

PATTERN FORMATION (I): THE KELLER-SEGEL MODEL

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ABSTRACT. We investigate nonlinear dynamics near an unstable constant equilibrium in the classical Keller-Segel model. Given any general perturbation of magnitude δ , we prove that its nonlinear evolution is dominated by the corresponding linear dynamics along a fixed finite number of fastest growing modes, over a time period of $\ln \frac{1}{\delta}$. Our result can be interpreted as a rigorous mathematical characterization for early pattern formation in the Keller-Segel model.

1. GROWING MODES IN THE KELLER-SEGEL MODEL

The goal of this section is to review the well-known instability criterion for the classical Keller-Segel model, which describes directed movement of microorganisms and cells stimulated by the chemical which they produce themselves. The Keller-Segel system takes the form

$$(1.1) \quad \begin{aligned} U_t &= -\nabla \cdot (-\mu \nabla U + \chi U \nabla V), \\ V_t &= \nabla \cdot (D \nabla V) + fU - kV, \end{aligned}$$

where $U(x, t)$ is the cell density, $V(x, t)$ the chemo-attractant, $\mu > 0$ the amoeboid motility, $\chi > 0$ the chemotactic sensitivity, $D > 0$ the diffusion rate of cAMP, $f > 0$ the rate of cAMP secretion per unit density of amoebae, $k > 0$ the rate of degradation of cAMP in environment.

We assume Neumann boundary conditions for $U(x, t)$ and $V(x, t)$, in a d -dimensional box $x \in \mathbb{T}^d = (0, \pi)^d$, $d = 1, 2, 3$, i.e.,

$$(1.2) \quad \frac{\partial U}{\partial x_i} = \frac{\partial V}{\partial x_i} = 0, \quad \text{at } x_i = 0, \pi, \text{ for } 1 \leq i \leq d.$$

A uniform constant solution

$$U(x, t) \equiv \bar{U}, \quad V(x, t) \equiv \bar{V}$$

forms a homogeneous steady state provided

$$(1.3) \quad f\bar{U} = k\bar{V}.$$

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In this article, we study the nonlinear evolution of a perturbation

$$u(x, t) = U(x, t) - \bar{U}, \quad v(x, t) = V(x, t) - \bar{V}$$

around $[\bar{U}, \bar{V}]$, which satisfies the equivalent Keller-Segel system:

$$(1.4) \quad u_t = \mu \nabla^2 u - \chi \bar{U} \nabla^2 v - \chi \nabla(u \nabla v),$$

$$(1.5) \quad v_t = D \nabla^2 v + f u - k v.$$

The corresponding linearized Keller-Segel system then takes the form

$$(1.6) \quad u_t = \mu \nabla^2 u - \chi \bar{U} \nabla^2 v,$$

$$(1.7) \quad v_t = D \nabla^2 v + f u - k v.$$

We use $[\cdot, \cdot]$ to denote a column vector, and let

$$\mathbf{w}(x, t) \equiv [u(x, t), v(x, t)].$$

Let $\mathbf{q} = (q_1, \dots, q_d) \in \Omega = (\mathbb{N} \cup \{0\})^d$ and let

$$e_{\mathbf{q}}(x) \equiv \prod_{i=1}^d \cos(q_i x_i),$$

Then $\{e_{\mathbf{q}}(x)\}_{\mathbf{q} \in \Omega}$ forms a basis of the space of functions in \mathbb{T}^d that satisfy Neumann boundary conditions (1.2). We look for a normal mode to the linear Keller-Segel system (1.6) and (1.7) of the following form:

$$(1.8) \quad \mathbf{w}(x, t) = \mathbf{r}_{\mathbf{q}} \exp(\lambda_{\mathbf{q}} t) e_{\mathbf{q}}(x),$$

where $\mathbf{r}_{\mathbf{q}}$ is a vector depending on \mathbf{q} . Plugging (1.8) into (1.6)-(1.7) yields

$$\lambda_{\mathbf{q}} \mathbf{r}_{\mathbf{q}} = \begin{pmatrix} -\mu q^2 & \chi \bar{U} q^2 \\ f & -D q^2 - k \end{pmatrix} \mathbf{r}_{\mathbf{q}},$$

where $q^2 = \sum_{i=1}^d q_i^2$. A nontrivial normal mode can be obtained by setting

$$\det \begin{pmatrix} \lambda_{\mathbf{q}} + \mu q^2 & -\chi \bar{U} q^2 \\ -f & \lambda_{\mathbf{q}} + D q^2 + k \end{pmatrix} = 0.$$

