Optimal Investment Models with Minimum Consumption Criteria

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Abstract

This paper considers a max-min formulation of multistage optimal investment and consumption problems, with uncertainties in the form of variable productivities of capital and interest rates. The criterion of control performance is minimum consumption over time, weighted by a coefficient which indicates the likelihood of possible disturbance sequences. A dynamic programming method is used. Explicit results for a max-min formulation of the Merton portfolio optimization problem are obtained. A production-consumption-debt model arising in international finance is also considered.

1 Introduction.

In this paper we consider multistage models for optimal investment and consumption in the presence of uncertainties. These uncertainties arise from varying rates of return on capital assets and perhaps also varying interest rates. A common approach to problems of this kind uses stochastic control methods. Uncertainties are modeled as random variables. The goal is to choose investment and consumption controls which maximize total expected utility of consumption, subject to constraints which are imposed. If the uncertainties from one time period to another are independent, then the dynamic programming method can be applied. An example is the Merton optimal portfolio problem with HARA utility. For this Merton problem, dynamic programming gives an explicit solution. Another class of examples comes from models in international finance, involving optimal investment consumption and foreign debt [11][12][16].

As an alternative to the stochastic control formulation we take a max-min approach, which is outlined in Sections 2 and 3. In this approach, a natural criterion of control performance is the minimum consumption over time, weighted by a coefficient which indicates the likelihood of any possible disturbance sequence. See (3.2)-(3.4). In the max-min formulation, a dynamic programming technique can again be used. In Sections 4 and 5, we apply this technique to the Merton problem. Explicit formulas are obtained for investment and consumption controls, which turn out to be constants. See (4.12), (5.11), (5.15). These constant controls are nearly optimal, when the Merton problem is considered over a long time horizon. These same controls also give a balance between consumption and growth of wealth over time, described at the end of Sections 4 and 5.

We also consider a max-min version of the kind of discrete time, international finance model introduced in [11]. See Example 3.1 and Appendix C. The minimum consumption criterion (3.3) seems a natural one in this context, since falls in national consumption to low levels may lead to societal unrest.

Risk-sensitive stochastic control theory provides a link between stochastic and max-min approaches. See [4][5][8]. Optimal investment-consumption problems with HARA utility can be regarded as risk sensitive stochastic control problems. If γ is the HARA parameter, then $1-\gamma$ is a measure of risk sensitivity. The max-min formulation of the optimal investment-consumption problem is obtained in the totally risk averse limit $\gamma \to -\infty$.

The max-min formulation which we use is similar to the robust control approach to disturbances in control systems [1][14]. This is sometimes also called the H-infinity control approach.

More detailed descriptions of some mathematical concepts and derivations are postponed to the Appendices. The totally risk averse expectation E^∞ is defined in Appendix A. It has properties similar to ordinary expectation. However, additivity of the operator E^∞ is with respect to "min-plus" addition rather than ordinary addition. The dynamic programming principle in Appendix B is conveniently expressed in terms of min-plus addition and E^∞ sense expectations. Appendix D summarizes some results needed for the continuous time max-min Merton problem.

2 Static optimization.

Let us begin by contrasting, at an abstract level, the max-min approach to optimization in the presence of uncertainty with the usual stochastic optimization approach. Max-min and stochastic approaches are related through the idea of risk sensitive stochastic optimization.

In the abstract formulation, u denotes a control, v an uncertainty and $\mathcal{J} = \mathcal{J}(u, v)$ some criterion. In a stochastic optimization model, v is random. For simplicity, let us assume that v has finitely many possible values, and let $p(v)$ denote the probability of v. The traditional stochastic optimization problem is to choose u which maximizes the expectation $E(\mathcal{J})$, where

$$E(\mathcal{J}) = \sum_v p(v)\mathcal{J}(u,v). \tag{2.1}$$

Max-min approach. The traditional max-min approach is to choose u to maximize $\min_v \mathcal{J}(u,v)$. Thus, u is chosen to get the best result when the worst uncertainty v occurs. This is ultraconservative, since the maximizing controller is guarding against all possible uncertainties v, which are considered "equally likely". We take a "less conservative" version of the max-min approach, in which $\mathcal{J}(u,v)$ is weighted by a factor $q(v)$. It is assumed that $q(v) \geq 1$, with $q(\bar{v}) = 1$ for at least one $\bar{v}$. Larger values of $q(v)$ mean that v is "less likely" and $\bar{v}$ is "most likely" if $q(\bar{v}) = 1$. The goal of the max-min optimization problem is to choose u which maximizes $\min_v q(v)\mathcal{J}(u,v)$.

Risk-sensitive maximization. A link between stochastic and max-min viewpoints is provided by considering risk-sensitive stochastic optimization. Let $F(z)$ be a smooth function such that $F'(z) > 0$, $|F''(z)| \neq 0$. The risk-sensitive stochastic optimization problem is to choose u which maximizes the expectation $E[F(\mathcal{J})]$, rather than the traditional criterion $E(\mathcal{J})$. The coefficient of risk sensitivity is defined as

$$r_F(z) = \frac{|F''(z)|}{F'(z)}. \tag{2.2}$$

For the traditional stochastic criterion, $F(z) = z$. This is the "risk neutral" case. Large $r_F(z)$ indicates great sensitivity to risk.

In this paper, we consider HARA functions F:

$$F(z) = \gamma^{-1} z^{\gamma}, \qquad \gamma < 1 \ (\gamma \neq 0) \tag{2.3}$$

or $F(z) = \log z$ if $\gamma = 0$. For HARA F,

$$r_F(z) = \frac{1-\gamma}{z}. \tag{2.4}$$

Thus there is great sensitivity to risk if $1-\gamma$ is large ($\gamma \to -\infty$).

