PATTERN FORMATION (II): THE TURING INSTABILITY

YAN GUO AND HYUNG JU HWANG

Abstract. We consider the classical Turing instability in a reaction-diffusion system as the secend part of our study on pattern formation. We prove that nonlinear dynamics of a general perturbation of the Turing instability is determined by the finite number of linear growing modes over a time scale of $\ln \frac{1}{\delta}$, where δ is the strength of the initial perturbation.

1. Growing modes in a reaction-diffusion system

In this section we summarize the classical linear Turing instability criterion for a reaction-diffusion system. Consider a reaction-diffusion system of 2-species as

$$\begin{aligned} \frac{\partial U}{\partial t} &= \nabla \cdot \left(D_1\left(U,V\right)\nabla U\right) + f\left(U,V\right), \\ \frac{\partial V}{\partial t} &= \nabla \cdot \left(D_2\left(U,V\right)\nabla V\right) + g\left(U,V\right), \end{aligned} \tag{1.1}$$

where $U\left(\mathbf{x},t\right), V\left(\mathbf{x},t\right)$ are concentration for species, D_1, D_2 diffusion coefficients, f, g reaction terms.

In this paper we consider a d-dimensional box $\mathbb{T}^d = \left(0,\pi\right)^d$, $d = 1,2,3$, with Neumann boundary conditions for U and V, i.e.,

$$\frac{\partial U}{\partial x_i} = \frac{\partial V}{\partial x_i} = 0 \quad \text{at } x_i = 0, \pi, \quad \text{for } 1 \leq i \leq d. \tag{1.2}$$

Homogeneous steady state $U = \bar{U}, V = \bar{V}$ forms a steady state provided

$$0 = f\left(\bar{U},\bar{V}\right) = g\left(\bar{U},\bar{V}\right). \tag{1.3}$$

In this article, we study the nonlinear evolution of a perturbation

$$u(x,t) = U(x,t) - \bar{U}, \ v(x,t) = V(x,t) - \bar{V}$$

1991 *Mathematics Subject Classification.*

around $[\bar{U},\bar{V}]$, which satisfies the equivalent reaction-diffusion system:

$$\frac{\partial u}{\partial t}=\nabla\cdot\left(D_1\left(u+\bar{U},v+\bar{V}\right)\nabla u\right)+f\left(u+\bar{U},v+\bar{V}\right), \tag{1.4}$$

$$\frac{\partial v}{\partial t}=\nabla\cdot\left(D_2\left(u+\bar{U},v+\bar{V}\right)\nabla v\right)+g\left(u+\bar{U},v+\bar{V}\right). \tag{1.5}$$

The corresponding linearized system then takes the form

$$u_t=\bar{D}_1\nabla^2u+\bar{f}_uu+\bar{f}_vv, \tag{1.6}$$

$$v_t=\bar{D}_2\nabla^2v+\bar{g}_uu+\bar{g}_vv, \tag{1.7}$$

where $\bar{D}_1=D_1\left(\bar{U},\bar{V}\right)$, $\bar{D}_2=D_2\left(\bar{U},\bar{V}\right)$, $\bar{f}_u=\frac{\partial f}{\partial u}\left(\bar{U},\bar{V}\right), \bar{f}_v=\frac{\partial f}{\partial v}\left(\bar{U},\bar{V}\right), \bar{g}_u=\frac{\partial g}{\partial u}\left(\bar{U},\bar{V}\right), \bar{g}_v=\frac{\partial g}{\partial v}\left(\bar{U},\bar{V}\right)$.

We use $[\cdot,\cdot]$ to denote a column vector, and let

$$\mathbf{w}(x,t)\equiv[u(x,t),v(x,t)],\ \bar{\mathbf{W}}=[\bar{U},\bar{V}].$$

Then the original nonlinear system (1.4) and (1.5) can be written in a matrix form:

$$\begin{aligned}\frac{\partial\mathbf{w}}{\partial t}&=\nabla\cdot(D\nabla\mathbf{w})+\mathbf{F}\\&=(\bar{D}\nabla^2\mathbf{w}+A\mathbf{w})+(\{\nabla\cdot(D\nabla\mathbf{w})-\bar{D}\nabla^2\mathbf{w}\}+\mathbf{F}-A\mathbf{w})\\&\equiv\mathcal{L}(\mathbf{w})+\mathcal{N}(\mathbf{w}).\end{aligned} \tag{1.8}$$

where

$$D=\begin{pmatrix}D_1\left(\mathbf{w}+\bar{\mathbf{W}}\right)&0\\0&D_2\left(\mathbf{w}+\bar{\mathbf{W}}\right)\end{pmatrix},\ \bar{D}=\begin{pmatrix}\bar{D}_1&0\\0&\bar{D}_2\end{pmatrix},$$

$$\mathbf{F}=\begin{pmatrix}f\left(\mathbf{w}+\bar{\mathbf{W}}\right)\\g\left(\mathbf{w}+\bar{\mathbf{W}}\right)\end{pmatrix},\ A=\begin{pmatrix}\bar{f}_u&\bar{f}_v\\\bar{g}_u&\bar{g}_v\end{pmatrix}.$$

Let $\mathbf{q}=(q_1,..,q_d)\in\Omega=(\mathbb{N}\cup\{0\})^d$ and let

$$e_{\mathbf{q}}(x)\equiv\prod_{i=1}^{d}\cos\left(q_ix_i\right),$$

where $\mathbf{q}\in\Omega$. Then $\{e_{\mathbf{q}}(x)\}_{\mathbf{q}\in\Omega}$ forms a basis of the space of functions in $\mathbb{T}^d$ that satisfy Neumann boundary condition (1.2).

