $a_{i,k}$ can be written as

$$-\sum_{i,k,\alpha}\left[w_{i,k}X^{p-1}_{i,k}(n) - w_{i,f(i,k,\alpha)}X^{p-1}_{i,f(i,k,a)}(n)\right] d_{i,k,\alpha}(n). \tag{2.2}$$

This can be written as

$$-\sum_{i,k} w_{i,k}X^{p-1}_{i,k}(n)\left[\sum_\alpha d_{i,k,\alpha}(n) - \sum_\beta d_{i,b(i,k,\beta),k}(n)\right]. \tag{2.3}$$

The lower order terms in $X_{i,k}(n)$ are nonlinear functions of the $d_{i,k,\alpha}(n)$ and higher conditional moments of the $a_{i,k}(n)$, and would be very hard to deal with. It turns out, as in [4], that is is enough to base the decisions on (2.2) or (2.3).

If the decisions at node k need not be coordinated with those at any other node, then the decision is a maximizer in

$$\max_{\{d_{i,k,\alpha}(n):i,\alpha\}} \sum_{i,\alpha}\left[w_{i,k}X^{p-1}_{i,k}(n) - w_{i,f(i,k,\alpha)}X^{p-1}_{i,f(i,k,\alpha)}(n)\right] d_{i,k,\alpha}(n), \tag{2.4}$$

subject to the local constraints. If there are constraints that involve the decisions at a set of nodes, then the decisions for such a set must be made together, and the decision rule is a maximizer in

$$\max_{\{d_{i,k,\alpha}(n);i,k,\alpha\}} \sum_{i,k,\alpha}\left[w_{i,k}X^{p-1}_{i,k}(n) - w_{i,f(i,k,\alpha)}X^{p-1}_{i,f(i,k,\alpha)}(n)\right] d_{i,k,\alpha}(n), \tag{2.5}$$

or, equivalently, in

$$\max_{\{d_{i,k,\alpha}(n);i,k,\alpha\}} \sum_{i,k} w_{i,k}X^{p-1}_{i,k}(n)\left[\sum_\alpha d_{i,k,\alpha}(n) - \sum_\beta d_{i,b(i,k,\beta),k}(n)\right], \tag{2.6}$$

subject to the constraints. If $\{X(n)\}$ is not a Markov process, then $V(X)$ cannot be a Liapunov function for the system. However, as shown in the next section, the perturbed Liapunov function method [4, 7, 8] can be used to prove that the maximizing rules (2.4), (2.5), or (2.6), assure stability under reasonable conditions.

Let $u_{i,k,\alpha}(j, X)$ denote the control function at queue i at node k for the transmission to node $f(i, k, \alpha)$. The $u_{i,k,\alpha}(j, X)$ represents the allocated resources (power, time, bandwidth, etc.) that are allocated at queue (i, k) to the link to node α. Also, unless otherwise noted, its dependence on the queues is only on X_k and the required queue values at the immediate upstream nodes, namely the $X_{i,f(i,k,\alpha)}$ for all i, α. If no route for source i uses node k, then ignore $u_{i,k,\alpha}(j, X)$. The amount of data that is sent from queue (i, k) to queue i at node α is determined by the allocated resources $u_{i,k,\alpha}(j, X)$ and the current channel state at node k. Let the function $g_{i,k,\alpha}(j, X_{i,k}, u_{i,k,\alpha}(j, X))$ denote the actual amount of data that is sent under current channel state j and control $u_{i,k,\alpha}(\cdot)$. This defines $d_{i,k,\alpha}(n)$; i.e., the channel rate for queue (i, k) on the link to node α, associated with channel state j and control $u_{i,k,\alpha}(j, X(n))$ is $d_{i,k,\alpha}(n) = g_{i,k,\alpha}(j, X_{i,k}(n), u_{i,k,\alpha}(j, X(n)))$. The $X_{i,k}$ appears as an argument of $g_{i,k,\alpha}(\cdot)$ only because the amount sent cannot be larger than the queue content.

Assumptions. The following assumptions are network analogs of those used in [4] and will be commented on further below.

A2.1. There are constraint sets U_k such that $\{u_{i,k,\alpha}(j, X); i, \alpha\} \in U_k$. It is always assumed that the maximizing constrained $d_{i,k,\alpha}(n)$ in (2.4), (2.5), or (2.6) exist and are Borel functions of the $\{X(n), L_k(n); i, k\}$.

A2.2. There is a constant K_1 such that $E_n|a_i(n)|^p \le K_1$. There are $\bar{\lambda}^a_{i,k}$ such that the sums

$$\delta V^a_{i,k}(n) = \sum_{l=n}^{v} \left[E_n \alpha_{i,k}(l) - \bar{\lambda}^a_{i,k}\right]$$

converge as $v \to \infty$, uniformly in n, ω.

It follows from the definitions that $\bar{\lambda}^a_{i,k} = 0$ if node k is not the source node $k(i)$ for source i. For future use, write $\bar{\lambda}^a_i = \bar{\lambda}^a_{i,k(i)}$ the mean input rate for source i (measured in packets per slot).

A2.3. For each node k there are $\Pi_{k,j} \ge 0$ such that $\sum_j \Pi_{k,j} = 1$ and $\sum_{l=n}^{v} \left[E_n I_{\{L_k(l)=j\}} - \Pi_{k,j}\right]$ converges as $v \to \infty$, uniformly in n, ω.

A2.4. Define $K_0 = \max_{i,k,j,u,\alpha,X} \left[g_{i,k,\alpha}(j, X_{i,k}, u_{i,k,\alpha}(j, X))\right]$. There is a resource allocation $\{\tilde{u}_{i,k,\alpha}(\cdot); i, k, \alpha\}$ such that the following holds under it. There are non-negative real numbers $\{\tilde{q}^j_{i,k,\alpha}; i, k, \alpha\}$ such that $\tilde{q}^j_{i,k,\alpha} =$

$g_{i,k,\alpha}(j, X_{i,k}(n), \tilde{u}_{i,k,\alpha}(j, X(n)))$ if $X_{i,k}(n) \geq K_0$. [5] Also, if $X_{i,k}(n) < K_0$, then $g_{i,k,\alpha}(j, X_{i,k}(n), \tilde{u}_{i,k,\alpha}(j, X(n))) \leq \tilde{q}^j_{i,k,\alpha}$. The $\tilde{q}^j_{i,k,\alpha}$ satisfy

$$\sum_{\beta,j} \tilde{q}^j_{i,b(i,k,\beta),k} \Pi_{b(i,k,\beta),j} \leq \bar{q}_{i,k} \equiv \sum_{j,\alpha} \tilde{q}^j_{i,k,\alpha} \Pi_{k,j}, \quad \text{each } i, k \neq k(i), \tag{2.7}$$

and, for $k = k(i)$, $\bar{\lambda}^a_i < \bar{q}_{i,k}$.