From risk sensitive to max-min. In the risk sensitive stochastic optimization model, let us now assume that the probability $p_\gamma(v)$ depends on the HARA parameter γ. If we write $F = F_\gamma$ in (2.3), then

$$\gamma E[F_\gamma(\mathcal{J})] = E(\mathcal{J}^\gamma) = \sum_v p_\gamma(v)[\mathcal{J}(u,v)]^\gamma. \tag{2.5}$$

For $\gamma < 0$, the goal in the risk sensitive problem is to choose u which minimizes $E(\mathcal{J}^\gamma)$, or equivalently which maximizes $[E(\mathcal{J}^\gamma)]^{\frac{1}{\gamma}}$. Suppose that the dependence of the probability

$p_\gamma(v)$ on the HARA parameter is such that

$$\lim_{\gamma\to-\infty}[p_\gamma(v)]^{\frac{1}{\gamma}} = q(v). \tag{2.6}$$

Thus, $p_\gamma(v) \sim [q(v)]^\gamma$ as $\gamma \to -\infty$. Since $0 < p_\gamma(v) \leq 1$, we have $q(v) \geq 1$. Moreover, $q(\bar{v}) = 1$ for at least one (most likely) $\bar{v}$ since the number of possible uncertainties v is finite. As $\gamma \to -\infty$,

$$E(\mathcal{J}^\gamma) \sim \sum_v [q(v)\mathcal{J}(u,v)]^\gamma.$$

The main contribution to the sum on the right side comes from those v which minimize $q(v)\mathcal{J}(u,v)$. This idea is expressed more precisely as follows. Let

$$E^\infty(\mathcal{J}) = \min_v [q(v)\mathcal{J}(u,v)]. \tag{2.7}$$

Then

$$E^\infty(\mathcal{J}) = \lim_{\gamma\to-\infty}[E(\mathcal{J}^\gamma)]^{\frac{1}{\gamma}}. \tag{2.8}$$

In the totally risk averse limit ($\gamma \to -\infty$), the goal is to choose u to maximize $E^\infty(\mathcal{J})$. This is the max-min optimization problem described above.

We call $E^\infty(\mathcal{J})$ the **totally risk averse expectation** of $\mathcal{J}$. Properties of the totally risk averse expectation operator E^∞ are described in Appendix A. The usual expectation operator E is linear with respect to sums of random variable and multiplication by scalars. The operator E^∞ has a similar property, provided usual addition $a + b$ is replaced by "min-plus" addition $a \oplus b$, defined by

$$a \oplus b = \min(a, b).$$

This property of E^∞ will be quite useful in considering a modified max-min approach to multi-period dynamic optimization problems in Section 3 and 4.

3 Multistage models with minimum consumption criteria.

In this section we describe in general terms a class of dynamic investment-consumption models, considered from the max-min optimization viewpoint outlined in Section 2. For the sake of less mathematically inclined readers, various details (including the dynamic programming technique) are deferred to Appendix B. Let us consider N discrete time periods $j = 1, 2, \ldots, N$ at each of which investment and consumption decisions are made. The choice of investment and consumption controls at time j is made knowing a quantity Y_j called the state. The states are updated according to difference equations of the form

$$Y_{j+1} = F(Y_j, u_j, v_j)\,, \qquad j = 1, 2, \cdots, N, \tag{3.1}$$

where u_j is the pair of investment and consumption controls and v_j the uncertainty. For instance, in the discrete time Merton-type model to be considered in Section 4, Y_j is the wealth X_j. In the production-consumption model mentioned later in this section (Example 3.1), Y_j is the pair (K_j, L_j) where K_j is capital and L_j is debt.

Max-min formulation. As in Section 2, we suppose that the possible values for the uncertainty v_j belong to a finite set and that there is a "likelihood function" $q(v)$ such that $q(v) \geq 1$ and $q(\bar{v}) = 1$ for some "most likely" $\bar{v}$. The likelihood of a sequence of uncertainties $\vec{v}_N = (v_1, v_2, \cdots, v_N)$ is defined to be

$$q_N(\vec{v}_N) = q(v_1)q(v_2)\cdots q(v_N). \tag{3.2}$$

Consider the criterion

$$\mathcal{J}_N = \min\{C_1, C_2, \cdots, C_N\}, \tag{3.3}$$

which is the minimum consumption over the N time periods. The consumption C_j in each period j depends on the state and controls: $C_j = G(Y_j, u_j)$. In the max-min formulation, the goal is to choose investment and consumption controls which maximize the totally risk averse expectation $E^\infty(\mathcal{J}_N)$, subject to constraints imposed on state and control variables. As in (2.7),

$$E^\infty(\mathcal{J}_N) = \min_{\vec{v}_N}\; q_N(\vec{v}_N)\mathcal{J}_N. \tag{3.4}$$

Note that $\mathcal{J}_N = \mathcal{J}_N(Y_1, \vec{u}_N, \vec{v}_N)$ depends on the initial state Y_1 and on the sequence $\vec{u}_N = (u_1, u_2, \cdots, u_N)$ of investment and consumption controls as well as the uncertainties $\vec{v}_N$. This optimization problem can be studied by the method of dynamic programming, outlined in Appendix B.

Risk-sensitive stochastic formulation. Suppose that the uncertainties are modeled as independent random variables, such that $p_\gamma(v)$ is the probability that $v_j = v$. Consider HARA utility of consumption, where as in (2.4) $1-\gamma$ is a measure of risk sensitivity. The goal is to choose investment and consumption controls to maximize total HARA utility of consumption over N time periods:

$$J_N(\gamma) = E[\gamma^{-1}\sum_{j=1}^{N} C_j^\gamma] \tag{3.5}$$

subject to constraints imposed on state and control variables.

The max-min formulation arises naturally from the risk-sensitive formulation by taking totally risk averse limits. Suppose that $p_\gamma(v) \sim [q(v)]^\gamma$ in the sense that (2.6) holds. Roughly speaking, as $\gamma \to -\infty$, the largest terms C_j^γ give the main contribution to the sum in (3.5). For $\gamma < 0$, C_j^γ is largest for those time periods j for which consumption C_j is smallest. This suggests that as $\gamma \to -\infty$,

$$\gamma J_N(\gamma) \sim [E^\infty(\mathcal{J}_N)]^\gamma.$$

In the totally risk averse limit, the problem of maximizing $J_N(\gamma)$ is replaced by maximizing $E^\infty(\mathcal{J}_N)$. This idea can be put on a mathematically more precise basis by comparing solutions to the corresponding dynamic programming equations. See Section 4 for the Merton problem.