We look for a normal mode to the linear reaction-diffusion system (1.6) and (1.7) of the following form:

$$\mathbf{w}\left(x,t\right)=\mathbf{r}_{\mathbf{q}}\exp\left(\lambda_{\mathbf{q}}t\right)e_{\mathbf{q}}(x), \tag{1.9}$$

where $\mathbf{r}_{\mathbf{q}}$ is a vector depending on $\mathbf{q}$. We substitute (1.9) into (1.6)-(1.7) to get

$$\lambda_{\mathbf{q}}\mathbf{r}_{\mathbf{q}}=\begin{pmatrix}\bar{f}_u-\bar{D}_1q^2&\bar{f}_v\\\bar{g}_u&\bar{g}_v-\bar{D}_2q^2\end{pmatrix}\mathbf{r}_{\mathbf{q}},$$

where $q^2 = \sum_{i=1}^d q_i^2$. A nontrivial normal mode can be obtained by setting

$$\det \begin{pmatrix} \lambda_{\mathbf{q}} - \bar{f}_u + \bar{D}_1 q^2 & -\bar{f}_v \\ -\bar{g}_u & \lambda_{\mathbf{q}} - \bar{g}_v + \bar{D}_2 q^2 \end{pmatrix} = 0.$$

This leads to the following dispersion formula for $\lambda_{\mathbf{q}}$:

$$\lambda_{\mathbf{q}}^2 + \{-\bar{f}_u + \bar{D}_1 q^2 - \bar{g}_v + \bar{D}_2 q^2\}\lambda_{\mathbf{q}} + \{\left(\bar{f}_u - \bar{D}_1 q^2\right)\left(\bar{g}_v - \bar{D}_2 q^2\right) - \bar{f}_v \bar{g}_u\} = 0. \tag{1.10}$$

We assume first that without diffusion, the $\lambda_{\mathbf{q}}$ has negative real part (stable):

$$\operatorname{tr} A = \bar{f}_u + \bar{g}_v < 0, \ \det A = \bar{f}_u \bar{g}_v - \bar{f}_v \bar{g}_u > 0, \tag{1.11}$$

On the other hand, in the presence of diffusion, we assume the following diffusion-driven (linear) instability criterion by requiring there exists a q such that

$$\left(\bar{f}_u - \bar{D}_1 q^2\right)\left(\bar{g}_v - \bar{D}_2 q^2\right) - \bar{f}_v \bar{g}_u < 0, \tag{1.12}$$

which ensures that (1.10) has at least one positive root $\lambda_{\mathbf{q}}$.

Remark 1. *To satisfy (1.11) and (1.12), the discriminant for the quadratic equation for q^2 in (1.12) must be positive:*

$$\left(\bar{f}_u \bar{D}_2 + \bar{g}_v \bar{D}_1\right) > 2\sqrt{\bar{D}_1 \bar{D}_2} \det A > 0, \tag{1.13}$$

which means the range of inhibition $\sqrt{\bar{D}_2 / |\bar{g}_v|}$ is larger than the range of activation $\sqrt{\bar{D}_1 / |\bar{f}_u|}$. From (1.11) and (1.13), it follows that

$$\bar{f}_u \bar{g}_v < 0, \quad \textit{and} \quad \bar{f}_v \bar{g}_u < 0, \tag{1.14}$$

and we have only two cases for A:

$$A = \begin{pmatrix} + & - \\ + & - \end{pmatrix} \quad \textit{or} \quad A = \begin{pmatrix} + & + \\ - & - \end{pmatrix},$$

where formal case is called activator-inhibitor (or predator-prey) and the latter positive feedback. It also follows from (1.11) that

$$\bar{D}_1 \neq \bar{D}_2.$$

For given $\mathbf{q} \in \Omega$, we denote the corresponding eigenvalues by $\lambda_\pm(\mathbf{q})$ and eigenvectors by $\mathbf{r}_\pm(\mathbf{q})$. We split into the three cases for the linear analysis:

(1) Generic case where we have two independent real eigenvectors and we denote

$$\Omega_{\text{generic}} \equiv \{\mathbf{q} \in \Omega \text{ such that } \mathbf{r}_+(\mathbf{q}) \neq \mathbf{r}_-(\mathbf{q})\}.$$

By an elementary computation of the discriminant of (1.10), we have, except for only *finitely* many q,

$$\left(\bar{D}_1 - \bar{D}_2\right) q^4 - \mathrm{tr}\ A\ \left(\bar{D}_1 + \bar{D}_2\right) q^2 + 4\left(\bar{f}_u \bar{D}_2 + \bar{g}_v \bar{D}_1\right) q^2 + (\mathrm{tr}\ A)^2 - 4\det A > 0,$$

since $\bar{D}_1 - \bar{D}_2 \neq 0$. Therefore, there are two distinct real roots such that

$$\lambda_-(\mathbf{q}) < \lambda_+(\mathbf{q})$$

for large q. Since $\bar{f}_v \neq 0$ in (1.14), the corresponding (linearly independent) eigenvectors $\mathbf{r}_-(\mathbf{q})$ and $\mathbf{r}_+(\mathbf{q})$ are given by

$$\mathbf{r}_\pm(\mathbf{q}) = \left[1, \frac{\lambda_\pm(\mathbf{q}) - \bar{f}_u + \bar{D}_1 q^2}{\bar{f}_v}\right]. \tag{1.15}$$

It is easy to see from (1.12) that there exist only finitely many $\mathbf{q}$ such that $\lambda_+(\mathbf{q}) > 0$. We therefore can denote the largest eigenvalue by $\lambda_{\max} > 0$ and define

$$\Omega_{\max} \equiv \{\mathbf{q} \in \Omega \text{ such that } \lambda_+(\mathbf{q}) = \lambda_{\max}\}.$$

We also denote $\nu > 0$ to be the gap between the $\lambda_{\max}$ and the rest. Moreover, there is one q^2 (possibly two) having $\lambda_{\mathbf{q}}^+(q^2) = \lambda_{\max}$ when we regard $\lambda_{\mathbf{q}}^+$ as a function of q^2.