Comments on the assumptions. (A2.1) simply states that there are constraints on the resources and allocations. (A2.2) and (A2.3) are mixing conditions on the data arrival and channel processes, resp., and do not appear to be restrictive. If the arrivals occur in batches, with the batches and intervals being mutually independent and each iid, then (A2.2) is just a constant times the residual time to the first arrival. See [4] for more discussion of this point. Both (A2.2) and (A2.3) say that the expectation of the future values of the random variables given the data in the remote past converges to the average in a "summable" way as the difference between the times goes to infinity. (A2.3) holds for the received signal power associated with Rayleigh fading. (A2.4) basically requires that there are controls under which the mean service rate at queue (i,k) for any i that uses node k is slightly greater than the mean data arrival rate $\bar{\lambda}^a_i$, if the queues remain large, for all (i,k). Similar conditions occur frequently in studies of stability in stochastic networks.

A variation of (A2.2) and (A2.3). The convergence of the sum in (A2.2) can be replaced by the condition that $E_n \alpha_{i,k}(l) - \bar{\lambda}^a_{i,k} \to 0$ uniformly in n, ω as $k - n \to \infty$. Then the perturbation (3.1) would be replaced by

$$w_{i,k} X^{p-1}_{i,k}(n) \sum_{l=n}^{m+n} E_n \left[a_{i,k}(l) - \bar{\lambda}^a_{i,k} \right]$$

for large enough m. The error terms in the proof are slightly different, but the method is the same. Analogous remarks hold for (A2.3) and the perturbations (3.2).

An equivalent form of (A2.4). Abusing terminology slightly, for $k \neq k(i)$, define $\bar{q}_{i,b(i,k)} = \sum_{\beta,j} \tilde{q}^j_{i,b(i,k,\beta),k} \Pi_{b(i,k,\beta),j}$, the average (over the channel variations) flow into (i,k) under the rates $\{\tilde{q}^j_{i,k,\alpha}; i,k,j,\alpha\}$. Then it is implied by (A2.4) that the $\tilde{q}^j_{i,k,\alpha}$ can be taken to satisfy

$$\sum_{j,\alpha} \tilde{q}^j_{i,k(i),\alpha} \Pi_{k(i),j} > \bar{\lambda}^a_i, \tag{2.8a}$$

[5] The lower bound K_0 is introduced in (A2.4) only because if the queue content is smaller than the maximum of what can be transmitted on a scheduling interval, then the mean (weighed with the $\Pi_{k,j}$) output might be too small to assure (2.7). For example if a queue is empty, then there are no departures.

and that there is $c_0 > 0$ such that for $k \neq k(i)$,

$$\text{average into } (i,k) - \text{average out of } (i,k) = \bar{q}_{i,b(i,k)} - \bar{q}_{i,k} \leq -c_0 < 0. \quad (2.8\text{b})$$

Section 5 gives a useful method for getting both the routing and the $\tilde{q}^j_{i,k,\alpha}$.

Example. Let the control be over bandwidth, with the rate proportional to bandwidth. Let the bandwidth allocated to (i,k) for transmission to node α be denoted by $B^j_{i,k,\alpha}$, let the constants of proportionality be $c^j_{i,k,\alpha}$ and define the rate $q^j_{i,k,\alpha} = c^j_{i,k,\alpha} B^j_{i,k,\alpha}$. There are the total bandwidth constraints $\sum_{i,\alpha} B^j_{i,k,\alpha} \leq B_k$ for each j,k. Suppose that the set of inequalities $\sum_{j,\alpha} q^j_{i,k,\alpha} \Pi_{k,j} > \bar{\lambda}^a_i$, all k, has a solution. Then the corresponding $q^j_{i,k,\alpha}$ satisfy (A2.4).

3. The Stability Theorem and Liapunov Function Perturbations. Suppose that, for a random process $\{x(n)\}$, we have $E_n V(x(n+1)) - V(x(n)) = c_n$, where $\{c_n\}$ is a random sequence that is "mixing" in the following sense. There is a constant $\bar{c} < 0$ such that $E_n[c_{n+m} - \bar{c}] \to 0$ fast enough as $m \to \infty$, for the sum $\delta V_n = \sum_{i=n} E_n[c_i - \bar{c}]$ to converge (and be bounded) uniformly in n, where E_n now denotes the expectation conditioned on $\{c_l, l \leq n\}$. Define $V_n = V(x(n\Delta)) + \delta V_n$. Then $E_n \delta V_{n+1} - \delta V_n = -(c_n - \bar{c})$ and $E_n V_{n+1} - V_n = c_n - [c_n - \bar{c}] = \bar{c} < 0$. The use of the perturbation has allowed us to replace c_n by a "mean." The perturbed Liapunov function method is an extension of this idea.

The perturbation $\delta V(n)$ that will be used will be a sum of components, one associated with each possible external input process, and one associated with each input link and one to each output link of each queue. The motivation for their form should be apparent from the way that they are used in the proof. See also [5, 7] for more motivation of the construction of the perturbations. Recall that $k(i)$ denotes the arrival node for source i. The perturbation associated with the arrivals from source i is

$$\delta V^a_{i,k}(n) = w_{i,k} X^{p-1}_{i,k}(n) \sum_{l=n}^{\infty} E_n \left[a_{i,k}(l) - \bar{\lambda}^a_{i,k} \right]. \quad (3.1)$$

This is zero if $k \neq k(i)$.