This leads to the following dispersion formula for $\lambda_{\mathbf{q}}$:

$$(1.9) \quad \lambda_{\mathbf{q}}^2 + \{q^2(\mu + D) + k\} \lambda_{\mathbf{q}} + q^2 \{\mu(D q^2 + k) - \chi \bar{U} f\} = 0.$$

Thus we deduce the following well-known aggregation (i.e., linear instability) criterion by requiring there exists a q such that

$$(1.10) \quad \mu(D q^2 + k) - \chi \bar{U} f < 0,$$

to ensure that (1.9) has at least one positive root $\lambda_{\mathbf{q}}$. This clearly implies that $\mu k - \chi \bar{U} f < 0$, and an elementary computation of the discriminant yields:

$$\begin{aligned} & \{q^2(\mu + D) + k\}^2 - 4q^2\{\mu(Dq^2 + k) - \chi \bar{U} f\} \\ &= q^4(\mu - D)^2 + k^2 + 2q^2(\mu + D)k + 4q^2\{-\mu k + \chi \bar{U} f\} \\ &> 0 \end{aligned}$$

for q . Therefore, there exist two distinct real roots for all \mathbf{q} to the quadratic equation (1.9), which we denote

$$\lambda_-(\mathbf{q}) < \lambda_+(\mathbf{q}).$$

We denote the corresponding (linearly independent) eigenvectors by $\mathbf{r}_-(\mathbf{q})$ and $\mathbf{r}_+(\mathbf{q})$, such that

$$(1.11) \quad \mathbf{r}_{\pm}(\mathbf{q}) = \left[\frac{\lambda_{\pm}(\mathbf{q}) + D\mathbf{q}^2 + k}{f}, 1 \right].$$

Clearly, for q large,

$$\mu(Dq^2 + k) - \chi \bar{U} f > 0.$$

Hence there are only finitely many \mathbf{q} such that $\lambda_+(\mathbf{q}) > 0$. We therefore denote the largest eigenvalue by $\lambda_{\max} > 0$ and define

$$\Omega_{\max} \equiv \{\mathbf{q} \in \Omega \text{ such that } \lambda_+(\mathbf{q}) = \lambda_{\max}\}.$$

It is easy to see that there is one q^2 (possibly two) having $\lambda_{\mathbf{q}}^+(q^2) = \lambda_{\max}$ when we regard $\lambda_{\mathbf{q}}^+$ as a function of q^2 . We also denote $\nu > 0$ to be the gap between the λ_{\max} and the rest.

Given any initial perturbation $\mathbf{w}(\mathbf{x}, 0)$, we can expand it as

$$\mathbf{w}(\mathbf{x}, 0) = \sum_{\mathbf{q} \in \Omega} \mathbf{w}_{\mathbf{q}} e_{\mathbf{q}}(x) = \sum_{\mathbf{q} \in \Omega} \{w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q})\} e_{\mathbf{q}}(x),$$

so that

$$(1.12) \quad \mathbf{w}_{\mathbf{q}} = w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}).$$

The unique solution $\mathbf{w}(x, t) = [u(x, t), v(x, t)]$ to (1.6)-(1.7) is given by

$$(1.13) \quad \begin{aligned} \mathbf{w}(x, t) &= \sum_{\mathbf{q} \in \Omega} \{w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) \exp(\lambda_{\mathbf{q}}^- t) + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) \exp(\lambda_{\mathbf{q}}^+ t)\} e_{\mathbf{q}}(x) \\ &\equiv e^{\mathcal{L}t} \mathbf{w}(x, 0). \end{aligned}$$

For any $\mathbf{u}(\cdot, t) \in [L^2(\mathbb{T}^d)]^2$, we denote $\|\mathbf{u}(\cdot, t)\| \equiv \|\mathbf{u}(\cdot, t)\|_{L^2}$. Our main result of this section is

Lemma 1. *Assume the instability criterion (1.10) is valid. Suppose*

$$\mathbf{w}(x, t) = [u(x, t), v(x, t)] \equiv e^{\mathcal{L}t} \mathbf{w}(x, 0)$$

as in (1.13) is a solution to the linearized KS system (1.6)-(1.7) with initial condition $\mathbf{w}(\mathbf{x}, 0)$. Then there exists a constant $C_1 \geq 1$ depending on $k, \bar{U}, D, \mu, f, \chi$, such that

$$\|\mathbf{w}(\cdot, t)\| \leq C_1 \exp(\lambda_{max}t) \|\mathbf{w}(\cdot, 0)\|,$$

for all $t \geq 0$.