Example 3.1. The following model arose in considering problems of national economic growth, consumption and foreign debt. An economic entity has productive capital and also liabilities in the form of debt. Let K_j denote the capital and L_j the debt at time period $j = 1, 2, \cdots$. In the context of international finance, the economic entity is a country. K_j is the nation's capital and L_j the debt owed to foreigners. See [11],[12]. If $L_j < 0$, then $S_j = -L_j$ is the amount loaned to foreigners.

Let I_j denote the amount of investment in capital and C_j the consumption in period j. Then capital and debt are updated according to the linear difference equations

$$K_{j+1} = K_j + I_j \tag{3.6}$$

$$L_{j+1} = (1+r_j)L_j + I_j + C_j - b_j K_j. \tag{3.7}$$

In (3.7), r_j is the interest rate and b_j the productivity of capital. Both r_j and b_j are uncertain, with $r^- \le r_j \le r^+$, $b^- \le b_j \le b^+$. The following constraints are imposed:

$$K_j > 0, \quad L_j \le \bar{\ell} K_j, \quad j = 1, 2, \cdots, N+1 \tag{3.8}$$

$$C_j \ge 0, \quad -\bar{i} K_j \le I_j \le \bar{i} K_j, \quad j = 1, 2, \cdots, N. \tag{3.9}$$

Thus, $\bar{\ell}$ is an upper bound for the allowable debt-to-capital ratio, and $\underline{i}, -\underline{i}$ are upper and lower bounds for the investment-to-capital ratio. We take $0 \le \underline{i} < 1$, which ensures that $K_{j+1} > 0$ whenever $K_j > 0$. We also take

$$0 < \bar{\ell} < \frac{b^- + \underline{i}}{r^+ + \underline{i}}. \tag{3.10}$$

The right side of (3.10) is the largest debt-to-capital ratio which can be guaranteed to hold under worst case interest rates $r_j = r^+$ and productivities $b_j = b^-$. See Appendix C. We assume that $b^- < r^+$, which implies that $\bar{\ell} < 1$.

In the dynamic optimization problem, the state variables are K_j, L_j. As controls we take $i_j = I_j/K_j$ and $c_j = C_j/K_j$. In the notation of (3.1), $Y_j = (K_j, L_j)$, $u_j = (i_j, c_j)$ and $v_j = (r_j, b_j)$. The function F in (3.1) is vector-valued: $F = (F_1, F_2)$, where by (3.6) and (3.7)

$$\begin{cases} F_1(K,i) = K(1+i) \\ \\ F_2(K,L,i,c,r,b) = (1+r)L + (i+c-b)K. \end{cases} \tag{3.11}$$

In Appendix C, the dynamic programming equations for the max-min formulation of this model are given.

4 Discrete-time Merton-type model.

To illustrate further the max-min approach outlined in Section 3, we now consider the classical Merton optimal consumption problem on a finite time horizon. For this simple model, the method of dynamic programming gives explicit recursive formulas for optimal investment and consumption controls. In a special case, there is an especially simple formula for the constant values to which these optimal controls tend as the number of time steps N tends to infinity (Example 4.1).

Let X_j denote an investor's wealth at time j. Let k_j and $1 - k_j$ denote the fraction of wealth invested in a risky and a riskless asset, respectively. Then

$$X_{j+1} = X_j(1 + r + (b_j - r)k_j - c_j), \tag{4.1}$$

where r is the riskless interest rate, b_j is the uncertain rate of return on the risky asset and $C_j = c_j X_j$ is consumption in time period $j = 1, 2, \cdots N$. We assume that b_j has finitely many values, the smallest and largest of which are b^- and b^+ respectively. We assume that

$$0 < b^- < r < b^+. \tag{4.2}$$

We call k_j and c_j the investment and consumption controls at time j. These controls are chosen knowing the wealth X_j but not the uncertain rate of return b_j. We require that $X_j > 0$ for all $j = 1, \cdots, N+1$. This holds provided that $X_1 > 0$, $k_j \geq 0$, $c_j \geq 0$ and $1 + r + (b_j - r)k_j - c_j \geq 0$ for $j = 1, \cdots, N$. In particular the last inequality must hold when $b_j = b^-$, the worst rate of return. Therefore, we require that

$$(k_j, c_j) \in \Gamma, \tag{4.3}$$

where Γ is the triangle in the (k, c) plane bounded by the lines $k = 0$, $c = 0$ and $1 + r = (r - b^-)k + c$. This triangle is called the "control space". The inequality $k_j \geq 0$ is a "no short selling" constraint.

Let us describe the solution to the max-min formulation of this Merton problem, with minimum consumption criterion. At the end of the section, the solution to this max-min formulation is seen to be the totally risk averse limit of the solution to the corresponding stochastic formulation of the finite horizon Merton problem with HARA utility.

In the notation of Section 3, the state at time j is $Y_j = X_j$, the control is the pair $u_j = (k_j, c_j)$ and the uncertainty is $v_j = b_j$. In the max-min formulation, the objective is to choose controls k_j, c_j for $j = 1, 2, \cdots, N$ which maximize $E^\infty(\mathcal{J}_N)$ where $\mathcal{J}_N$ is the minimum consumption over these N time periods and E^∞ is the totally risk averse expectation. Let $Z_N(x)$ be the maximum of $E^\infty(\mathcal{J}_N)$, considered as a function of the initial wealth $x = X_1$. It satisfies the dynamic programming equation (see Appendix B)

$$Z_N(X_1) = \max_{(k_1,c_1)} \left\{ (c_1 X_1) \oplus E^\infty \left[Z_{N-1}(X_2) \right] \right\}, \quad N = 2, 3, \cdots \tag{4.4}$$

where $\oplus$ is min-plus addition, namely, $a \oplus b = \min(a, b)$ and by (4.1) with $N = 1$

$$X_2 = X_1(1 + r + (b_1 - r)k_1 - c_1). \tag{4.5}$$

For a one-state problem ($N = 1$), one should take $c_1 = 1 + r$ and $k_1 = 0$. Hence, the initial condition the difference equation (4.4) is $Z_1(x) = (1 + r)x$. The value function Z_N is linear: $Z_N(x) = B_N x$. From (4.4), the constants B_N satisfy the recursive formula

$$B_N = \max_{k_1,c_1} \left[c_1 \oplus B_{N-1} \min_{b_1} q(b_1)(1 + r + (b_1 - r)k_1 - c_1) \right] \tag{4.6}$$

with $B_1 = 1 + r$. The max in (4.6) is taken over the triangle Γ, and $q(b_1)$ indicates the likelihood of rate of return b_1 on the risky asset in the first time period $j = 1$. Let $k_1 = k_1^*(N-1)$, $c_1 = c_1^*(N-1)$ give the maximum in (4.6), for $N = 2, 3, \cdots$. For $N = 1$, $k_1^*(0) = 0$ and $c_1^*(0) = 1 + r$. These controls are optimal choices in the initial time period for the N period Merton problem, in the max-min formulation. Similarly, the optimal controls for the jth time period are

$$k_j = k_1^*(N - j), \quad c_j = c_1^*(N - j), \quad j = 1, 2, \cdots, N. \tag{4.7}$$