The function $\delta V^{d,+}_{i,k,j,\alpha}(n)$ defined in (3.2) is concerned with the effects of the departure of packets from queue (i,k), via link $f(i,k,\alpha)$, on the value of $E_n X^p_{i,k}(n+1) - X^p_{i,k}(n)$ when j is the vector-valued channel state at node k, and under the fixed rate $\tilde{q}^j_{i,k,\alpha}$ defined in (A2.4). The $\delta V^{d,-}_{i,k,j,\beta}(n)$ is concerned with the effects on $E_n X^p_{i,k}(n+1) - X^p_{i,k}(n)$ of the inputs to (i,k) from the link leading to it from node $b(i,k,\beta)$, when the vector-valued channel state at node $b(i,k,\beta)$ is j, and under the fixed rate $\tilde{q}^j_{i,b(i,k,\beta),k}$.

$$\begin{aligned}
\delta V^{d,+}_{i,k,j,\alpha}(n) &= -w_{i,k} X^{p-1}_{i,k}(n) \tilde{q}^j_{i,k,\alpha} \sum_{l=n}^{\infty} E_n \left[I_{\{L_k(l)=j\}} - \Pi_{k,j} \right], \\
\delta V^{d,-}_{i,k,j,\beta}(n) & \\
&= w_{i,k} X^{p-1}_{i,k}(n) \tilde{q}^j_{i,b(i,k,\beta),k} \sum_{l=n}^{\infty} E_n \left[I_{\{L_{b(i,k,\beta)}(l)=j\}} - \Pi_{b(i,k,\beta),j} \right].
\end{aligned} \tag{3.2}$$

The complete perturbation and time-dependent Liapunov function are, resp.,

$$\begin{aligned}
\delta V(n) = \sum_{i,k} \delta V^a_{i,k}(n) + \sum_{i,k,j,\alpha} \delta V^{d,+}_{i,k,j,\alpha}(n) + \sum_{i,k,j,\beta} \delta V^{d,-}_{i,k,j,\beta}(n), \\
\tilde{V}(n) = V(X(n)) + \delta V(n).
\end{aligned} \tag{3.3}$$

Theorem 3.1. *The system is stable under* (A2.1)–(A2.4).

Proof. The function $\tilde{V}(n)$ is the (time-varying) Liapunov function that is to be used. We need to show that $\tilde{V}(n)$ is a local supermartingale, when the queue values are large. In particular, we need to show that there is $c > 0$ such that for large X, we have $E_n \tilde{V}(n+1) - \tilde{V}(n) \le -c$, and then to show that this inequality can be used to get (2.1). Thus, we need to evaluate

$$\begin{aligned}
E_n \tilde{V}(n+1) - \tilde{V}(n) &= \sum_{i,k} w_{i,k} E_n \left[X^p_{i,k}(n+1) - X^p_{i,k}(n) \right] \\
&+ \sum_{i,k} E_n \left[\delta V^a_{i,k}(n+1) - \delta V^a_{i,k}(n) \right] \\
&+ \sum_{i,k,j,\alpha} E_n \left[\delta V^{d,+}_{i,k,j,\alpha}(n+1) - \delta V^{d,+}_{i,k,j,\alpha}(n) \right] \\
&+ \sum_{i,k,j,\beta} E_n \left[\delta V^{d,-}_{i,k,j,\beta}(n+1) - \delta V^{d,-}_{i,k,j,\beta}(n) \right].
\end{aligned}$$

The components will be evaluated separately, and then the results summed. The summation will effectively cancel various "undesirable" terms, and replace them by averages. This is the key idea of the method. In the expansions to follow, K denotes a constant whose value might vary from usage to usage. A first order Taylor expansion yields

$$\begin{aligned}
&\sum_{i,k} w_{i,k} E_n \left[X^p_{i,k}(n+1) - X^p_{i,k}(n) \right] = \\
&\sum_{i,k} w_{i,k} X^{p-1}_{i,k}(n) \left[E_n a_{i,k}(n) - \sum_{\alpha} d_{i,k,\alpha}(n) + \sum_{\beta} d_{i,b(i,k,\beta),k}(n) \right] \\
&+ O(|X^{p-2}(n)|) + K.
\end{aligned} \tag{3.4}$$

Now consider (3.1) for any i,k. Recall that $\delta V^a_{i,k}(n) = 0$ if $k \neq k(i)$, the origin node for source i. If $k = k(i)$, then a first order expansion yields

$$\begin{aligned} &E_n \delta V^a_{i,k}(n+1) - \delta V^a_{i,k}(n) \\ &\quad = -w_{i,k} X^{p-1}_{i,k}(n) \left[E_n a_{i,k}(n) - \bar{\lambda}^a_{i,k} \right] + O(|X^{p-2}(n)|) + K. \end{aligned}$$

Thus,

$$\begin{aligned} &\sum_{i,k} E_n \left[\delta V^a_{i,k}(n+1) - \delta V^a_{i,k}(n) \right] \\ &\quad = -\sum_{i,k} w_{i,k} X^{p-1}_{i,k}(n) \left[E_n a_{i,k}(n) - \bar{\lambda}^a_{i,k} \right] + O(|X^{p-2}(n)|) + K. \end{aligned} \tag{3.5}$$

Let us see what has been accomplished so far. On adding (3.4) and (3.5), we see that the terms $w_{i,k} X^{p-1}_{i,k}(n) E_n a_{i,k}(n)$ are cancelled, and a "mean value" term $w_{i,k} X^{p-1}_{i,k}(n) \bar{\lambda}^a_{i,k}$ appears, together with a term of order $p-2$. These lower order terms will be dominated by the terms of order $p-1$ for large values of the queue state. The desire for such cancellations and replacements by mean values determined the form of (3.1).

Let us now consider the perturbation defined by the first term in (3.2), which will facilitate dominating the $d_{i,k,\alpha}(n)$ in (3.4) by a term that can be effectively averaged. The definition (3.2) yields

$$\begin{aligned} E_n &\left[\delta V^{d,+}_{i,k,j,\alpha}(n+1) - \delta V^{d,+}_{i,k,j,\alpha}(n) \right] = \\ &\qquad -w_{i,k} E_n X^{p-1}_{i,k}(n+1) \tilde{q}^j_{i,k,\alpha} \sum_{l=n+1}^{\infty} E_{n+1} \left[I_{\{L_k(l)=j\}} - \Pi_{k,j} \right] \\ &\qquad +w_{i,k} X^{p-1}_{i,k}(n) \tilde{q}^j_{i,k,\alpha} \sum_{l=n}^{\infty} E_n \left[I_{\{L_k(l)=j\}} - \Pi_{k,j} \right]. \end{aligned} \tag{3.6}$$