Proof. We first consider the case for $t \geq 1$. By analyzing (1.9), for q large, we have

$$\lim_{q \rightarrow \infty} \frac{\lambda_{\mathbf{q}}^{\pm}}{q^2} = -\mu, -D$$

respectively. Notice that from the quadratic formula for (1.9),

$$\frac{\lambda_{\mathbf{q}}^+ - \lambda_{\mathbf{q}}^-}{q^2} \geq \frac{2\sqrt{-\mu k + \chi \bar{U} f}}{q}.$$

From solving (1.12)

$$\begin{aligned} |w_{\mathbf{q}}^{\pm}| &\leq \frac{1}{\det[\mathbf{r}_-(\mathbf{q}), \mathbf{r}_+(\mathbf{q})]} |\mathbf{r}_{\pm}(\mathbf{q})| \times |\mathbf{w}_{\mathbf{q}}| \\ &\leq Cq |\mathbf{w}_{\mathbf{q}}|, \end{aligned}$$

we deduce that for $t \geq 1$ and q large,

$$|w_{\mathbf{q}}^{\pm} \mathbf{r}_{\pm}(\mathbf{q}) \exp(\lambda_{\mathbf{q}}^{\pm} t)| \leq Cq |\mathbf{w}_{\mathbf{q}}| \exp(-\min\{\mu, D\}q^2 t) \leq C |\mathbf{w}_{\mathbf{q}}|.$$

Thus we deduce the Lemma on the linear growth rate for $t \geq 1$ by the formula (1.13).

On the other hand, for finite time $t \leq 1$, it suffices to derive the standard energy estimate in L^2 . From the Neumann boundary conditions, we can take $u \times$ (1.6) and add $Av \times$ of (1.7) to get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} \{|u|^2 + A|v|^2\} \\ &+ \int_{\mathbb{T}^d} \{\mu |\nabla u|^2 + AD |\nabla v|^2 - \chi \bar{U} \nabla v \nabla u\} + Ak \int_{\mathbb{T}^d} |v|^2 \\ &= \int_{\mathbb{T}^d} A f u v. \end{aligned}$$

The integrand of the second integral can be chosen non-negative

$$(1.14) \quad \mu |\nabla u|^2 + AD |\nabla v|^2 - \chi \bar{U} \nabla v \nabla u \geq \frac{\mu}{2} |\nabla u|^2 + \frac{(\bar{U} \chi)^2 |\nabla v|^2}{2\mu} \geq 0,$$

if the constant A is

$$(1.15) \quad A = \frac{(\bar{U}\chi)^2}{D\mu}.$$

It thus follows that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} \{|u|^2 + A|v|^2\} \leq \frac{Af}{2} \int_{\mathbb{T}^d} \{|u|^2 + |v|^2\},$$

and the Gronwall inequality implies

$$\|\mathbf{w}(\cdot, t)\| \leq C \exp(Ct) \|\mathbf{w}(\cdot, 0)\|,$$

for some $C > 0$. This immediately implies our lemma when $t \leq 1$. \square

2. MAIN RESULT

Let θ be a small fixed constant, and λ_{\max} be the dominant eigenvalue which is the maximal growth rate. We also denote the gap between the largest growth rate λ_{\max} and the rest by $\nu > 0$. Then for $\delta > 0$ arbitrary small, we define the escape time T^δ by

$$(2.1) \quad \theta = \delta \exp(\lambda_{\max} T^\delta),$$

or equivalently

$$T^\delta = \frac{1}{\lambda_{\max}} \ln \frac{\theta}{\delta}.$$

Our main theorem is

Theorem 1. *Assume that the set of $q^2 = \sum_{i=1}^d q_i^2$ satisfying instability criterion (1.10) is not empty for given parameters μ, D, k, χ, f and \bar{U} . Let*

$$\mathbf{w}_0(x) = \sum_{\mathbf{q} \in \Omega} \{w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q})\} e_{\mathbf{q}}(x).$$