Since minimum consumption over time cannot increase as the number of time periods increases, $\mathcal{J}_{N+1} \le \mathcal{J}_N$ Hence, $B_{N+1} \le B_N$. The sequence B_N tends to a limit B as $N \to \infty$. The limit B satisfies the steady state version of (4.6):

$$B = \max_{(k,c)} \left[c \oplus B \min_b q(b)(1 + r + (b - r)k - c) \right] \tag{4.8}$$

where for notational convenience we now write k_1, c_1, b_1 in (4.6) as k, c, b. We claim that $B \ge r$. To see this, consider the particular control $k_j = 0$ and $c_j = r$. By (4.1) $X_{j+1} =$

$X_j = X_1 = x$ and hence the consumption is $C_j = rx$ for all j, for these particular (extremely cautious) controls. Therefore $Z_N(x) \geq rx$ which implies $B_N \geq r$ for every N. Hence $B \geq r$.

Suppose that the maximum on the right side of (4.8) occurs at a unique point (k^*, c^*) of the triangle Γ. Then for fixed j, the optimal controls k_j, c_j in (4.7) tend to k^*, c^* as $N \to \infty$. When this happens, the constant controls k^*, c^* are approximately optimal for a large number of time steps N. In particular, the desired uniqueness of (k^*, c^*) holds in the following example, in which there are explicit formulas for k^* and c^*.

Example 4.1. Suppose that the risky asset return rates in each time period have only two possible values $b_j = b^+$ or $b_j = b^-$, with $b^- < r < b^+$. Assume that $q(b^+) = 1$ and $q(b^-) = q^- > 1$. Thus the favorable return rate b^+ is more likely than the unfavorable return rate b^-. Equation (4.8) now has the form

$$B = \max_{(k,c)} \min \left[c, B(1 + r + (b^+ - r)k - c), Bq^-(1 + r + (b^- - r)k - c) \right]. \tag{4.9}$$

The right side of (4.9) is the maximum over the triangle Γ of the minimum of three linear functions of k, c. First, fix c and consider the max over k of the minimum of the last two linear functions. Since $b^+ - r > 0$ and $b^- - r < 0$, these linear functions must be equal at the maximizing k:

$$1 + r + (b^+ - r)k - c = q^-(1 + r + (b^- - r)k - c). \tag{4.10}$$

From (4.10), $k = \lambda(c)$ where λ is a linear, decreasing function of c. By (4.9), B equals the maximum over c of $\min[c, B(1 + r + (b^+ - r)\lambda(c) - c)]$. At the maximum, $B = c$ and

$$1 = 1 + r + (b^+ - r)k - c = q^-(1 + r + (b^- - r)k - c). \tag{4.11}$$

We solve equations to obtain k^* and c^*:

$$(4.12) \qquad \begin{aligned} &(a) \qquad k^* = \tfrac{1 - (q^-)^{-1}}{b^+ - b^-} \\ &(b) \qquad c^* = r + \tfrac{b^+ - r}{b^+ - b^-} \left(1 - (q^-)^{-1}\right). \end{aligned}$$

Note that k^* does not depend on the interest rate r in this example. Also note that $B = c^*$.

Trade off between growth and consumption. At an intuitive level, there is a tradeoff between the growth of wealth X_j and consumption C_j. Higher consumption levels reduce wealth. The changes in wealth according to (4.1) are affected by the uncertain risky asset return rates b_j. We "average out" the uncertainties by considering the growth of $E^\infty(X_j)$ as j increases. For simplicity, let us assume that k_j and c_j are constant: $k_j = k,\ c_j = c$ for all j. From (4.1) and the product form of (3.2) with $v_j = b_j$

$$E^\infty(X_{j+1}) = (1 + g(k, c))E^\infty(X_j), \tag{4.13}$$

(4.14) $$g(k,c) = \min_b q(b)(1 + r + (b - r)k - c).$$

By induction on j, $E^\infty(X_j) = (1 + g(k,c))^{j-1}x$, where $x = X_1$ is the initial wealth. We call $g(k,c)$ the *average growth rate* of wealth, in the sense of E^∞ expectations. In particular, suppose that (k^*, c^*) maximizes the right side of (4.8). Then

(4.15) $$B = \min\left(c^*, B(1 + g(k^*, c^*)\right).$$

A modification of the argument used in Example 4.1 shows that the terms on the right side must be equal. Hence $B = c^*$ and $g(k^*, c^*) = 0$. The constant investment and consumption controls k^*, c^* give an exact balance between expected growth and consumption, in the sense that

(4.16) $$E^\infty(C_j) = c^* E^\infty(X_j) = c^* x$$

remains constant over time and the average growth rate of wealth is 0.