(2) Defective case where we have the repeated real eigenvalues and eigenvectors:

Note that there may be possibly one q^2 (so finitely many $\mathbf{q}$) such that from (1.11)

$$\lambda_+(\mathbf{q}) = \lambda_-(\mathbf{q}) \equiv \lambda(\mathbf{q}) = \{\bar{f}_u + \bar{g}_v - \left(\bar{D}_1 + \bar{D}_2\right) q^2\}/2 < 0 \tag{1.16}$$

and $\mathbf{r}_+(\mathbf{q}) = \mathbf{r}_-(\mathbf{q}) \equiv \mathbf{r}(\mathbf{q})$ and we denote

$$\Omega_{\text{defective}} \equiv \{\mathbf{q} \in \Omega \text{ such that } \mathbf{r}_+(\mathbf{q}) = \mathbf{r}_-(\mathbf{q})\}.$$

In this case we find another independent vector

$$\mathbf{r}'(\mathbf{q}) = [0, \frac{1}{\bar{f}_v}]$$

satisfying $(A - \lambda(\mathbf{q})I)\,\mathbf{r}'(\mathbf{q}) = \mathbf{r}(\mathbf{q})$.

(3) Complex case where we have complex eigenvalues for q and we denote it by $\Omega_{\text{complex}} \equiv \Omega - (\Omega_{\text{generic}} \cup \Omega_{\text{defective}})$. For $\mathbf{q} \in \Omega_{\text{complex}}$, we denote $\lambda_+(\mathbf{q}) \equiv \mathrm{Re}\,\lambda(\mathbf{q}) + i\,\mathrm{Im}\,\lambda(\mathbf{q})$ and $\mathbf{r}_+(\mathbf{q}) \equiv \mathrm{Re}\,\mathbf{r}(\mathbf{q}) + i\,\mathrm{Im}\,\mathbf{r}(\mathbf{q})$. Then we have $\lambda_-(\mathbf{q}) \equiv \mathrm{Re}\,\lambda(\mathbf{q}) - i\,\mathrm{Im}\,\lambda(\mathbf{q})$ and $\mathbf{r}_-(\mathbf{q}) \equiv \mathrm{Re}\,\mathbf{r}(\mathbf{q}) - i\,\mathrm{Im}\,\mathbf{r}(\mathbf{q})$. Notice that $\mathrm{Re}\,\lambda(\mathbf{q}) < 0$ as in (1.16), and $\mathrm{Re}\,\mathbf{r}(\mathbf{q})$ and $\mathrm{Im}\,\mathbf{r}(\mathbf{q})$ are linearly independent vectors.

Given any initial perturbation $\mathbf{w}(\mathbf{x},0)$, we can expand it as

$$\begin{aligned}\mathbf{w}(\mathbf{x},0) = \sum_{\mathbf{q}\in\Omega}\mathbf{w}_{\mathbf{q}}e_{\mathbf{q}}(x) &= \sum_{\mathbf{q}\in\Omega_{\text{generic}}}\{w_{\mathbf{q}}^{-}\mathbf{r}_{-}(\mathbf{q}) + w_{\mathbf{q}}^{+}\mathbf{r}_{+}(\mathbf{q})\}e_{\mathbf{q}}(x)\\ &+ \sum_{\mathbf{q}\in\Omega_{\text{defective}}}\{w_{\mathbf{q}}\mathbf{r}(\mathbf{q}) + w'_{\mathbf{q}}\mathbf{r}'(\mathbf{q})\}e_{\mathbf{q}}(x)\\ &+ \sum_{\mathbf{q}\in\Omega_{\text{complex}}}\{w_{\mathbf{q}}^{\text{Re}}\operatorname{Re}\mathbf{r}(\mathbf{q}) + w_{\mathbf{q}}^{\text{Im}}\operatorname{Im}\mathbf{r}(\mathbf{q})\}e_{\mathbf{q}}(x),\end{aligned}$$

so that

$$\begin{aligned}\mathbf{w}_{\mathbf{q}} &= w_{\mathbf{q}}^{-}\mathbf{r}_{-}(\mathbf{q}) + w_{\mathbf{q}}^{+}\mathbf{r}_{+}(\mathbf{q}) \quad \text{for} \quad \mathbf{q}\in\Omega_{\text{generic}},\\ \mathbf{w}_{\mathbf{q}} &= w_{\mathbf{q}}\mathbf{r}(\mathbf{q}) + w'_{\mathbf{q}}\mathbf{r}'(\mathbf{q}) \qquad \text{for} \quad \mathbf{q}\in\Omega_{\text{defective}},\\ \mathbf{w}_{\mathbf{q}} &= w_{\mathbf{q}}^{\text{Re}}\operatorname{Re}\mathbf{r}(\mathbf{q}) + w_{\mathbf{q}}^{\text{Im}}\operatorname{Im}\mathbf{r}(\mathbf{q}) \text{ for} \quad \mathbf{q}\in\Omega_{\text{complex}}.\end{aligned} \tag{1.17}$$

The unique solution $\mathbf{w}(x,t) = [u(x,t), v(x,t)]$ to (1.6)-(1.7) is given by