$\in H^2$ such that $\|\mathbf{w}_0\| = 1$. Then there exist constants $\delta_0 > 0$, $C > 0$, and $\theta > 0$, depending on $k, \bar{U}, D, \mu, f, \chi$, such that for all $0 < \delta \leq \delta_0$, if the initial perturbation of the steady state $[\bar{U}, \bar{V}]$ in (1.3) is

$$\mathbf{w}^\delta(\mathbf{x}, 0) = \delta \mathbf{w}_0,$$

then its nonlinear evolution $\mathbf{w}^\delta(t, x)$ satisfies

$$\begin{aligned} & \|\mathbf{w}^\delta(t, x) - \delta e^{\lambda_{\max} t} \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}(x)\| \\ & \leq C \{e^{-\nu t} + \delta \|w_0\|_{H^2}^2 + \delta e^{\lambda_{\max} t}\} \delta e^{\lambda_{\max} t} \end{aligned}$$

for $0 \leq t \leq T^\delta$, and $\nu > 0$ is the gap between λ_{\max} and the rest of $\lambda_{\mathbf{q}}$ in (1.9).

We notice that for $0 \leq t \leq T^\delta$, $\delta e^{\lambda_{\max} t} \leq \theta$, is sufficiently small. As long as $w_{\mathbf{q}_0}^+ \neq 0$ for at least one $\mathbf{q}_0 \in \Omega_{\max}$, which is generic for perturbations, the corresponding fastest growing modes

$$\|\delta e^{\lambda_{\max} t} \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}\| \geq \delta e^{\lambda_{\max} t} |w_{\mathbf{q}_0}^+| |\mathbf{r}_+(\mathbf{q}_0)|,$$

have the dominant leading order of $\delta e^{\lambda_{\max} t}$. Our theorem implies that the dynamics of a general perturbation is characterized by such linear dynamics over a long time period of $\varepsilon T^\delta \leq t \leq T^\delta$, for any $\varepsilon > 0$. In particular, choose a fixed $\mathbf{q}_0 \in \Omega_{\max}$ and let

$$w_0(x) = \frac{\mathbf{r}_+(\mathbf{q}_0)}{|\mathbf{r}_+(\mathbf{q}_0)|} e_{\mathbf{q}_0}(x)$$

then if $t = T^\delta$,

$$\left\| \mathbf{w}^\delta(t, \cdot) - \delta e^{\lambda_{\max} T^\delta} \frac{\mathbf{r}_+(\mathbf{q}_0)}{|\mathbf{r}_+(\mathbf{q}_0)|} e_{\mathbf{q}_0}(\cdot) \right\| \leq C\{\delta^{\nu/\lambda_{\max}} + \theta^2\},$$

hence

$$\|\mathbf{w}^\delta(t, \cdot)\| \geq \theta - C\{\delta^{\nu/\lambda_{\max}} + \theta^2\} \geq \theta/2 > 0,$$

which implies nonlinear instability as $\delta \rightarrow 0$. The instability occurs before the possible blow-up time.

In the early work of Keller and Segel [15] in 1970, they formulated the advection-diffusion system (1.1) which consists of two parabolic equations and viewed the initiation of Slime mold aggregation as instability. Linearized system was used to analyze early stage of pattern formation and its instability around homogeneous steady states. This Keller-Segel model has since received much attention and there have been many contributions on this subject such as aggregations, dynamics of blow-ups, travelling waves. See [1],[2],[3],[9],[10],[11],[8],[13],[16],[17],[19],[20],[21] for related results. Linear stability and instability of stationary solutions with more general nonlinearity was studied in [22] using bifurcation analysis. However, nonlinear evolution of the pattern formation has yet been fully understood for the Keller-Segel model, to the authors' knowledge.