Totally risk averse limits. To compare the max-min approach to the discrete time Merton problem with the more familiar stochastic formulation, let us now suppose that be risky asset return rates $b_1, b_2, \cdots$ are independent random variables, with $p_\gamma(b)$ the probability that $b_j = b$. As in (2.6) we assume that

(4.17) $$\lim_{\gamma \to -\infty} [p_\gamma(b)]^{\frac{1}{\gamma}} = q(b).$$

The goal in the stochastic model is to choose investment and consumption controls k_j, c_j for $j = 1, 2, \cdots, N$ to maximize the expectation

(4.18) $$E\left(\frac{1}{\gamma}\sum_{j=1}^{N} C_j^\gamma\right),$$

subject to the constraints $(k_j, c_j) \in \Gamma$. The solution is found by dynamic programming. Let $V_N(X_1, \gamma)$ denote the value function, which is the maximum of (4.18) considered as a function of the initial wealth X_1 and the number N of time periods. Then $V_N(x, \gamma) = \gamma^{-1} A_N(\gamma) x^\gamma$, where

(4.19) $$A_N(\gamma) = \min_{(k,c)} \left[c^\gamma + A_{N-1}\psi(k,c,\gamma)\right], \qquad N \geq 2,$$

(4.20) $$\psi(k,c,\gamma) = \sum_b p_\gamma(b)\,(1 + r + (b - r)k - c)^\gamma.$$

The initial data for (4.19) is $A_1(\gamma) = (1+r)^\gamma$. Let $B_N(\gamma) = [A_N(\gamma)]^{\frac{1}{\gamma}}$. Then $B_N(\gamma)$ tends to B_N as $N \to \infty$, where $B_1, B_2, \cdots$ are the constants in the solution to the max-min version of the Merton problem above.

5 Continuous time Merton Problems.

The continuous time Merton portfolio optimization problem is as follows. Let $X_t > 0$ denote an investor's wealth at time $t \geq 0$. Let k_t be the fraction of wealth in a risky asset and $1-k_t$ the fraction in a riskless asset. Consumption is at rate $C_t > 0$. The risky asset price obeys a logarithmic Brownian motion, with the mean rate of return μ and volatility σ. Then X_t obeys the Ito sense stochastic differential equation

$$dX_t = X_t \left[(r + (\mu - r)k_t - c_t)\, dt + \sigma k_t dw_t \right] \tag{5.1}$$

where $c_t = C_t/X_t$ and w_t is a Brownian motion. The goal is to choose the controls k_t, c_t to maximize total expected HARA utility of consumption

$$J = E\left[\frac{1}{\gamma} \int_0^T C_t^\gamma dt \right], \tag{5.2}$$

where either T is finite or $T = \infty$. We assume that $\gamma < 0$. For $0 < \gamma < 1$, discount factor should be included in (5.2) in the infinite horizon case $T = \infty$.

The method of dynamic programming gives an explicit solution to the Merton problem. See [9,p. 160] if $T < \infty$ and [10,p. 174] if $T = \infty$. In both cases, the optimal investment control is constant: $k_t = k^*$ for all t, where

$$k^* = \frac{\mu - r}{\sigma^2(1-\gamma)}. \tag{5.3}$$

We write down the optimal investment control only for $T = \infty$. In that case, $c_t = c^*$ for all t, where

$$\begin{cases} c^* = \frac{\gamma\Lambda}{\gamma - 1} \\ \\ \Lambda = \frac{(\mu-r)^2}{2\sigma^2(1-\gamma)} + r. \end{cases} \tag{5.4}$$

The constant Λ has the following interpretation. If consumption is omitted from the model ($c_t = 0$ in (5.1)), then Λ is the optimal growth rate of expected HARA utility of wealth $\gamma^{-1}E(X_T^\gamma)$ as time T increases.

The purpose of this section is to revisit the Merton problem, from the max-min viewpoint taken for discrete time problems in Sections 3 and 4. In the max-min model, we denote the wealth by x_t, which is assumed to satisfy the differential equation

$$\frac{dx_t}{dt} = x_t \left[r + (\mu - r)k_t - c_t + \alpha k_t v_t \right], \tag{5.5}$$

where $\alpha > 0$ and v_t denotes an uncertainty in the risky asset return rate. Consider a finite time interval $0 \le t \le T$. The max-min version of the Merton problem is to choose controls k_t, c_t to maximize $E^\infty(\mathcal{J}_T)$, where $\mathcal{J}_T$ is minimum consumption over time:

$$\mathcal{J}_T = \min_{0 \le t \le T} C_t. \tag{5.6}$$

These controls are allowed to depend on the state x_t. In defining the totally risk averse expectation, we must choose a suitable likelihood function $q_T(v_\cdot)$, where $v_\cdot = \{v_t : 0 \le t \le T\}$ has the same role as the vector $\vec{v}_N$ in (3.2) for the discrete time case. Motivated by the theory of large deviations for small random perturbations of dynamical systems, we choose $q_T(v_\cdot)$ such that

$$\log q_T(v_\cdot) = \frac{1}{2} \int_0^T v_s^2 ds. \tag{5.7}$$

See [3][13].

This finite horizon Merton problem can be solved by dynamic programming. We will not give the solution here, since the method involves ideas from differential games and first order nonlinear partial differential equations [5]. Instead, let us consider only constant controls $k_t = k$, $c_t = c$. Let us find controls $\hat{k}$, $\hat{c}$ which maximize $E^\infty(\mathcal{J}_T)$ among all constant controls k, c, if T is chosen large enough. For constant controls, (5.5) gives

$$\log x_t = \log x_0 + (r + (\mu - r)k - c)\, t + \alpha k \int_0^t v_s ds, \tag{5.8}$$

where x_0 is the initial wealth. Moreover,

$$E^\infty(\mathcal{J}_T) = \min_{0 \le t \le T} E^\infty(C_t) = c \min_{0 \le t \le T} E^\infty(x_t). \tag{5.9}$$

We also have (Appendix D)

$$\log E^\infty(x_t) = \min_{v_\cdot} \left[\log x_t + \frac{1}{2} \int_0^t v_s^2 ds \right] = \log x_0 + \left[r - c + (\mu - r)k - \frac{\alpha^2}{2} k^2 \right] t. \tag{5.10}$$

The right side is maximized, as a function of k, when $k = \hat{k}$ where

$$\hat{k} = \frac{\mu - r}{\alpha^2}. \tag{5.11}$$

When $k = \hat{k}$, then (5.10) becomes

$$\log E^\infty(x_t) = \log x_0 + (\hat{\Lambda} - c)t \tag{5.12}$$

(5.13) $$\hat{\Lambda} = \frac{(\mu - r)^2}{2\alpha^2} + r.$$

Note that (5.11), (5.13) have the same form as for the stochastic Merton problem, with σ replaced by α and with $\gamma = 0$ (log utility). From (5.9) and (5.12) we have

(5.14) $$\log E^\infty(\mathcal{J}_T) = \log c + \log x_0 + \min_{0 \le t \le T} (\hat{\Lambda} - c)t.$$

The minimum on the right side of (5.14) is 0 if $\hat{\Lambda} \ge c$ and is $(\hat{\Lambda} - c)T$ if $\hat{\Lambda} < c$. Let us suppose that T is chosen large enough that $T^{-1} < \hat{\Lambda}$. Then the maximum over c of (5.14) occurs for $c = \hat{c}$, where

(5.15) $$\hat{c} = \hat{\Lambda}.$$

Since $E^\infty(\mathcal{J}_T)$ is maximized when $\log E^\infty(\mathcal{J}_T)$, the choice $c = \hat{\Lambda}$ also maximizes $E^\infty(\mathcal{J}_T)$ as a function of c, where $\mathcal{J}_T$ is the minimum consumption in (5.6).