(1.18)

$$\begin{aligned}\mathbf{w}(x,t) = &\sum_{\mathbf{q}\in\Omega_{\text{generic}}}\{w_{\mathbf{q}}^{-}\mathbf{r}_{-}(\mathbf{q})\exp\left(\lambda_{\mathbf{q}}^{-}t\right) + w_{\mathbf{q}}^{+}\mathbf{r}_{+}(\mathbf{q})\exp\left(\lambda_{\mathbf{q}}^{+}t\right)\}e_{\mathbf{q}}(x)\\ &+ \sum_{\mathbf{q}\in\Omega_{\text{defective}}}\{\left(w_{\mathbf{q}}\mathbf{r}(\mathbf{q}) + w'_{\mathbf{q}}\mathbf{r}'(\mathbf{q})\right) + w'_{\mathbf{q}}\mathbf{r}(\mathbf{q})t\}\exp\left(\lambda_{\mathbf{q}}t\right)e_{\mathbf{q}}(x)\\ &+ \sum_{\mathbf{q}\in\Omega_{\text{complex}}}\{w_{\mathbf{q}}^{\text{Re}}\left(\operatorname{Re}\mathbf{r}(\mathbf{q})\cos\left[\left(\operatorname{Im}\lambda_{\mathbf{q}}\right)t\right] - \operatorname{Im}\mathbf{r}(\mathbf{q})\sin\left[\left(\operatorname{Im}\lambda_{\mathbf{q}}\right)t\right]\right)\\ &+ w_{\mathbf{q}}^{\text{Im}}\left(\operatorname{Re}\mathbf{r}(\mathbf{q})\sin\left[\left(\operatorname{Im}\lambda_{\mathbf{q}}\right)t\right] + \operatorname{Im}\mathbf{r}(\mathbf{q})\cos\left[\left(\operatorname{Im}\lambda_{\mathbf{q}}\right)t\right]\right)\}\exp[\left(\operatorname{Re}\lambda_{\mathbf{q}}\right)t]e_{\mathbf{q}}(x)\\ \equiv\ & e^{\mathcal{L}t}\mathbf{w}(x,0).\end{aligned}$$

For any $\mathbf{u}(\cdot,t) \in \left[L^2\left(\mathbb{T}^d\right)\right]^2$, we denote $\|\mathbf{u}(\cdot,t)\| \equiv \|\mathbf{u}(\cdot,t)\|_{L^2}$. Our main result of this section is

Lemma 1. *Assume that (1.11) and the instability criterion (1.12) are valid. Suppose*

$$\mathbf{w}(x,t) = [u(x,t), v(x,t)] \equiv e^{\mathcal{L}t}\mathbf{w}(x,0)$$

as in (1.18) is a solution to the linearized reaction-diffusion system (1.6)-(1.7) with initial condition $\mathbf{w}(\mathbf{x},0)$*. Then there exists a constant* $C_1 \geq 1$ *depending on* $\bar{U}, \bar{V}, \bar{D}_1, \bar{D}_2, A$ *such that*

$$\|\mathbf{w}(\cdot,t)\| \leq C_1\exp\left(\lambda_{max}t\right)\|\mathbf{w}(\cdot,0)\|,$$

for all $t \geq 0$.

Proof. We first notice that from the quadratic formula for (1.10), for q large,

$$|\det[\mathbf{r}_-(\mathbf{q}), \mathbf{r}_+(\mathbf{q})]| = \frac{\lambda^+_{\mathbf{q}} - \lambda^-_{\mathbf{q}}}{\left|\bar{f}_v\right|} \geq c\frac{\left|\bar{D}_1 - \bar{D}_2\right|}{\left|\bar{f}_v\right|}q^2.$$

Thus solving (1.17) yields, due to $\bar{D}_1 \neq \bar{D}_2$,

$$\begin{aligned} |w^{\pm}_{\mathbf{q}}| &\leq \frac{1}{\det[\mathbf{r}_-(\mathbf{q}), \mathbf{r}_+(\mathbf{q})]}|\mathbf{r}_\pm(\mathbf{q})| \times |\mathbf{w}_{\mathbf{q}}| \\ &\leq C|\mathbf{w}_{\mathbf{q}}|, \end{aligned}$$

Since $\lambda_{\mathbf{q}} < 0$, for $q \in \Omega_{\text{defective}}$, we have

$$t\exp(\lambda_{\mathbf{q}}t) \leq C.$$

Moreover, recall $\operatorname{Re}\lambda(\mathbf{q}) < 0$ for $q \in \Omega_{\text{complex}}$. Thus we deduce the Lemma on the linear growth rate by the formula (1.18). ☐

2. Main Result

Let θ be a small fixed constant, and $\lambda_{\max}$ be the dominant eigenvalue which is the maximal growth rate. We also denote the gap between the largest growth rate $\lambda_{\max}$ and the rest by $\nu > 0$. Then for $\delta > 0$ arbitrary small, we define the escape time T^δ by

$$\theta = \delta\exp\left(\lambda_{\max}T^\delta\right), \tag{2.1}$$

or equivalently

$$T^\delta = \frac{1}{\lambda_{\max}}\ln\frac{\theta}{\delta}.$$

Our main theorem is

Theorem 1. *Assume (1.11) and that there exists $q^2 = \sum_{i=1}^d q_i^2$ satisfying instability criterion (1.12). Let*

$$\begin{aligned} \mathbf{w}_0(x) = &\sum_{\mathbf{q}\in\Omega}\{w^-_{\mathbf{q}}\mathbf{r}_-(\mathbf{q}) + w^+_{\mathbf{q}}\mathbf{r}_+(\mathbf{q})\}e_{\mathbf{q}}(x) \\ &+ \sum_{\mathbf{q}\in\Omega_{defective}}\{w_{\mathbf{q}}\mathbf{r}(\mathbf{q}) + w'_{\mathbf{q}}\mathbf{r}'(\mathbf{q})\}e_{\mathbf{q}}(x) \\ &+ \sum_{\mathbf{q}\in\Omega_{complex}}\{w^{\text{Re}}_{\mathbf{q}}\operatorname{Re}\mathbf{r}(\mathbf{q}) + w^{\text{Im}}_{\mathbf{q}}\operatorname{Im}\mathbf{r}(\mathbf{q})\}e_{\mathbf{q}}(x). \end{aligned}$$