We rigorously prove that linear fastest growing modes determine unstable patterns for the full Keller-Segel system (1.4) and (1.5), over a time period of the order $\ln \frac{1}{\delta}$. Each initial perturbation certainly can behaves drastically differently from another, which gives rise to the richness of patterns. On the other hand, the dominating linear dynamics over a fixed finite dimensional space of maximal growing modes ensures that there is a common characteristic pattern for a general class initial data. Therefore, we believe that our result indeed provide a

mathematical description for the pattern formation in the Keller-Segel model.

Our paper stems from a program to study various nonlinear instabilities for non-dissipative systems arising in mathematical physics [5],[6],[7],[12], where severe higher order perturbations (unbounded in the L^2 norms, for instance) occur. Indeed, for many such systems without dissipation, the passage from linear instability to nonlinear instability is very delicate. If there is a dominant eigenvalue, then a bootstrap argument was developed by Strauss and the first author to prove nonlinear instability, for the perturbation initially along the dominant eigenfunction. The key is to try to control the nonlinear growth of higher-order energy norm for the perturbation by the linear growth rate, up to the time T^δ . Very recently in [4], based upon a precise linear analysis, dynamics of general perturbation can be characterized by the linear dynamics of fastest growing modes for unstable Kirchhoff ellipses. This marks a beginning of a quantitative description of instability.

Our research is inspired by the work [4]. In the presence of dissipation, continuum spectra are absent in bounded domain, which leads to finite number of dominant growing modes. Moreover, natural higher-order energy estimate now can be easily combined with the bootstrap idea to control the nonlinear term $-\chi \nabla(u \nabla v)$ in the L^2 space. Since our method is general, we believe that such kind of pattern formation should exist for a wide class of systems with dissipation.

3. BOOTSTRAP LEMMA

We state existence of local-in-time solutions for (1.4)-(1.5).

Lemma 2. *(Local existence) For $s \geq 1$ ($d = 1$) and $s \geq 2$ ($d = 2, 3$), there exist a $T > 0$ and a constant C depending on $k, \bar{U}, \bar{V}, D, \mu, f, \chi$ such that*

$$\|\mathbf{w}(t)\|_{H^s} \leq C \|\mathbf{w}(0)\|_{H^s}.$$

We now derive the following energy estimates for d -dimensional chemotaxis model with $d = 1, 2, 3$.

Lemma 3. *Suppose that $[u(x, t), v(x, t)]$ is a solution to the full system (1.4)-(1.5). Then*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \sum_{|\partial|=2} \int_{\mathbb{T}^d} \left\{ |\partial u|^2 + \frac{(\bar{U}\chi)^2}{D\mu} |\partial v|^2 \right\} d\mathbf{x} \\ & + \sum_{|\partial|=2} \int_{\mathbb{T}^d} \left\{ \frac{\mu}{4} |\nabla \partial u|^2 + \frac{(\bar{U}\chi)^2}{2\mu} |\nabla \partial v|^2 \right\} d\mathbf{x} + \frac{Ak}{2} \sum_{|\alpha|=2} \int_{\mathbb{T}^d} |\partial v|^2 \\ & \leq C_0 \|\mathbf{w}\|_{H^2} \|\nabla^3 \mathbf{w}\|^2 + C_2 \|u\|^2. \end{aligned}$$

where C_0 is the universal constant while $C_2 = \frac{\bar{U}^6 \chi^6 f^6}{2D^3 \mu^5 k^3}$.

Proof. We first notice that the Keller-Segel equation preserves the evenness of the solution $\mathbf{w}(x, t)$, i.e., if $\mathbf{w}(x, t)$ is a solution, then $\mathbf{w}(-x_i, t)$ is also a solution. We can regard the Neumann problem as a special case with evenness of the periodic problem by standard way of even extension $\mathbf{w}(x, t)$ with respect to one of the x_i . For this reason we may assume periodicity at the boundary of the extended $2\mathbf{T}^3 \equiv (-\pi, \pi)^d$. Since now there is no contributions from the boundaries, we can take second order ∂ -derivative of (1.4) and add $A \times \partial$ of (1.5) to get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{2\mathbb{T}^d} \{ |\partial u|^2 + A |\partial v|^2 \} \\ & + \int_{2\mathbb{T}^d} \{ \mu |\nabla \partial u|^2 + AD |\nabla \partial v|^2 - \chi \bar{U} \nabla \partial v \nabla \partial u \} + Ak \int_{2\mathbb{T}^d} |\partial v|^2 \\ & = \chi \int_{2\mathbb{T}^d} \partial \{ u \nabla v \} \nabla \partial u + Af \int_{2\mathbb{T}^d} \partial u \partial v \\ & \equiv I_1 + I_2, \end{aligned}$$

where the constant A is given in (1.15). As in (1.14), the second integrand is bounded below by

$$\frac{\mu}{2} |\nabla \partial u|^2 + \frac{(\bar{U}\chi)^2}{2\mu} |\nabla \partial v|^2.$$