Tradeoff between growth and consumption. The controls $k_t = \hat{k}$, $c_t = \hat{c}$ give a balance between growth of wealth and consumption, in a way similar to the discrete time case discussed in Section 4. Let us take $k_t = \hat{k}$ as in (5.11) and $c_t = c$. Then by (5.12) we can regard $\hat{\Lambda} - c$ as the growth rate of $E^\infty(x_t)$. According to (5.15), the choice $c_t = \hat{c}$ makes the growth rate equal to 0. It is in this sense that we say that: **the controls $\hat{k}, \hat{c}$ exactly balance growth of wealth and consumption**. This balance can also be stated in terms of inequalities which hold for all possible uncertainties v_t in the risky asset return rates. When $k_t = \hat{k}$ and $c_t = c$,

(5.16) $$\log x_t + \frac{1}{2}\int_0^t v_s^2 ds \ge \log x_0 + (\hat{\Lambda} - c)t.$$

See Appendix D. Note that $\hat{\Lambda} - c$ would be the growth rate of wealth x_t itself, not just of $E^\infty(x_t)$ if there were no uncertainties ($v_t = 0$ for all t). Equality holds in (5.16) for the constant disturbances $v_s = -\alpha\hat{k}$. Note that when $c = \hat{c}$, the right side of (5.16) is the constant $\log x_0$ by (5.15). The values of $\log x_t$ deviate from the initial from the initial value $\log x_0$, depending on the uncertainties v_s for $0 \le s \le t$. For $c_t = \hat{c}$, inequality (5.16) gives a lower bound for $\log x_t$:

(5.17) $$\log x_t \ge \log x_0 - \frac{1}{2}\int_0^t v_s^2 ds.$$

Totally risk averse limits. In the stochastic differential equation model (5.1) for wealth dynamics, let us now assume that $\sigma = \sigma_\gamma$ depends on the HARA parameter γ and that

(5.18) $$\sigma_\gamma \to 0, \qquad \sigma_\gamma^2(1-\gamma) \to \alpha^2 \ \text{ as } \ \gamma \to -\infty.$$

If we again assume that $k_t = k$ and $c_t = c$ are constant controls, then a direct calculation using the Ito differential rule gives (see Appendix D)

$$E^{\infty}(x_t) = \lim_{\gamma \to -\infty} \left[E\left(X_t^{\gamma} \right) \right]^{\frac{1}{\gamma}} . \tag{5.19}$$

This is in accord with (2.8), if we take $\mathcal{J} = X_t$. If in (5.3) and (5.4) we write k_{γ}^{*}, c_{γ}^{*}, Λ_{γ}^{*} to indicate dependence on γ, then these tend respectively to $\hat{k}, \hat{c}$ and $\hat{\Lambda}$ as $\gamma \to -\infty$. Thus, $\hat{k}, \hat{c}$ **are the totally risk averse limits of the optimal controls for the infinite horizon stochastic Merton problem.**

Extensions. The approach taken in this section can readily be extended to the kind of production-debt-consumption model considered in [11]. In that model, random Brownian motion type fluctuations in both productivity of capital and interest rates are allowed. In the max-min formulation, these two Brownian motions are replaced by two corresponding uncertainty functions v_{1t}, v_{2t}. In [7] portfolio optimization models are considered, in which interest rates are stochastic but mean reverting. In the totally risk averse limit, the optimal investment and consumption controls exactly balance growth and consumption. The result mentioned above for the Merton model is a special case.

Appendices

A Totally risk averse expectations.

We define the notion of totally averse expectation as follows. Let Ω be a set, the elements of which are denoted by ω. Let q be a real valued function of Ω such that

$$(A.1) \qquad \inf_{\omega} q(\omega) = 1.$$

For any non-negative function ϕ on Ω, the risk sensitive expectation of ϕ is defined as

$$(A.2) \qquad E^{\infty}(\phi) = \inf_{\omega} q(\omega)\phi(\omega).$$

The following properties are immediate from the definition:

$$(A.3) \qquad \begin{aligned} &(i) \;\; \text{For any finite number of nonnegative functions } \phi_1, \cdots, \phi_m \\ &E^{\infty}(\phi_1 \oplus \cdots \oplus \phi_m) = E^{\infty}(\phi_1) \oplus \cdots \oplus E^{\infty}(\phi_m); \\ &(ii) \;\; E^{\infty}(\lambda\phi) = \lambda E^{\infty}(\phi) \text{ if } \lambda \geq 0; \\ &(iii) \;\; E^{\infty}(\phi) \leq E^{\infty}(\psi) \text{ if } \phi \leq \psi. \end{aligned}$$

In (i), $\oplus$ denotes "min-plus addition", namely

$$(A.4) \qquad (\phi_1 \oplus \cdots \oplus \phi_m)(\omega) = \min_{1 \leq i \leq m} \phi_i(\omega).$$

In Section 2, Ω is a finite set the elements ω of which are denoted there by v. In Section 3, $\omega = \vec{v}_N$ and $q_N(\vec{v}_N)$ has the role of $q(\omega)$. In Section 5, Ω is the set of square integrable functions $v_.$ on the interval $0 \leq t \leq T$ with $q(\omega) = q_T(v_.)$ defined by (5.7).