$\in H^2$ such that $||\mathbf{w}_0|| = 1$. Assume D_1, D_2 $f, g \in C^2$ near $\bar{W}$, so that there exists $\eta > 0$
(2.2)

$$C_\eta \equiv \max_{||w||_\infty \leq \eta} \{\sum_{i=1}^{2} ||D_i(\bar{W}+w)||_{C^2} + ||f(\bar{W}+w)||_{C^2} + ||g(\bar{W}+w)||_{C^2} < \infty.$$

Then there exist constants $\delta_0 > 0$, $C > 0$, and $\theta > 0$, depending on $\bar{U}, \bar{V}, \bar{D}_1, \bar{D}_2, f, g$, such that for all $0 < \delta \leq \delta_0$, if the initial perturbation of the steady state $[\bar{U}, \bar{V}]$ in (1.3) is

$$\mathbf{w}^\delta(\mathbf{x}, 0) = \delta \mathbf{w}_0,$$

then its nonlinear evolution $\mathbf{w}^\delta(t, x)$ satisfies

$$\begin{aligned} (2.3) \qquad & ||\mathbf{w}^\delta(t,x) - \delta e^{\lambda_{\max} t} \sum_{\mathbf{q} \in \Omega_{max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}(x)|| \\ & \leq C\{e^{-\nu t} + \delta ||w_0||_{H^2}^2 + \delta e^{\lambda_{\max} t}\} \delta e^{\lambda_{\max} t} \end{aligned}$$

for $0 \leq t \leq T^\delta$, and $\nu > 0$ is the gap between $\lambda_{\max}$ and the rest of $\operatorname{Re} \lambda_{\mathbf{q}}$ in (1.10).

We notice that for $0 \leq t \leq T^\delta$, $\delta e^{\lambda_{\max} t} \leq \theta$, is sufficiently small. The initial profile $\mathbf{w}_0$ is any H^2 function. In particular, as long as $w_{\mathbf{q}_0}^+ \neq 0$ for at least one $\mathbf{q}_0 \in \Omega_{\max}$ (generic for a general H^2 perturbation), the part of its fastest growing modes satisfies

$$||\delta e^{\lambda_{\max} t} \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}|| \geq \delta e^{\lambda_{\max} t} |w_{\mathbf{q}_0}^+| |\mathbf{r}_+(\mathbf{q}_0)|,$$

which has the dominant leading order of $\delta e^{\lambda_{\max} t}$. Our estimate (2.3) implies that the dynamics of a general perturbation can be characterized by such linear dynamics over a long time period of $\varepsilon T^\delta \leq t \leq T^\delta$, for any fixed constant $\varepsilon > 0$. In particular, choose a fixed $\mathbf{q}_0 \in \Omega_{\max}$ and let

$$w_0(x) = \frac{\mathbf{r}_+(\mathbf{q}_0)}{|\mathbf{r}_+(\mathbf{q}_0)|} e_{\mathbf{q}_0}(x)$$

then if $t = T^\delta$,

$$\left\| \mathbf{w}^\delta(t, \cdot) - \delta e^{\lambda_{\max} T^\delta} \frac{\mathbf{r}_+(\mathbf{q}_0)}{|\mathbf{r}_+(\mathbf{q}_0)|} e_{\mathbf{q}_0}(\cdot) \right\| \leq C\{\delta^{\nu/\lambda_{\max}} + \theta^2\},$$

hence

$$\left\| \mathbf{w}^\delta(t, \cdot) \right\| \geq \theta - C\{\delta^{\nu/\lambda_{\max}} + \theta^2\} \geq \theta/2 > 0,$$

which implies nonlinear instability as $\delta \to 0$. The instability occurs before the possible blow-up time.

Reaction-diffusion systems are often employed to study chemical and biological pattern formation and have received much attention from scientists [3], [4], [14], [13], [16], since the pioneering work of Turing [17] in 1951. This symmetry breaking instability is called diffusion-driven instability, since the presence of diffusion and the difference of diffusion coefficients are essential for the instability mechanism and nonuniform patterns formation. After some experimental results such as in [2], [12], [15], more extensive and serious works began towards this Turing-like pattern formation across many fields of study. Our result can be interpreted as a mathematical description of early pattern formation. Each initial perturbation can be drastically different from another, which gives rise to the richness of the pattern; on the other hand, the finite number maximal growing modes determine the common characteristics of the pattern, over the time scale of $\ln \frac{1}{\delta}$. In comparision with an earlier *different* result along this direction [18]: First of all, the reaction-diffusion system considered here is *not* scaled. Secondly, our initial perturbation is more general, need not to be close to the space of finite number of maximal growing modes. Thirdly, a precise estimate of the time scale ($\ln \frac{1}{\delta}$) for pattern formation is given here, without an *a-priori* assumption for the smallness of the perturbation later in time as in [18]. Lastly, based on Guo-Strauss' bootstrap argument, our proof is much simpler and direct

3. Bootstrap Lemma

We state existence of local-in-time solutions for (1.4)-(1.5).

Lemma 2. *(Local existence) For $s \geq 1$ $(d=1)$ and $s \geq 2$ $(d=2,3)$, there exist a $T>0$ and a constant C depending on $\bar{U}, \bar{V}, D_1, D_2, f, g$ such that $\|\mathbf{w}(t)\|_{H^s}$ is continuous in $[0,T)$, and*

$$\|\mathbf{w}(t)\|_{H^s} \leq C \|\mathbf{w}(0)\|_{H^s}.$$

We now derive the following energy estimates for d-dimensional reaction-diffusion system with $d = 1, 2, 3$.