The nonlinear term I_1 is bounded by

$$\begin{aligned} I_1 & = \int |\partial (u \nabla v) \cdot \nabla \partial u| d\mathbf{x} \\ & \leq \|u\|_{L^\infty} \|\nabla \partial v\| \|\nabla \partial u\| + \|\nabla u\|_{L^\infty} \|\partial v\| \|\nabla \partial u\| \\ & \quad + \|\partial u\| \|\nabla v\|_{L^\infty} \|\nabla \partial u\|. \end{aligned}$$

We apply the following the Sobolev imbedding to control $\|u\|_{L^\infty}$

$$(3.1) \quad \|g\|_{L^\infty(2\mathbb{T}^d)} \leq C_0 \|g\|_{H^2(2\mathbb{T}^d)},$$

for $d \leq 3$. Moreover, from the periodic boundary conditions,

$$\int_{2\mathbb{T}^d} \nabla u = \int_{2\mathbb{T}^d} \nabla v = 0,$$

we also use the Poincare inequality

$$(3.2) \quad \|g\| \leq \|g\|_{L^4(2\mathbb{T}^d)} \leq C_0 \|\nabla g\| \quad \text{if } d \leq 3,$$

to further get

$$\|\nabla u\|_{L^\infty} + \|\nabla v\|_{L^\infty} \leq C_0 \{\|\nabla u\|_{H^2} + \|\nabla v\|_{H^2}\} \leq C_0 \sum_{|\partial|=2} \|\partial \nabla \mathbf{w}\|,$$

where C_0 is a universal constant. Hence $I_1 \leq C_0 \|\mathbf{w}\|_{H^2} \|\nabla^3 \mathbf{w}\|^2$ as desired.

Finally, I_2 is simply bounded by

$$I_2 = Af \int \partial u \partial v \leq \frac{Af^2}{2k} \|\partial u\|^2 + \frac{Ak}{2} \|\partial v\|^2$$

By the interpolation between $\|\nabla \partial u\|$ and $\|u\|$, the first term above is bounded by

$$\frac{Af^2}{2k} \left\{ a \|\nabla \partial u\|^2 + \frac{1}{4a^2} \|u\|^2 \right\}$$

for any $a > 0$. We can choose a such that $\frac{Af^2}{2k} a = \frac{1}{4}\mu$. Collecting terms, we conclude the proof. \square

We are now ready to establish the bootstrap lemma, which controls the H^2 growth of $\mathbf{w}(x, t)$ in term of its L^2 growth.

Lemma 4. *Suppose that $\mathbf{w}(x, t)$ is a solution to the full system (1.4)-(1.5) such that for $0 \leq t \leq T$,*

$$\|\mathbf{w}(\cdot, t)\|_{H^2} \leq \frac{1}{C_0} \min \left\{ \frac{\mu}{4}, \frac{(\bar{U}\chi)^2}{2\mu} \right\}$$

and

$$(3.3) \quad \|\mathbf{w}(\cdot, t)\| \leq 2C_1 e^{\lambda_{\max} t} \|\mathbf{w}(\cdot, 0)\|,$$

then we have for $0 \leq t \leq T$,

$$\|\mathbf{w}(t)\|_{H^2}^2 \leq C_3 \{ \|\mathbf{w}(0)\|_{H^2}^2 + e^{2\lambda_{\max} t} \|\mathbf{w}(\cdot, 0)\|^2 \}$$

where $C_3 = C_1^2 \max \left\{ \frac{(\bar{U}\chi)^2}{D\mu}, \frac{D\mu}{(\bar{U}\chi)^2} \right\} \times \max \left\{ \frac{4C_2}{\lambda_{\max}}, 1 \right\} \geq 1$.