Rewrite (3.6) by splitting out the lowest summand of the last term to get

$$\begin{aligned} &w_{i,k} X^{p-1}_{i,k}(n) \tilde{q}^j_{i,k,\alpha} \left[I_{\{L_k(n)=j\}} - \Pi_{k,j} \right] \\ &\quad -w_{i,k} E_n X^{p-1}_{i,k}(n+1) \tilde{q}^j_{i,k,\alpha} \sum_{l=n+1}^{\infty} E_{n+1} \left[I_{\{L_k(l)=j\}} - \Pi_{k,j} \right] \\ &\quad +w_{i,k} X^{p-1}_{i,k}(n) \tilde{q}^j_{i,k,\alpha} \sum_{l=n+1}^{\infty} E_n \left[I_{\{L_k(l)=j\}} - \Pi_{k,j} \right]. \end{aligned} \tag{3.7}$$

By expanding $X^{p-1}_{i,k}(n+1) - X^{p-1}_{i,k}(n)$ we can represent (3.7) as

$$\begin{aligned} &E_n \left[\delta V^{d,+}_{i,k,j}(n+1) - \delta V^{d,+}_{i,k,j}(n) \right] \\ &\quad = w_{i,k} X^{p-1}_{i,k}(n) \tilde{q}^j_{i,k,\alpha} \left[I_{\{L_k(n)=j\}} - \Pi_{k,j} \right] + O(|X^{p-2}(n)|) + K. \end{aligned} \tag{3.8}$$

An analogous procedure yields that

$$E_n\left[\delta V^{d,-}_{i,k,j,\beta}(n+1)-\delta V^{d,-}_{i,k,j,\beta}(n)\right] = \\ -w_{i,k}X^{p-1}_{i,k}(n)\tilde q^{j}_{i,b(i,k,\beta),k}\left[I_{\{L_{b(i,k,\beta)}(n)=j\}}-\Pi_{b(i,k,\beta),j}\right] \\ +O(|X^{p-2}(n)|)+K. \tag{3.9}$$

Now add the expansions (3.4), (3.5), (3.8), and (3.9), over i,k,j,α,β. Some terms in one expansion are the negative of terms in some other expansions. Adding the expansions and canceling such terms yields the expression

$$\begin{aligned} E_n\tilde V(n+1)-\tilde V(n) &= \sum_{i,k} w_{i,k}X^{p-1}_{i,k}(n)\bar\lambda^a_{i,k} \\ &+\sum_{i,k}\left[-w_{i,k}X^{p-1}_{i,k}(n)\sum_\alpha d_{i,k,\alpha}(n)+w_{i,k}X^{p-1}_{i,k}(n)\sum_\beta d_{i,b(i,k,\beta),k}(n)\right] \\ &+\sum_{i,k,j} w_{i,k}X^{p-1}_{i,k}(n)\sum_\alpha \tilde q^j_{i,k,\alpha}\left[I_{\{L_k(n)=j\}}-\Pi_{k,j}\right] \\ &-\sum_{i,k,j} w_{i,k}X^{p-1}_{i,k}(n)\sum_\beta \tilde q^j_{i,b(i,k,\beta),k}\left[I_{\{L_{b(i,k,\beta)}(n)=j\}}-\Pi_{b(i,k,\beta),j}\right] \\ &+O(|X^{p-2}(n)|)+K. \end{aligned} \tag{3.10}$$

The terms in the second, third and fourth lines of (3.10) that do not involve the $\Pi_{k,j}$ variables are

$$\begin{aligned} &-\sum_{i,k} w_{i,k}X^{p-1}_{i,k}(n)\left[\sum_\alpha d_{i,k,\alpha}(n)-\sum_\beta d_{i,b(i,k,\beta),k}(n)\right] \\ &+\sum_{i,k} w_{i,k}X^{p-1}_{i,k}(n)\left[\sum_{\alpha,j}\tilde q^j_{i,k,\alpha}I_{\{L_k(n)=j\}}-\sum_{\beta,j}\tilde q^j_{i,b(i,k,\beta),k}I_{\{L_{b(i,k,\beta)}(n)=j\}}\right]. \end{aligned}$$

For each k,α,β, the indicator functions in the above sums over j select the actual current channel state $j=L_k(n)$ or $j=L_{b(i,k,\beta)}(n)$, as appropriate. Hence, the previous expression can be rewritten as

$$\begin{aligned} &-\sum_{i,k} w_{i,k}X^{p-1}_{i,k}(n)\left[\sum_\alpha d_{i,k,\alpha}(n)-\sum_\beta d_{i,b(i,k,\beta),k}(n)\right] \\ &+\sum_{i,k} w_{i,k}X^{p-1}_{i,k}(n)\left[\sum_\alpha \tilde q^{L_k(n)}_{i,k,\alpha}-\sum_\beta \tilde q^{L_{b(i,k,\beta)}(n)}_{i,b(i,k,\beta),k}\right]. \end{aligned} \tag{3.11}$$

It is simpler to complete the proof first under the assumption that $X_{i,k}(n)\ge K_0$ for all i,k, and then to add the few details for the general case. If all $X_{i,k}(n)\ge K_0$ then by (A2.4) there are resource allocations $\{\tilde u_{i,k,\alpha}(\cdot);i,k,\alpha\}$ such that, for channel state j, the output from queue (i,k) to queue $(i,f(i,k,\alpha))$ will be $g_{i,k,\alpha}(j,X_{i,k}(n),\tilde u_{i,k,\alpha}(j,X(n)))=\tilde q^j_{i,k,\alpha}$. The

$d_{i,k,\alpha}(n)$ are chosen by either the maximization rule (2.4), or by the rules (2.5) or (2.6) (which are equivalent to each other). The rule (2.4) is implied (2.5) and by (2.6). On the other hand, the $\tilde{q}^j_{i,k,\alpha}$ defined in (A2.4) are not necessarily maximizers in (2.6). Hence the expression (3.11) is non-positive. Using this non-positivity in (3.10) together with the definitions of $\bar{q}_{i,k,\alpha}$, $\bar{q}_{i,k}$ and $\bar{q}_{i,b(i,k)}$ yields the following upper bound to (3.10):

$$\sum_{i,k} w_{i,k} X^{p-1}_{i,k}(n) \left[\bar{\lambda}^a_{i,k} - \bar{q}_{i,k} + \bar{q}_{i,b(i,k)}\right] + O(|X^{p-2}(n)|) + K. \tag{3.12}$$

By (2.8), the terms in the brackets in the first line of (3.12) are $\leq -c_0 < 0$. Thus we have proved that

$$E_n\tilde{V}(n+1) - \tilde{V}(n) \leq -c_0 \sum_{i,k} w_{i,k} X^{p-1}_{i,k}(n) + O(|X(n)|^{p-2}) + K, \tag{3.13}$$