Lemma 3. *Suppose that $[u(x,t), v(x,t)]$ is a solution to the full system (1.4)-(1.5). Then for $||w(t)||_{H^2} \leq \eta$,*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{|\partial|=2}\int_{\mathbb{T}^d}\{|\partial u|^2 + |\partial v|^2\}d\mathbf{x} \\
&+ \sum_{|\partial|=2}\int_{\mathbb{T}^d}\left\{\frac{\bar{D}_1}{2}|\nabla\partial u|^2 + \bar{D}_2|\nabla\partial v|^2\right\}d\mathbf{x} + \frac{|\bar{g}_v|}{2}\sum_{|\alpha|=2}\int_{\mathbb{T}^d}|\partial v|^2 \\
&\leq C_0C_1||\mathbf{w}||_{H^2}||\nabla^3\mathbf{w}||^2 + C_2||u||^2.
\end{aligned}$$

where C_0 is the universal constant while $C_1 = C_0 C_\eta(1+\eta)$ and

$$C_2 = \frac{\left(\frac{\left(\bar{f}_v+\bar{g}_u\right)^2}{2|\bar{g}_v|} + \bar{f}_u\right)^3}{\bar{D}_1^2}.$$

Proof. We first notice that the reaction-diffusion system (1.4)-(1.5) preserves the evenness of the solution $\mathbf{w}(x,t)$, i.e., if $\mathbf{w}(x,t)$ is a solution, then $\mathbf{w}(-x_i,t)$ is also a solution. We can regard the Neumann problem as a special case with evenness of the periodic problem by standard way of even extension $\mathbf{w}(x,t)$ with respect to one of the x_i. For this reason we may assume periodicity at the boundary of the extended periodic box $2\mathbf{T}^3 \equiv (-\pi,\pi)^d$. Since now there is no contributions from the boundaries, we can take second order ∂-derivative of (1.8) to get

$$\frac{1}{2}\frac{d}{dt}\int_{2\mathbb{T}^d} |\partial \mathbf{w}|^2 = \int_{2\mathbb{T}^d} \partial \mathbf{w}^T \partial \mathcal{L}(\mathbf{w}) + \int_{2\mathbb{T}^d} \partial \mathbf{w}^T \partial \mathcal{N}(\mathbf{w}). \tag{3.1}$$

We first treat the last nonlinear term:

$$\begin{aligned}
&-\int_{2\mathbb{T}^d} \{\nabla\partial\mathbf{w}\}^T [\partial\{D\left(\mathbf{w}+\bar{\mathbf{W}}\right)\nabla\mathbf{w}\} + \bar{D}\nabla\partial\mathbf{w}] + \{\nabla\partial\mathbf{w}\}^T \partial\left(\mathbf{F}-A\mathbf{w}\right)\\
\le& C\left\|D\left(\mathbf{w}+\bar{\mathbf{W}}\right)-\bar{D}\right\|_\infty \|\nabla\partial\mathbf{w}\|^2 + C\left\|(\nabla D)\left(\mathbf{w}+\bar{\mathbf{W}}\right)\right\|_\infty \|\nabla\mathbf{w}\|_\infty \|\partial\mathbf{w}\| \|\nabla\partial\mathbf{w}\|\\
&+ C\left\|(\partial D)\left(\mathbf{w}+\bar{\mathbf{W}}\right)\right\|_\infty \|\nabla\mathbf{w}\|_{L^4}^2 \|\nabla\mathbf{w}\|_\infty \|\nabla\partial\mathbf{w}\|\\
&+ C\left\|(\partial\mathbf{F})\left(\mathbf{w}+\bar{\mathbf{W}}\right)\right\|_\infty \|\nabla\mathbf{w}\|_\infty \|\nabla\mathbf{w}\| \|\nabla\partial\mathbf{w}\| + C\left\|\nabla\mathbf{F}\left(\mathbf{w}+\bar{\mathbf{W}}\right)-A\right\|_\infty \|\partial\mathbf{w}\| \|\nabla\partial\mathbf{w}\|.
\end{aligned}$$

We apply the following the Sobolev imbedding to control $||w||_\infty$

$$\|g\|_{L^\infty\left(2\mathbb{T}^d\right)} \le C_0 \|g\|_{H^2\left(2\mathbb{T}^d\right)}, \tag{3.2}$$

for $d \le 3$. Moreover, from the periodic boundary conditions,

$$\int_{2\mathbb{T}^d} \nabla u = \int_{2\mathbb{T}^d} \nabla v = 0,$$

we also use the Poincare inequality

$$||g|| \le \|g\|_{L^4(2\mathbb{T}^d)} \le C_0 \|\nabla g\| \quad \text{if } d \le 3. \tag{3.3}$$

to further get

$$||\nabla\mathbf{w}||_\infty \le C_0 \|\nabla\mathbf{w}\|_{H^2} \le C_0 \sum_{|\partial|=2} ||\partial\nabla\mathbf{w}||.$$

where C_0 is a universal constant. From (2.2) and the assumption $||\mathbf{w}||_{H^2} \le \eta$, the last nonlinear term in (3.1) is bounded by

$$C_0 C_\eta (1+\eta) \|\mathbf{w}\|_{H^2} \|\nabla\partial\mathbf{w}\|^2.$$

We now estimate the second quadratic term in (3.1)

$$-\int_{2\mathbb{T}^d}\{\bar{D}_1\left|\nabla\partial u\right|^2+\bar{D}_2\left|\nabla\partial v\right|^2\}+\bar{g}_v\int_{2\mathbb{T}^d}\left|\partial v\right|^2$$
$$+\left(\bar{f}_v+\bar{g}_u\right)\int_{2\mathbb{T}^d}\partial u\partial v+\bar{f}_u\int_{2\mathbb{T}^d}\left|\partial u\right|^2.$$

The last two terms are bounded by

$$\left(\bar{f}_v+\bar{g}_u\right)\int_{2\mathbb{T}^d}\partial u\partial v+\bar{f}_u\int_{2\mathbb{T}^d}\left|\partial u\right|^2$$
$$\leq\frac{\left|\bar{g}_v\right|}{2}\int_{2\mathbb{T}^d}\left|\partial v\right|^2+\{\frac{\left(\bar{f}_v+\bar{g}_u\right)^2}{2\left|\bar{g}_v\right|}+\bar{f}_u\}\int_{2\mathbb{T}^d}\left|\partial u\right|^2.$$