In the setting of Section 3, the totally risk averse expectation is obtained as a limit from ordinary expectations as follows. Suppose that $v_1, \cdots, v_N$ are modeled as independent random variables, with $p_\gamma(v)$ the probability that $v_j = v$. The expectation of ϕ^γ is

$$E(\phi^\gamma) = \sum_{\vec{v}_N} \left[p_{\gamma N}(\vec{v}_N)^{\frac{1}{\gamma}} \phi(\vec{v}_N) \right]^\gamma$$

$$p_{\gamma N}(\vec{v}_N) = p_\gamma(v_1) \cdots p_\gamma(v_N).$$

If we assume (2.6), then

$$(A.5) \qquad q_N(\vec{v}_N) = \lim_{\gamma \to -\infty} \left[p_{\gamma N}(\vec{v}_N) \right]^{\frac{1}{\gamma}}.$$

This implies

$$(A.6) \qquad E^{\infty}(\phi) = \lim_{\gamma \to -\infty} \left[E(\phi^{\gamma})\right]^{\frac{1}{\gamma}}.$$

Similarly, given any finite collection of functions $\phi_1(\vec{v}_N), \cdots, \phi_m(\vec{v}_N)$

$$(A.7) \qquad \lim_{\gamma \to -\infty} \left[E\left(\sum_{i=1}^{m} \phi_i^{\gamma}\right)\right]^{\frac{1}{\gamma}} = E^{\infty}\left(\min_{1 \le i \le m} \phi_m\right) = E^{\infty}(\phi_1) \oplus \cdots \oplus E^{\infty}(\phi_m).$$

The last equality is by (A.3)(i) and (A.4).

B Dynamic programming.

We outline the method of dynamic programming for the class of max-min optimization models in Section 3, with state dynamics (3.1) and with $E^{\infty}(\mathcal{J}_N)$ to be maximized subject to the constraints imposed on the problem. There are state constraints of the form $Y_j \in \Sigma$ for all $j = 1, \cdots, N$. The control constraints may be state dependent. We assume that $u_j \in \Gamma(Y_j)$, where the sets $\Gamma(y)$ are chosen such that $y \in \Gamma$ implies that $F(y, u, v) \in \Sigma$ for all $u \in \Gamma(y)$ and all possible uncertainties v. This implies that the state constraints $Y_j \in \Sigma$ always hold, provided that $Y_1 \in \Sigma$. Consumption depends on state and control: $C_j = G(Y_j, u_j)$, where $G \ge 0$.

Example. For the Merton problem in Section 4, $Y_j = X_j$ and $u_j = (k_j, c_j)$. The set Σ consists of all $x > 0$, and Γ is the triangle described in Section 4. In this example, Σ is not state dependent. Consumption is $C_j = c_j x_j$ corresponding to $G(x, c) = cx$.

In the method of dynamic programming, the value function $Z_N(y)$ is the maximum of $E^{\infty}(\mathcal{J}_N)$, considered as a function of the initial state $Y_1 = y$ and the number N of time periods. The value function satisfies, at least formally, the difference equation

$$(B.1) \qquad Z_N(Y_1) = \max_{u_1 \in \Gamma(Y_1)} \left[G(Y_1, u_1) \oplus E^{\infty}(Z_{N-1}(Y_2))\right], Y_2 = F(Y_1, u_1, v_1), \qquad N \ge 2.$$

Equivalently, (B.1) can be written as

$$(B.1') \qquad Z_N(Y_1) = \max_{u_1 \in \Gamma(Y_1)} \left[C_1 \oplus \min_{v_1} q(v_1) Z_{N-1}(Y_2)\right].$$

The initial data for (B.1) are $Z_1(y) = \max_{u \in \Gamma(y)} G(y, u)$. Still proceeding formally, optimal controls are obtained from the max-min dynamic programming equation (B.1) as follows. Let $u_1 = u_1^*(Y_1, N-1)$ give the maximum in (B.1). Then u_1 is the optimal control in the first time period. Similarly, $u_j = u_1^*(Y_j, N-j)$ is optimal in the jth time period. For $N = 1$, $u_1^*(Y_1, 0)$ maximizes $G(Y_1, u)$ over $\Gamma(Y_1)$.

The dynamic programming method formalized above can be put on a rigorous mathematical basis, under suitable assumptions on the functions F, G and the constraint sets. The arguments are similar to ones used in discrete time stochastic dynamic programming [2]. In particular, the dynamic programming method applies to the discrete time Merton problem, and to the model in Example 3.1 and Appendix C.

We also note that the max-min optimization problem can be regarded as a dynamic two-player zero-sum game. At each time j the maximizing player (or controller) chooses u_j knowing the state Y_j. The minimizing player then chooses the uncertainties v_j, knowing both Y_j and u_j. The game payoff is the minimum consumption $\mathcal{J}_N$ in (3.3).

C Capital-debt-consumption model.

Consider the model in Example 3.1. In the notation of Section 3 and Appendix B, the state is now $X_j = (K_j, L_j)$. According to (3.8) it is constrained to lie in the sector Σ of the (K, L) plane below the line $L = \bar{\ell}K$ and to the right of the line $K = 0$. The controls are $u_j = (i_j, c_j)$ and $v_j = (r_j, b_j)$ is the uncertainty at time j. $F = (F_1, F_2)$ is given by (3.11) and $G(Y_j, u_j) = K_j c_j = C_j$, where C_j is consumption. The control constraints are $(i_j, c_j) \in \Gamma(K_j, L_j)$, where $\Gamma(K, L)$ is defined below.

Let us first derive the inequality (3.10). Assume that $\bar{\ell} > 0$ and that $L_1 \leq \bar{\ell}K_1$. There must exists controls (i_1, c_1) such that $L_2 \leq \bar{\ell}K_2$ for all possible values of r_1 and b_1. If $L_1 \leq 0$, we can take for instance $i_1 = c_1 = 0$. For $L_1 > 0$, the worst case is $r_1 = r^+$, $b_1 = b^-$. In (3.6), (3.7) we take $j = 1$ and the worst case to obtain

$$(C.1) \qquad (1 + r^+)\ell_1 + i_1 + c_1 - b^- \leq \bar{\ell}(1 + i_1).$$

If we assume (3.10) and $b^- < r^+$, then (C.1) is satisfied with $i_1 = -\underline{i}$ and $c_1 = 0$ whenever $0 < \ell_1 \leq \bar{\ell}$. Similarly, when (3.10) holds $L_j \leq \bar{\ell}K_j$ implies $L_{j+1} \leq \bar{\ell}K_{j+1}$, for $j = 2, \cdots, N$ and all possible uncertainties (r_j, b_j).