$\delta V(n)$ satisfies

$$|\delta V(n)| = O(|X(n)|^{p-1}) + K \tag{3.14}$$

and, by (3.13),

$$E_n\tilde{V}(n+1) - \tilde{V}(n) \to -\infty, \text{ uniformly in } n, \omega \text{ as } X(n) \to \infty. \tag{3.15}$$

By (3.15), there are $c_1 > 0$ and $q_0 > 0$, such that, for $|X(n)| \geq q_0$,

$$E_n\tilde{V}(n+1) - \tilde{V}(n) \leq -c_1. \tag{3.16}$$

Given small $\delta > 0$, (3.14) implies that for q_0 sufficiently large,

$$|V(X(n)) - \tilde{V}(n)| \leq \delta(1 + V(X(n)). \tag{3.17}$$

Let σ_0 be a stopping time for which $|X(\sigma_0)| = c_2 > q_0$, and define the stopping time $\sigma_1 = \min\{n > \sigma_0 : |X(n)| \leq q_0\}$. Then, by (3.16), we have

$$E_{\sigma_0}\tilde{V}(\sigma_1) - \tilde{V}(\sigma_0) \leq -c_1 E_{\sigma_0}[\sigma_1 - \sigma_0]. \tag{3.18}$$

Using (3.18) and the bound (3.17) on $\tilde{V}(n) - V(X(n))$ to bound $\tilde{V}(\sigma_i) - V(X(\sigma_i))$, $i = 0, 1$, yields

$$\begin{aligned} &-\delta E_{\sigma_0}[1 + V(X(\sigma_1))] + E_{\sigma_0}V(X(\sigma_1)) \\ &\qquad \leq E_{\sigma_0}\tilde{V}(\sigma_1) \leq -c_1 E_{\sigma_0}(\sigma_1 - \sigma_0) + [\delta + V(X(\sigma_0))(1 + \delta)] \end{aligned}$$

or

$$E_{\sigma_0}(\sigma_1 - \sigma_0) \leq \frac{2\delta + V(X(\sigma_0))(1 + \delta) + \delta E_{\sigma_0}V(X(\sigma_1))}{c_1}$$

which implies that the definition of stability (2.1) holds since $V(X(\sigma_1)) \leq \sup_{|x| \leq q_0} V(x)$.

Finally, we complete the details when some components of $X(n)$ are less than K_0. Recall the definition of $\tilde{u}^j_{i,k,\alpha}(\cdot)$ and $\tilde{q}^j_{i,k,\alpha}$ in (A2.4). Define

$$\tilde{g}_{i,k,\alpha}(L_k(n), X(n)) = g_{i,k,\alpha}(L_k(n), X_{i,k}(n), \tilde{u}_{i,k,\alpha}(L_k(n), X(n))).$$

For $X_{i,k}(n) \geq K_0$, we have $\tilde{g}_{i,k,\alpha}(L_k(n), X(n)) = \tilde{q}^{L_k(n)}_{i,k,\alpha}$ by the definition of the $\tilde{q}^{L_k(n)}_{i,k,\alpha}$ in (A2.4). If $X_{i,k}(n) \leq K_0$ then, also by (A2.4), $\tilde{g}_{i,k,\alpha}(L_k(n), X(n)) \leq \tilde{q}^{L_k(n)}_{i,k,\alpha}$. Rewrite (3.11) as follows.

$$\begin{aligned}
&-\sum_{i,k} w_{i,k} X^{p-1}_{i,k}(n)\Big[\sum_\alpha d_{i,k,\alpha}(n) - \sum_\beta d_{i,b(i,k,\beta),k}(n)\Big] \\
&+\sum_{i,k} w_{i,k} X^{p-1}_{i,k}(n)\Big[\sum_\alpha \tilde{g}_{i,k,\alpha}(L_k(n), X(n)) \\
&\qquad\qquad - \sum_\beta \tilde{g}_{i,b(i,k,\beta),k}(L_{b(i,k,\beta)}(n), X(n))\Big] \\
&+\sum_{i,k: X_{i,k}(n)\geq K_0} w_{i,k} X^{p-1}_{i,k}(n) \sum_\alpha \Big[\tilde{q}^{L_k(n)}_{i,k,\alpha} - \tilde{g}_{i,k,\alpha}(L_k(n), X(n))\Big] \\
&+\sum_{i,k: X_{i,k}(n)< K_0} w_{i,k} X^{p-1}_{i,k}(n) \Big[\sum_\alpha \tilde{q}^{L_k(n)}_{i,k,\alpha} - \sum_\alpha \tilde{g}_{i,k,\alpha}(L_k(n), X(n))\Big] \\
&-\sum_{i,k} w_{i,k} X^{p-1}_{i,k}(n)\Big[\sum_\beta \tilde{q}^{L_{b(i,k,\beta)}(n)}_{i,b(i,k,\beta),k} - \sum_\beta \tilde{g}_{i,b(i,k,\beta),k}(L_{b(i,k,\beta)}(n), X(n))\Big].
\end{aligned}$$

As was argued for the case where all $X_{i,k} \geq K_0$, the sum of the first three lines is non-positive, since the $\{d_{i,k,\alpha}(n); i, k\}$ are chosen by the maximization rule. The two terms in each bracket in the fourth line are equal by the definition of the $\tilde{g}_{i,k,\alpha}$ when $X_{i,k}(n) \geq K_0$. Hence this term is zero. By (A2.4), the bracketed terms in the last line are non-negative, hence the last line is non-positive. Thus the only possible positive term is the next to last line, and this is $O(1)$ since it is a sum over i, k for which $X_{i,k}(n) \leq K_0$. Thus (3.11) is $O(1)$. From this point on the proof is completed just as for the case where all $X_{i,k}(n) \geq K_0$. ■

Notes. The decision rule (2.4) requires that each node k know the value of the $X_{i,k}(n)$ and $X_{i,f(i,ka)}(n)$ for all i, α that are relevant at node k. In fact, if the value of the $X_{i,f(i,k,\alpha)}(n)$ were known only subject to a bounded error, then the proof still goes through under the same conditions. So, only an occasional approximate estimate of the queues at the upstream nodes is needed. Suppose that some links are preempted by priority users from time to time, where the intervals of availability are defined by a renewal process that is independent of the arrival and channel rate processes. Then it can be shown that the results continue to hold, but with the $\bar{q}_{i,k}$ multiplied by the fraction of time that the channel is available, so the capacity must be sufficient to handle the average down times. Under the other assumptions, condition (A2.4) is sufficient but not necessary for stability. But it is "nearly" necessary in the sense that if for each choice of the $\{\tilde{q}^{j}_{i,k,\alpha}\}$, there is some (i_0, k_0) such that $\bar{q}_{i_0,b(i_0,k_0)} - \bar{q}_{i_0,k_0} > 0$, then the system would not be stable.