Thus we can bound the linear term in (3.1) by ($\bar{g}_v<0$)

$$-\int_{2\mathbb{T}^d}\{\bar{D}_1\left|\nabla\partial u\right|^2+\bar{D}_2\left|\nabla\partial v\right|^2\}-\frac{\left|\bar{g}_v\right|}{2}\int_{2\mathbb{T}^d}\left|\partial v\right|^2$$
$$+\{\frac{\left(\bar{f}_v+\bar{g}_u\right)^2}{2\left|\bar{g}_v\right|}+\bar{f}_u\}\int_{2\mathbb{T}^d}\left|\partial u\right|^2.$$

By the interpolation between $\|\nabla\partial u\|$ and $||u||$, the last term above is bounded by

$$\{\frac{\left(\bar{f}_v+\bar{g}_u\right)^2}{2\left|\bar{g}_v\right|}+\bar{f}_u\}\{a\int_{2\mathbb{T}^d}\|\nabla\partial u\|^2+\frac{1}{4a^2}\int_{2\mathbb{T}^d}||u||^2\}$$

for any $a>0$. We can choose a such that

$$\{\frac{\left(\bar{f}_v+\bar{g}_u\right)^2}{2\left|\bar{g}_v\right|}+\bar{f}_u\}a=\frac{1}{2}\bar{D}_1.$$

Collecting terms, we conclude the proof. □

We are now ready to establish the bootstrap lemma, which controls the H^2 growth of $\mathbf{w}(x,t)$ in term of its L^2 growth nonlinearly.

Lemma 4. *Suppose that* $\mathbf{w}(x,t)$ *is a solution to the full system (1.4)-(1.5) such that for* $0\leq t\leq T$

$$||\mathbf{w}(\cdot,t)||_{H^2}\leq\min\left\{\eta,\frac{\bar{D}_1}{2C_0C_1},\frac{\bar{D}_2}{C_0C_1}\right\}$$

and

$$||\mathbf{w}(\cdot,t)||\leq 2C_1e^{\lambda_{\max}t}||\mathbf{w}(\cdot,0)||, \tag{3.4}$$

then we have for $0\leq t\leq T$

$$||\mathbf{w}(t)||_{H^2}^2\leq C_3\{||\mathbf{w}(0)||_{H^2}^2+e^{2\lambda_{\max}t}||\mathbf{w}(\cdot,0)||^2\}$$

where $C_3 = C_1^2 \max\{\frac{4C_2}{\lambda_{max}}, 1\} \geq 1$.

Proof. It suffices to only consider the second-order derivatives of $\mathbf{w}(x,t)$. From the previous lemma and our assumption for $||\mathbf{w}||_{H^2}$, we deduce that

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=2}\int_{\mathbb{T}^d}\left\{|\partial u|^2 + |\partial v|^2\right\} d\mathbf{x} \leq C_2||u||^2.$$

So that by (3.4) and an integration from 0 to $t \leq T$, we have

$$\begin{aligned}&\sum_{|\partial|=2}\int_{\mathbb{T}^d}\left\{|\partial u(t)|^2 + |\partial v(t)|^2\right\}\\ &\leq \sum_{|\partial|=2}\int_{\mathbb{T}^d}\left\{|\partial u(0)|^2 + |\partial v(0)|^2\right\} + \frac{4C_2C_1^2}{\lambda_{\max}}e^{2\lambda_{\max}t}||\mathbf{w}(\cdot,0)||^2.\end{aligned}$$

Thus our lemma follows. □

4. Nonlinear instability and pattern formation

We now prove our main Theorem 1:

Proof. Let $\mathbf{w}^\delta(x,t)$ be the family of solutions to the reaction-diffusion system (1.4)-(1.5) with initial data $\mathbf{w}^\delta(x,0) = \delta\mathbf{w}_0$. Define T^* by

$$T^* = \sup\left\{t \mid \left\|\mathbf{w}^\delta(t) - \delta e^{\mathcal{L}t}\mathbf{w}_0\right\| \leq \frac{C_1}{2}\delta\exp\left(\lambda_{\max}t\right)\right\}.$$

Note that T^* is well defined. We also define

$$T^{**} = \sup\left\{t \mid ||\mathbf{w}(t)||_{H^2} \leq \min\left\{\eta, \frac{\bar{D}_1}{2C_0C_1}, \frac{\bar{D}_2}{C_0C_1}\right\}\right\}.$$

We now derive estimates for H^2 norm of $\mathbf{w}^\delta(x,t)$ for $0 \leq t \leq \min\{T^*, T^{**}\}$. First of all, by the definition of T^*, for $t \leq T^*$ and Lemma 1

$$\left\|\mathbf{w}^\delta(t)\right\| \leq \frac{3C_1}{2}\delta\exp\left(\lambda_{\max}t\right).$$

Moreover, using Lemma 4 and applying a bootstrap argument yields

$$\left\|\mathbf{w}^\delta(t)\right\|_{H^2} \leq \sqrt{C_3}\{\delta||\mathbf{w}_0||_{H^2} + \delta e^{\lambda_{\max}t}\}. \tag{4.1}$$

We now estimate the L^2 norm of $\mathbf{w}^\delta(x,t)$ for $0 \leq t \leq \min\{T^*, T^{**}\}$. We apply Duhamel's principle to obtain

$$\mathbf{w}^\delta(t) = \delta e^{\mathcal{L}t}\mathbf{w}_0 - \int_0^t e^{\mathcal{L}(t-\tau)}\mathcal{N}\left(\mathbf{w}^\delta(\tau)\right)d\tau,$$