Proof. It suffices to only consider the second-order derivatives of $\mathbf{w}(x, t)$. From the previous lemma and our assumption for $\|\mathbf{w}\|_{H^2}$, we deduce that for $0 \leq t \leq T$

$$\frac{1}{2} \frac{d}{dt} \sum_{|\partial|=2} \int_{\mathbb{T}^d} \left\{ |\partial u|^2 + \frac{(\bar{U}\chi)^2}{D\mu} |\partial v|^2 \right\} d\mathbf{x} \leq C_2 \|u\|^2.$$

So that by (3.3) and an integration from 0 to t , we have

$$\begin{aligned} & \sum_{|\partial|=2} \int_{\mathbb{T}^d} \left\{ |\partial u(t)|^2 + \frac{(\bar{U}\chi)^2}{D\mu} |\partial v(t)|^2 \right\} \\ & \leq \sum_{|\partial|=2} \int_{\mathbb{T}^d} \left\{ |\partial u(0)|^2 + \frac{(\bar{U}\chi)^2}{D\mu} |\partial v(0)|^2 \right\} + \frac{4C_2 C_1^2}{\lambda_{\max}} e^{2\lambda_{\max} t} \|\mathbf{w}(\cdot, 0)\|^2, \end{aligned}$$

for $0 \leq t \leq T$. Now our lemma follows directly by separating the cases of $\frac{(\bar{U}\chi)^2}{D\mu} \geq 1$ and $\frac{(\bar{U}\chi)^2}{D\mu} < 1$. \square

4. NONLINEAR INSTABILITY AND PATTERN FORMATION

We now prove our main Theorem 1:

Proof. Let $\mathbf{w}^\delta(x, t)$ be the family of solutions to the Keller-Segel system (1.4)-(1.5) with initial data $\mathbf{w}^\delta(x, 0) = \delta \mathbf{w}_0$. Define T^* by

$$T^* = \sup \left\{ t \mid \|\mathbf{w}^\delta(t) - \delta e^{\mathcal{L}t} \mathbf{w}_0\| \leq \frac{C_1}{2} \delta \exp(\lambda_{\max} t) \right\}.$$

Note that T^* is well defined. We also define

$$T^{**} = \sup \left\{ t \mid \|\mathbf{w}^\delta(t)\|_{H^2} \leq \frac{1}{C_0} \min \left\{ \frac{\mu}{4}, \frac{(\bar{U}\chi)^2}{2\mu} \right\} \right\}.$$

We recall T^δ in (2.1) where θ is chosen such that

$$(4.1) \quad C_0 C_3 \theta < \min \left\{ \frac{\lambda_{\max}}{4}, \frac{\mu}{8}, \frac{(\bar{U}\chi)^2}{4\mu} \right\},$$

We now derive estimates for H^2 norm of $\mathbf{w}^\delta(x, t)$ for $0 \leq t \leq \min\{T^*, T^\delta, T^{**}\}$. First of all, by the definition of T^* , for $t \leq T^*$ and Lemma 1

$$\|\mathbf{w}^\delta(t)\| \leq \frac{3C_1}{2} \delta \exp(\lambda_{\max} t).$$

Moreover, using Lemma 4 and applying a bootstrap argument yields

$$(4.2) \quad \|\mathbf{w}^\delta(t)\|_{H^2} \leq \sqrt{C_3} \{ \delta \|\mathbf{w}_0\|_{H^2} + \delta e^{\lambda_{\max} t} \}.$$

We now establish a sharper L^2 estimate for $\mathbf{w}^\delta(x, t)$, for $0 \leq t \leq \min\{T^{**}, T^\delta, T^*\}$. We first apply Duhamel's principle to obtain

$$\mathbf{w}^\delta(t) = \delta e^{\mathcal{L}t} \mathbf{w}_0 - \int_0^t e^{\mathcal{L}(t-\tau)} [\nabla \cdot (u^\delta(\tau) \nabla v^\delta(\tau)), 0] d\tau,$$