4. Some Extensions. The basic approach to scheduling and stability can be extended in many ways, and the examples described below illustrate some of the possibilities.

A. Acknowledgments of receipt required for each link. The foregoing development did not require that received packets be acknowledged. Suppose that packets on the link from any queue (i, k) to node α that are not acknowledged within a window $W_{i,k,\alpha}$ of scheduling intervals will need to be requeued at (i, k) and retransmitted. The treatment of the acknowledgment and loss processes involves a more complicated notation and an additional perturbation to the Liapunov function. In order to keep the notation reasonable, we will suppose that the routing is unique for each source-destination pair. Thus, the indices α, β can and will be dropped. The approach for the non-unique routing case is essentially the same, with analogous results. The acks are sent back to the previous node when a packet is received, subject to a possible delay.

If we fully accounted for the possibility that the packet loss or non-ack process depended on the traffic in the channel, and the channel characteristics, the resulting problem would be very difficult. Because of this, it is often assumed that the loss is a consequence of uncontrolled additional traffic in the channels. We will take the following often used approach. For each link, an ack for each received packet is sent to the node from which it just came. If a packet sent from queue (i, k) at time n is not acknowledged by time $n + W_{i,k}$, then that packet will be requeued at (i, k). The development in Section 3 can be readily modified to accommodate these changes. The development in [6] supposed that acks for source i data are sent only to the origin node $k(i)$, and that packets lost anywhere must be retransmitted from that node. Here acks are required for each link.

Until the end of the example, we suppose that the packet loss process is random. Thus, the events that packets are lost are independent among the inks, iid for the packets on each link, and independent of the channel states, decisions, and arrivals. Let $\tilde{\zeta}_{i,k}(n)$ denote the fraction of packets sent from queue (i, k) at time n that were not received at queue $(i, f(i, k))$. These would not be acknowledged by the end of the waiting period $W_{i,k}$, and must be requeued and retransmitted at that time. Let $\mathcal{F}_n$ now measure the $\tilde{\zeta}_{i,k}(l), l < n$ for all i, k ,as well. Define $p_{i,k} = E_n \zeta_{i,k}(n) = E\zeta_{i,k}(n)$.

The queue dynamics are now

$$X_{i,k}(n+1) = X_{i,k}(n) + a_{i,k}(n) - d_{i,k}(n) + (1-\tilde{\zeta}_{i,b(i,k)}(n))d_{i,b(i,k)}(n) \\ +d_{i,k}(n-W_{i,k})\tilde{\zeta}_{i,k}(n-W_{i,k}),$$

The last term on the right are the requeued packets, and the next to last term are the packets sent from $(i,b(i,k))$ to (i,k) that were received. We have

$$\begin{aligned} & X^p_{i,k}(n+1) = \\ & X^{p-1}_{i,k}(n)\big[-d_{i,k}(n) + (1-\tilde{\zeta}_{i,b(i,k)}(n))d_{i,b(i,k)}(n) \\ & \qquad +d_{i,k}(n-W_{i,k})\tilde{\zeta}_{i,k}(n-W_{i,k})\big] \\ & +X^{p-1}_{i,k}(n)a_{i,k}(n) + O(|X^{p-2}_{i,k}(n)|) + K, \end{aligned} \tag{4.1}$$

where K is a constant whose value might change from usage to usage. The additional Liapunov function perturbation component

$$\delta V^W(n) = \sum_{i,k} w_{i,k}X^{p-1}_{i,k}(n) \sum_{m=n-W_{i,k}}^{n-1} d_{i,k}(l)\tilde{\zeta}_{i,k}(l). \tag{4.2}$$

will help us deal with averaging the increases in the various queues due to not receiving an ack in time.

Recall that if $k = k(i)$, the origin node for source i, then $d_{i,b(i,k)}(n) = 0$. Noting that, for $k \neq k(i)$, $E_n\tilde{\zeta}_{i,b(i,k)}(n) = p_{i,b(i,k)}$, we can write

$$\begin{aligned} & E_n[V(X(n+1)) - V(X(n))] + E_n[\delta V^W(n+1) - \delta V^W(n)] \\ & = \sum_{i,k} w_{i,k}X^{p-1}_{i,k}(n)E_n a_{i,k}(n) \\ & \quad + \sum_{i,k} w_{i,k}X^{p-1}_{i,k}(n)\big[-d_{i,k}(n) + (1-p_{i,b(i,k)})d_{i,b(i,k)}(n) \\ & \qquad +\tilde{\zeta}_{i,k}(n-W_{i,k})d_{i,k}(n-W_{i,k})\big] \\ & \quad + \sum_{i,k} w_{i,k}X^{p-1}_{i,k}(n)\left[p_{i,k}d_{i,k}(n) - \tilde{\zeta}_{i,k}(n-W_{i,k})d_{i,k}(n-W_{i,k})\right] \\ & \quad +O(|X^{p-2}(n)|) + K. \end{aligned} \tag{4.3}$$

The second, third and fourth lines contain the highest order terms in $E_nV(X(n+1)) - V(X(n))$, and the next to last line is the highest order term in the expansion of $E_n[\delta V^W(n+1) - \delta V^W(n)]$. The terms with $d_{i,k}(n-W_{i,k})$ lines cancel each other, and we drop them now.