We define $\Gamma(K, L)$ to be the set of all (i, c) which satisfy the inequalities

$$(C.2) \qquad \begin{aligned} &(a) \qquad c \geq 0, \quad -\underline{i} \leq i \leq \bar{i}; \\ &(b) \qquad (1 + \tilde{r})\ell + i + c - b^- \leq \bar{\ell}(1 + i), \qquad \ell = L/K, \end{aligned}$$

where $\tilde{r} = r^+$ if $\ell > 0$ and $\tilde{r} = r^-$ if $\ell < 0$.

Then the state constraint $L_j \leq \bar{\ell}K_j$ is satisfied for all j, provided $L_1 \leq \bar{\ell}K_1$, and $(i_j, c_j) \in \Gamma(K_j, L_j)$ for all j.

The dynamic programming equation (B.1′) is now

$$(C.3) \qquad Z_N(K_1, L_1) = \max_{(i_1, c_1)} \left[(c_1 K_1) \oplus \min_{(r_1, b_1)} q(r_1, b_1) Z_{N-1}(K_2, L_2) \right]$$

The initial data for (C.3) are $Z_1(K_1, L_1) = K_1 \bar{c}_1(\ell_1)$, where $\bar{c}_1(\ell_1)$ is the largest possible value of the consumption control c_1. To find $\bar{c}_1(\ell_1)$, we take $i_1 = -\underline{i}$ and require that (C.2) holds:

$$\bar{c}_1(\ell_1) = \bar{\ell} - \ell_1(1 + \tilde{r}) + b^- + \underline{i}(1 - \bar{\ell}). \tag{C.4}$$

While the dynamic programming equation cannot be solved explicitly, it can be reduced to an equation with the one dimensional state $\ell_j = L_j/K_j$. From the linearity of the dynamics (3.6), (3.7) and the inequalities (3.8), (3.9) it follows that $Z_N(K_1, L_1) = K_1 \hat{Z}_N(\ell_1)$. From (C.3), $\hat{Z}_N$ satisfies the "reduced" dynamic programming equation

$$\hat{Z}_N(\ell_1) = \max_{(i_1,c_1)} \left[c_1 \oplus \min_{(r_1,b_1)} q_1(r_1, b_1) \hat{Z}_{N-1}(\ell_2) \right], \tag{C.5}$$

where from (3.6) and (3.7)

$$\ell_2 = \ell_1 + \frac{r_1 \ell_1 + i_1(1 - \ell_1) + c_1 - b_1}{1 + i_1}. \tag{C.6}$$

The initial data for (C.5) are $\hat{Z}_1(\ell_1) = \bar{c}_1(\ell_1)$ as in (C.4).

It can be shown by induction on N that (C.5) has a continuous solution $\hat{Z}_N(\ell)$ for $\ell \leq \bar{\ell}$. If $i_1 = i_1^*(\ell_1, N-1)$, $c_1 = c_1^*(\ell_1, N-1)$ give the maximum in (C.5), then these controls are optimal in the first time period. Similarly, $i_j = i_1^*(\ell_j, N-j)$, $c_j = c_1^*(\ell_j, N-j)$ are optimal in the jth time period.

D Addenda for continuous time Merton problem.

We first derive formula (5.10). By the definition (A.2) of the operator E^∞:

$$E^\infty(x_t) = \min_{v.} \left[q_T(v.) x_t \right], \qquad 0 \leq t \leq T, \tag{D.1}$$

with $q_T(v.)$ as in (5.7). In (D.1), $q_T(v.)$ can be replaced by $q_t(v.)$, by taking $v_s = 0$ for $t < s \leq T$. Hence

$$\begin{aligned} \log E^\infty(x_t) &= \log \min_{v.} [q_t(v.) x_t] \\ &= \min_{v.} \log [q_t(v.) x_t] \\ &= \min_{v.} \left[\log x_t + \tfrac{1}{2} \int_0^t v_s^2 ds \right]. \end{aligned} \tag{D.2}$$

We substitute the right side of (5.8) for $\log x_t$. The minimum in (D.2) is the same as among constant functions $v_s = v$ for all s. This is true because $\log x_t$ is the same if in (5.8) v_s is replaced by its time average $\bar{v}$ over $0 \le s \le t$, and the integral in (D.2) is no less than $\frac{1}{2}\bar{v}^2 t$. Therefore

$$\log E^\infty(x_t) = \log x_0 + \left[r - c + (\mu - r)k + \min_v (\alpha k v + \tfrac{1}{2} v^2) \right] t$$

$$= \log x_0 + [r - c + (\mu - r)k - \tfrac{\alpha^2}{2} k^2] t.$$

This is (5.10).

For $k = \hat{k}$, (5.10) becomes (5.12); and (5.16) is immediate from (D.2) and (5.12).

Derivation of (5.19). In (5.1) we take $\sigma = \sigma_\gamma$ and $k_t = k$, $c_t = c$ constants. Let $\nu = r + (\mu - r)k - c$. Then

$$dx_t = X_t(\nu dt + k\sigma_\gamma dw_t).$$

A direct calculation using the Ito differential rule gives

$$E(X_t^\gamma) = x_0^\gamma \exp\left[\gamma \left(\nu - \frac{(1-\gamma)k^2\sigma_\gamma^2}{2} \right) t \right]$$

By (5.18)

$$(D.3) \qquad \log \lim_{\gamma \to \infty} [E(X_t^\gamma)]^{\frac{1}{\gamma}} = \log x_0 + \left(\nu - \frac{k^2\alpha^2}{2} \right) t.$$

By (5.10) the right side is $\log E^\infty(x_t)$.

Remark. Instead of the totally risk averse expectation operator E^∞, one can consider the related operator E^- with the property that for any positive function ϕ

$$(D.4) \qquad \log E^\infty(\phi) = E^-(\log \phi).$$

E^- is called the min-plus expectation operator. It is linear with respect to "min-plus" addition and multiplication: $a \oplus b = \min(a, b)$, $a \otimes b = a + b$. See [6].

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