Using Lemma 1, (3.1), (3.2), and Lemma 4 yields, for $0 \leq t \leq \min\{T^\delta, T^{**}, T^*\}$

$$\begin{aligned} & \|\mathbf{w}^\delta(t) - \delta e^{\mathcal{L}t} \mathbf{w}_0\| \\ & \leq C_1 \int_0^t e^{\lambda_{\max}(t-\tau)} \|\nabla \cdot (u^\delta(\tau) \nabla v^\delta(\tau))\| d\tau \\ & \leq C_1 \int_0^t e^{\lambda_{\max}(t-\tau)} \|u^\delta(\tau)\|_{L^\infty} \|\nabla^2 v^\delta(\tau)\| d\tau \\ & \quad + C_1 \int_0^t e^{\lambda_{\max}(t-\tau)} \|\nabla u^\delta(\tau)\|_{L^4} \|\nabla v^\delta(\tau)\|_{L^4} d\tau \\ & \leq C_1 C_0 \int_0^t e^{\lambda_{\max}(t-\tau)} \|\mathbf{w}^\delta(\tau)\|_{H^2}^2 d\tau. \end{aligned}$$

By our choice of $t \leq \min\{T^*, T^{**}, T^\delta\}$, it is further bounded by

$$\begin{aligned} (4.3) \quad & \|\mathbf{w}^\delta(t) - \delta e^{\mathcal{L}t} \mathbf{w}_0\| \\ & \leq C_1 C_0 C_3 \int_0^t e^{\lambda_{\max}(t-\tau)} \{\delta^2 \|\mathbf{w}_0\|_{H^2}^2 + \delta^2 e^{2\lambda_{\max}\tau}\} d\tau \\ & \leq C_1 C_0 C_3 \left\{ \frac{\|\mathbf{w}_0\|_{H^2}^2 \delta}{\lambda_{\max}} + \frac{1}{\lambda_{\max}} \delta e^{\lambda_{\max}t} \right\} \delta e^{\lambda_{\max}t}. \end{aligned}$$

We now prove by contradiction that for δ sufficiently small,

$$T^\delta = \min\{T^\delta, T^*, T^{**}\},$$

and therefore our theorem follows by further separating $\mathbf{q} \in \Omega_{\max}$ and move $\mathbf{q} \notin \Omega_{\max}$ in (1.13) to the right hand side.

If T^{**} is the smallest, we can let $t = T^{**} \leq T^\delta$ in (4.2)

$$\begin{aligned} \|\mathbf{w}^\delta(T^{**})\|_{H^2} & \leq \sqrt{C_3} \{\delta \|\mathbf{w}_0\|_{H^2} + \delta e^{\lambda_{\max} T^\delta}\} \\ & = \sqrt{C_3} \{\delta \|\mathbf{w}_0\|_{H^2} + \theta\} \\ & < \frac{1}{C_0} \min\left\{\frac{\mu}{4}, \frac{(\bar{U}\chi)^2}{2\mu}\right\}, \end{aligned}$$

for $\sqrt{C_3} \delta \|\mathbf{w}_0\|_{H^2} \leq \frac{1}{2C_0} \min\left\{\frac{\mu}{4}, \frac{(\bar{U}\chi)^2}{2\mu}\right\}$, by our choice of θ in (4.1) with $C_3 \geq 1$. This is a contradiction to the definition of T^{**} .

On the other hand, if T^* is the smallest, we let $t = T^*$ in (4.3) to get

$$\begin{aligned} & \|\mathbf{w}^\delta(T^*) - \delta e^{\mathcal{L}T^*} \mathbf{w}_0\| \\ & \leq C_1 C_0 C_3 \left\{ \frac{\|\mathbf{w}_0\|_{H^2}^2 \delta}{\lambda_{\max}} + \frac{1}{\lambda_{\max}} \delta e^{\lambda_{\max} T^\delta} \right\} \delta e^{\lambda_{\max} T^*} \\ & \leq C_1 C_0 C_3 \left\{ \frac{\|\mathbf{w}_0\|_{H^2}^2 \delta}{\lambda_{\max}} + \frac{\theta}{\lambda_{\max}} \right\} \delta e^{\lambda_{\max} T^*} \\ & < \frac{C_1}{2} \delta e^{\lambda_{\max} T^*}, \end{aligned}$$

for $C_0 C_3 \frac{\|\mathbf{w}_0\|_{H^2}^2 \delta}{\lambda_{\max}} < 1/4$ for δ small, by our choice of θ in (4.1). This again contradicts the definition of T^* and our theorem follows. \square

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