The decision rule that replaces (2.4) is

$$\max_{\{d_{i,k}(n):i\}} \sum_i (1-p_{i,k})\left[w_{i,k}X^{p-1}_{i,k}(n) - w_{i,f(i,k)}X^{p-1}_{i,f(i,k)}(n)\right]d_{i,k}(n).$$

The rules (2.5), and (2.6) are modified similarly. The full new perturbed Liapunov function is

$$\tilde{V}^W(n) = V(X(n)) + \delta V^W(n) + \sum_{i,k} \delta V^a_{i,k}(n) + \sum_{i,k}(1-p_{i,k})\delta V^{d,+}_{i,k,j}(n) + \sum_{i,k,j}(1-p_{i,b(i,k)})\delta V^{d,-}_{i,k,j}(n). \tag{4.4}$$

Then, using (4.1), (4.3), (3.5), (3.8), and (3.9),

$$\begin{aligned}
E_n\tilde{V}^W(n+1) - \tilde{V}^W(n) &= \sum_{i,k} w_{i,k}X^{p-1}_{i,k}(n)\bar{\lambda}^a_{i,k} \\
&+ \sum_{i,k}\Big[-(1-p_{i,k})w_{i,k}X^{p-1}_{i,k}(n)d_{i,k}(n) \\
&\quad +(1-p_{i,b(i,k)})w_{i,k}X^{p-1}_{i,k}(n)d_{i,b(i,k)}(n)\Big] \\
&+ \sum_{i,k}(1-p_{i,k})w_{i,k}X^{p-1}_{i,k}(n)\tilde{q}^j_{i,k}\left[I_{\{L_k(n)=j\}} - \Pi_{k,j}\right] \\
&- \sum_{i,k,j}(1-p_{i,b(i,k)})w_{i,k}X^{p-1}_{i,k}(n)\tilde{q}^j_{i,b(i,k)}\Big[I_{\{L_{b(i,k)}(n)=j\}} - \Pi_{b(i,k),j}\Big] \\
&+O(|X^{p-2}(n)|) + K.
\end{aligned}$$

The second and third lines are due to (the non-arrival parts of) the third–fifth lines of (4.3). The fourth line is due to $(1-p_{i,k})\delta V^{d,+}_{i,k,j}(n)$ and the next to last line to $(1-p_{i,b(i,k)})\delta V^{d,-}_{i,k,j}(n)$. Dominating terms as in the part of the proof of the theorem concerning (3.11) yields the following upper bound to the last expression:

$$\begin{aligned}
&\sum_{i,k} w_{i,k}X^{p-1}_{i,k}(n)\left[\bar{\lambda}^a_{i,k} - (1-p_{i,k})\bar{q}_{i,k} + (1-p_{i,b(i,k)})\bar{q}_{i,b(i,k)}\right] \\
&+O(|X^{p-2}(n)|) + K.
\end{aligned}$$

At most one of $\bar{q}_{i,b(i,k)}$ and $\bar{\lambda}^a_{i,k}$ can be non-zero for any i, k.

The condition (A2.4) is modified to read, for all i,

$$\begin{aligned}
&(1-p_{i,k})\bar{q}_{i,k} > \bar{\lambda}_i, && \text{for } k = k(i), \\
&(1-p_{i,k})\bar{q}_{i,k} - (1-p_{i,b(i,k)})\bar{q}_{i,b(i,k)} > 0, && \text{for } k \neq k(i).
\end{aligned}$$

The proof is completed as in Theorem 3.1.

If the packet loss process for the link out of (i, k) is correlated, then the process $\{\tilde{\zeta}_{i,k}(n), n\}$ is correlated and another perturbation is required to average it. Suppose that there are $p_{i,k}$ such that the sums in

$$\delta V^{C,+}_{i,k}(n) = w_{i,k}X^{p-1}_{i,k}(n)\sum_{l=n}^{\infty} E_n\left[\tilde{\zeta}_{i,k}(l) - p_{i,k}\right]d_{i,k}(l),$$

$$\delta V_{i,k}^{C,-}(n) = -w_{i,k} X_{i,k}^{p-1}(n) \sum_{l=n}^{\infty} E_n \left[\tilde{\zeta}_{i,b(i,k)}(l) - p_{i,b(i,k)} \right] d_{i,b(i,k)}(l),$$

are well defined and bounded, uniformly in n, ω. Then add the $\delta V_{i,k}^{C,\pm}(\cdot)$ to $V^W(\cdot)$. The conclusion is unchanged.

B. Multicasting. Suppose that some sources have multiple destinations, with a unique route for each source-destination pair. Let the route network for each source form a tree, with the source as the root and the final destinations as the end branches. Suppose that if the tree branches at node k, then transmissions must be done to all of the branches simultaneously, as is commonly required in multicasting. If the route for source i uses node k, then redefine $b(i, k, \gamma)$ to denote the nodes at the end of the branches of the tree out of queue (i, k), where the dimension of the index parameter γ is the number of branches.

Then (2.4) is replaced by

$$\max_{\{d_{i,k}(n):i\}} \sum_{i,\gamma} \left[w_{i,k} X_{i,k}^{p-1}(n) - \sum_{\gamma} w_{i,f(i,k,\gamma)} X_{i,f(i,k,\gamma)}^{p-1}(n) \right] d_{i,k}(n),$$

subject to the constraints at node k. Modify (2.5) and (2.6) analogously. The criterion (A2.4) is modified in an obvious manner to take account of the new flows.

C. Variable number of sources and destinations. When the number of sources, nodes and destinations vary randomly, the modeling problem can be quite vexing. For example, if a node disappears slowly as its links fade, what happens to its still untransmitted data ? We will take a simple approach, by supposing that there is a backbone network, with an unchanging number of nodes, although the associated links in the backbone will still vary randomly. There is a large and randomly varying number of sources that send data to the nodes in the backbone. The arriving packets from the randomly changing number of sources are multiplexed on arrival. These packets are assigned priority values and at the backbone nodes, the data is queued according to both priority and the node to which that packet would be sent to next on its route to its final destination.

Owing to the multiplexing and the large number of sources, it is assumed that the total arrival processes (per slot) from the exterior to the various queues (i, k) are mutually independent, and the elements of each are iid, with bounded variances, and means denoted by $\bar{\lambda}_{i,k}^a$. The index i denotes the ith queue at backbone node k, and that queue is associated with both priority and the next node, and might contain packets from many different sources. Let $\xi_{i,k;v,\gamma}(n)$ denote the fraction of the number of packets that are sent at time n from queue (i,k) to node γ will be assigned to queue v there. Again, owing to the multiplexing and the large number of sources, we suppose a conditional independence in routing in that there are $p_{i,k;v,\gamma}$ such that $E_n\xi_{i,k;v,\gamma}(n) = p_{i,k;v,\gamma}$, where E_n denotes the expectation conditioned on the data to time n.