Using Lemma 1, (3.2), (3.3), and Lemma 4 yields, for $0 \le t \le \min\{T^\delta, T^*, T^{**}\}$

$$\begin{aligned}
&\left\|\mathbf{w}^\delta(t) - \delta e^{\mathcal{L}t}\mathbf{w}_0\right\| \\
\le& C_1 \int_0^t e^{\lambda_{\max}(t-\tau)} \left\|\{\nabla\cdot\left(D\nabla\mathbf{w}^\delta\right) - \bar{D}\nabla^2\mathbf{w}^\delta\} + \mathbf{F} - A\mathbf{w}^\delta\right\| d\tau \\
\le& C_1 \int_0^t e^{\lambda_{\max}(t-\tau)} ||D||_{C^1} \left\|\mathbf{w}^\delta(\tau)\right\|_\infty \left\|\mathbf{w}^\delta(\tau)\right\|_{H^2} d\tau \\
&+ C_1 \int_0^t e^{\lambda_{\max}(t-\tau)} ||D||_{C^1} \left\|\nabla\mathbf{w}^\delta(\tau)\right\|_{L^4} \left\|\nabla\mathbf{w}^\delta(\tau)\right\|_{L^4} d\tau \\
&+ C_1 \int_0^t e^{\lambda_{\max}(t-\tau)} ||F||_{C^2} \left\|\mathbf{w}^\delta(\tau)\right\|_\infty \left\|\mathbf{w}^\delta(\tau)\right\| d\tau \\
\le& C_1 C_0^2 C_\eta \int_0^t e^{\lambda_{\max}(t-\tau)} \left\|\mathbf{w}^\delta(\tau)\right\|_{H^2}^2 d\tau.
\end{aligned}$$

from assumption (2.2) with $||w||_{H^2} \le \eta$. We plug (4.1) with $t = \tau$ to further obtain

$$\begin{aligned}
&\left\|\mathbf{w}^\delta(t) - \delta e^{\mathcal{L}t}\mathbf{w}_0\right\| \\
&\le C_1 C_0^2 C_\eta C_3 \int_0^t e^{\lambda_{\max}(t-\tau)} \{\delta^2 ||\mathbf{w}_0||_{H^2}^2 + \delta^2 e^{2\lambda_{\max}\tau}\} d\tau \\
&\le C_1 C_0^2 C_\eta C_3 \{\frac{||\mathbf{w}_0||_{H^2}^2 \delta}{\lambda_{\max}} + \frac{1}{\lambda_{\max}} \delta e^{\lambda_{\max}t}\} \delta e^{\lambda_{\max}t}.
\end{aligned} \tag{4.2}$$

We now choose θ in T^δ in (2.1) to satisfy

$$C_0^2 C_3 C_\eta \theta < \frac{\lambda_{\max}}{4}, \tag{4.3}$$

$$2\sqrt{C_3}\theta < \min\left\{\eta, \frac{\bar{D}_1}{2C_0C_1}, \frac{\bar{D}_2}{C_0C_1}\right\}. \tag{4.4}$$

We now prove by contradiction that for δ sufficiently small,

$$T^\delta \le \min\{T^*, T^{**}\},$$

and therefore our theorem follows from (**??**), by further separating $\mathbf{q} \in \Omega_{\max}$ and move $\mathbf{q} \notin \Omega_{\max}$ in (1.18) to the right hand side .

If T^{**} is the smallest among T^δ, T^* and T^{**}, we can let $t = T^{**} < T^\delta$ in (4.1)

$$\begin{aligned}
\left\|\mathbf{w}^\delta(T^{**})\right\|_{H^2} &< \sqrt{C_3}\{\delta||\mathbf{w}_0||_{H^2} + \delta e^{\lambda_{\max}T^\delta}\} \\
&= \sqrt{C_3}\{\delta||\mathbf{w}_0||_{H^2} + \theta\} \le 2\sqrt{C_3}\theta,
\end{aligned}$$

for small δ such that $\delta||\mathbf{w}_0||_{H^2} \leq \theta$. By the choice of θ in (4.4), we have

$$||\mathbf{w}(T^{**})||_{H^2} < \min\left\{\eta, \frac{\bar{D}_1}{2C_0C_1}, \frac{\bar{D}_2}{C_0C_1}\right\}.$$

This is a contradiction to the definition of T^{**}.

On the other hand, if T^* is the smallest among among T^δ, T^* and T^{**}, we can let $t = T^*$ in (4.2) to get

$$\begin{aligned}
&\left\|\mathbf{w}^\delta\left(T^*\right) - \delta e^{\mathcal{L}t}\mathbf{w}_0\right\| \\
&\leq C_1C_0^2C_3C_\eta\{\frac{||\mathbf{w}_0||_{H^2}^2\delta}{\lambda_{\max}} + \frac{1}{\lambda_{\max}}\delta e^{\lambda_{\max}T^\delta}\}\delta e^{\lambda_{\max}T^*} \\
&\leq C_1C_0^2C_3C_\eta\{\frac{||\mathbf{w}_0||_{H^2}^2\delta}{\lambda_{\max}} + \frac{\theta}{\lambda_{\max}}\}\delta e^{\lambda_{\max}T^*} \\
&< \frac{C_1}{2}\delta e^{\lambda_{\max}T^*},
\end{aligned}$$

for $C_0^2C_3C_\eta\frac{||\mathbf{w}_0||_{H^2}^2\delta}{\lambda_{\max}} < 1/4$ for δ small, by our choice of θ in (4.3). This again contradicts the definition of T and our theorem follows. $\square$

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DIVISION OF APPLIED MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI 02912, USA
E-mail address: `guoy@dam.brown.edu`

SCHOOL OF MATHEMATICS, TRINITIY COLLEGE DUBLIN, DUBLIN 2, IRELAND
E-mail address: `hjhwang@maths.tcd.ie`