The queue dynamics are

$$X_{i,k}(n+1) = X_{i,k}(n) + a_{i,k}(n) - d_{i,k}(n) + \sum_{v,\gamma} d_{v,\gamma}(n)\xi_{v,\gamma;i,k}(n)$$

Condition (A2.4) is changed to require the existence of $\{\tilde{q}^j_{i,k}; i,k,j\}$ such that

$$\bar{\lambda}^a_{i,k} + \sum_{j,v,\gamma} \tilde{q}^j_{v,\gamma} p_{v,\gamma:i,k}\Pi_{\gamma,j} - \sum_j \tilde{q}^j_{i,k}\Pi_{k,j} < 0$$

for each i,k. The proof follows the lines of that of Theorem 3.1. The decision rule (2.4) is replaced by, for each node k and channel state j,

$$\max_{\{d_{i,k};i\}} \sum_i \left[X^{p-1}_{i,k}(n) - \sum_{v,\gamma} X^{p-1}_{i,k} p^j_{i,k;v,\gamma} \right] d_{i,k}(n).$$

5. An A Priori Routing Selection. A potentially useful approach for getting the routing and the $\tilde{u}(\cdot)$ functions is based on a type of fluid controlled-flow approximation. In applications the algorithm would be run periodically to produce new routings as conditions change. The example is intended to be illustrative of the possibilities only. Suppose that power only is to be allocated. Let $\bar{p}^j_{i,k,\alpha}$ denote the power assigned to queue (i,k) for data transmitted to node α, when the channel state at node k is j. The associated channel rate is $C^j_{i,k,\alpha}(\bar{p}^j_{i,k,\alpha}) \equiv q^j_{i,k,\alpha}$. The routes to be given might depend on the channel states. But the development in Section 3 is readily modified to account for this dependence.

Suppose that there are upper bounds Q_i such that for each i,j,

$$\sum_{i,\alpha} C^j_{i,k,\alpha}(\bar{p}^j_{i,k,\alpha}) \le Q_k. \tag{5.1}$$

This might reflects the fact that each packet takes a minimal time. Suppose that each node k has a constraint of the form

$$\sum_{i,\alpha} \bar{p}^j_{i,k,\alpha} \le P_k, \quad \text{each } j, \tag{5.2}$$

where P_k is the total energy/slot available at node k. We also need a constraint that assures that the average output for each non-source node equals the average input, and we write this as follows, for each $i, k \neq k(i)$:

$$\overline{\text{out}} = \sum_{\alpha,j} C^j_{i,k,\alpha}(\bar{p}^j_{i,k,\alpha})\Pi_{k,j} \ge \sum_{l,j} C^j_{i,l,k}(\bar{p}^j_{i,l,k})\Pi_{l,j} = \overline{\text{in}}. \tag{5.3}$$

If node $k(i)$ is the input node for source i, then replace (5.3) by

$$\overline{\text{out}} = \sum_{\alpha,j} C^j_{i,k(i),\alpha}(\bar{p}^j_{i,k(i),\alpha})\Pi_{k(i),j} = \bar{\lambda}^a_i + \epsilon. \tag{5.4}$$

The (arbitrarily small) $\epsilon > 0$ is used to assure slight overcapacity so that (A2.4) will hold and the stability argument of Theorem 3.1 can be used. Suppose that $c(i)$ is the destination node for source i. Then to assure that all packets end up where they are intended, for each i use the constraint

$$\sum_{k,j} C^j_{i,k,c(i)}(\bar{p}^j_{i,k,c(i)})\Pi_{k,j} = \bar{\lambda}^a_i + \epsilon. \tag{5.5}$$

Any $q^j_{i,k,\alpha} \equiv C^j_{i,k,\alpha}(\bar{p}^j_{i,k,\alpha})$ that satisfy the constraints (5.1)–(5.5) will yield an acceptable a priori route. But one might wish to select one via an optimization problem. One possible cost criterion is the total average power given by

$$\sum_{i,k,\alpha,j} \bar{p}^j_{i,k,\alpha}\Pi_{k,j}. \tag{5.6}$$

Minimize (5.6), subject to (5.1)–(5.5). The above approach to getting the a priori routes might yield a distributed flow for some sources. However, given these routes, the maximization rules (2.4), (2.5), or (2.6), still work. Replace (2.4) by

$$\max_{\{d_{i,k,\alpha}(n);i,\alpha\}} \sum_{i,\alpha} \left[w_{i,k}X^{p-1}_{i,k}(n) - w_{i,f(j,i,k,\alpha)}X^{p-1}_{i,f(j,i,k,\alpha)}(n) \right] d_{i,k,\alpha}(n),$$

where for each i, j, k, $f(j,i,k,\alpha)$ indexes the links for which $\bar{p}^j_{i,k,\alpha} > 0$ and $d_{i,k,\alpha}(n)$ is the amount sent to node α from queue (i,k).

For multicasting, use (5.5) for all destination nodes for source i.

The criterion (5.6) is concerned with total power. An alternative is to strive for maximum stability. To do this rewrite (5.3) as

$$\sum_{\alpha,j} C^j_{i,k,\alpha}(\bar{p}^j_{i,k,\alpha})\Pi_{k,j} - \sum_{l,j} C^j_{i,l,k}(\bar{p}^j_{i,l,k})\Pi_{l,j} = b_{i,k},$$

where $b_{i,k} > 0$. With appropriate definitions, this can be made to include (5.3) and (5.4). Then either maximize $\sum_{i,k} b_{i,k}$, or seek $\max\min_{i,k} b_{i,k}$. This approach will get routes and $\tilde{q}^j_{i,k,m}$ that yield the best c_0 in (A2.4). In addition, the dual variables associated with the constraints provide "price" guidelines, that tell us the places where an increase in the resources would do the most good (in the sense of the mathematical programming formulation).

The example in [6, Section 5] was concerned with a simpler model, where each packet that was transmitted was required to have a minimum

S/N ratio at the receiver, and the final form of the optimization problem was a linear program.

Comment on another case: bandwidth allocation. Suppose that the basic control is over bandwidth allocation, with the number of packets/slot being proportional to bandwidth as $q^j_{i.k,\alpha} = b^j_{i,k,\alpha} p^j_{i,k,\alpha}$, where the $p^j_{i,k,\alpha}$ are the constants of proportionality and $b^j_{i,k,\alpha}$ is the assigned bandwidth. There would be a total BW constraint of the form $\sum_{i,\alpha} b^j_{i,k,\alpha} \le B_k$ at each node, replacing (5.2). Input-output constraints analogous to (5.3), (5.4), and (5.5), are still to hold. To get the routes, one could either strive for maximum stability or minimize the total average bandwidth, which is

$$\sum_{i,k,\alpha,j} b^j_{i,k,\alpha} \Pi_{k,j}.$$

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