Max-plus Stochastic Processes

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Abstract

This paper is concerned with processes which are max-plus counterparts of Markov diffusion processes governed by Ito sense stochastic differential equations. Concepts of max-plus martingale and max-plus stochastic differential equation are introduced. The max-plus counterparts of backward and forward PDEs for Markov diffusions turn out to be first order PDEs of Hamilton-Jacobi-Bellman type. Max-plus additive integrals and a max-plus additive dynamic programming principle are considered. This leads to variational inequalities of Hamilton-Jacobi-Bellman type.

1 Introduction.

The Maslov idempotent calculus provides a framework in which a variety of asymptotic problems, including large deviations for stochastic processes, can be considered [MS]. The asymptotic limit is typically described through a deterministic optimization problem. However, the limit still retains a "stochastic" interpretation, if probabilities are assigned which are additive with respect to "max-plus" addition and expectations are defined so as to be linear with respect to max-plus addition and scalar multiplication. Instead of "idempotent probability" we will use the term "max-plus probability". There is an extensive literature on max-plus probability and max-plus stochastic processes. See [A][AQV][BCOQ][DMD][LM][MS][P1,2][Qu] and references cited there. The max-plus framework is also important for certain problems in discrete mathematics, and in computer science applications [BCOQ][LM].

In this paper we are concerned with processes which are max-plus counterparts of Markov diffusion processes governed by Ito sense stochastic differential equations. Such max-plus stochastic processes are of a type called decision diffusion processes in [BCOQ] and Bellman-Maslov processes in [DMD]. We begin in Section 2 with a quick review of elementary aspects of max-plus probability theory, which are sufficient for purposes of this paper. Then in Section 3, a concept of max-plus martingale is defined, which is similar to the concept of exponential maxingale as defined in [P1, Appendix A]. In Sections 4 and 5 we introduce the idea of max-plus stochastic differential equation, and of associated backward and forward partial differential equations (PDEs). The results presented here are intended as initial steps toward a max-plus stochastic calculus.

As is known, the backward and forward PDEs are of first order and of Hamilton-Jacobi-Bellman type. These PDEs are nonlinear when considered with respect to the usual arithmetical operations, but are linear with respect to max-plus addition and scalar multiplication. Solutions should be considered in the viscosity sense, since classical (smooth) solutions typically do not exist. Solutions to the backward PDE (5.6) correspond to value functions of associated control problems. A dynamic programming principle replaces the Markov property of solutions to Ito sense stochastic differential equations.

In Section 6 we consider some ideas related to H_∞-control theory from the perspective of Sections 4 and 5. A dissipation inequality (6.2) which insures that the H_∞-norm is finite turns out to be equivalent to the existence of a corresponding max-plus supermartingale (6.3). The minimal such supermartingale is a max-plus martingale, described by (6.6)-(6.7).

In Section 7 we consider max-plus expectations of max-plus additive time integrals. This leads to a max-plus additive dynamic programming principle (7.4), and to a corresponding variational inequality (7.8). The ideas in Section 7 are used in Section 8 to obtain infinite horizon sup norm bounds, in a way analogous to the H_∞ bounds discussed previously in Section 6. Related results have been obtained recently in [DM] and [HuJ]. Finally, in Section 9 the max-plus expectations in Sections 5 and 7 are obtained as Freidlin-Wentzell type large deviations limits of corresponding expectations for solutions of Ito sense stochastic differential equations. This assumes a particular (quadratic) choice of max-plus likelihood function.

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2 Max-plus probability.

We begin by reviewing some elementary aspects of max-plus probability. For background we refer to [AQV][BCOQ][DMD][HJ][MS]. Let $I\!R$ denote the real numbers and $I\!R^- = I\!R \cup \{-\infty\}$. For $a, b \in I\!R^-$, the max-plus sum is $\max\{a, b\}$, and the max-plus product is the usual sum $a + b$. Max-plus sums and products are denoted by $a \oplus b$ and $a \otimes b$ respectively. Let Ω be some "sample space", and let Q be a $I\!R^-$-valued function on Ω with

$$\sup_{\omega \in \Omega} Q(\omega) = 0.$$

For any $A \subset \Omega$, the max-plus probability of A is

$$P^+(A) = \sup_{\omega \in A} Q(\omega). \tag{2.1}$$

We call $Q(\omega)$ the *likelihood* of ω, and Q the *max-plus probability density function.* A max-plus random variable is any $I\!R^-$-valued function Z on Ω. The max-plus expectation of Z is

$$E^+(Z) = \sup_{\omega\in\Omega}[Z(\omega) + Q(\omega)]. \tag{2.2}$$

If $E^+(Z) < +\infty$, then Z is max-plus integrable. It is immediate that E^+ is linear under max-plus addition and scalar multiplication. Moreover, if $Z_n(\omega) \uparrow Z(\omega)$ as $n \to \infty$ then $E^+(Z_n) \uparrow E^+(Z)$ as $n \to \infty$.

Remark 2.1 In this paper, no topological structure on Ω is assumed. Max-plus additivity of P^+ is immediate from (2.1), and by assumption $P^+(A) \leq 0$, $P^+(\Omega) = 0$. Further properties of the max-plus probability measure P^+ (which we will not need) are obtained under further assumptions. For instance, if Ω is a Hausdorff topological space and Q is upper semicontinuous on Ω, then $P^+(A)$ is the infimum of $P^+(G)$ among open sets G containing A. See for example [HJ, Appendix C].

Max-plus conditional expectations. For any A such that $P^+(A) > -\infty$, the conditional max-plus probability density given A is

$$Q(\omega|A) = \begin{cases} Q(\omega) - \sup_{\alpha\in A} Q(\alpha), & \omega \in A \\ -\infty \ , \ \omega \notin A. \end{cases}$$

The conditional expectation of a max-plus integrable random variable Z is

$$E^+(Z|A) = \sup_{\omega}[Z(\omega) + Q(\omega|A)].$$

In particular, let π map Ω onto another Ω' and take $A(\omega') = \pi^{-1}(\omega')$. Let $Q(\omega|\omega')$ denote the corresponding max-plus conditional density. Thus, provided $P^+[A(\omega')] > -\infty$:

$$Q(\omega|\omega') = \begin{cases} Q(\omega) - Q'(\omega'), & \text{if } \omega' = \pi(\omega) \\ -\infty \ , & \text{if } \omega' \neq \pi(\omega), \end{cases} \tag{2.3}$$

$$Q'(\omega') = \sup_{\omega^\theta = \pi(\omega)} Q(\omega). \tag{2.4}$$

$Q'(\omega')$ is the density of the max-plus probability measure on Ω' induced by the mapping π. Similarly, the max-plus conditional expectation is denoted by $E^+(Z|\omega')$. It is immediate that $E^+(Z|\omega')$ is max-plus integrable and

$$E^+[E^+(Z|\omega')] = E^+(Z). \tag{2.5}$$

If $\tilde{\pi}$ maps Ω' onto some Ω'', then the composition $\tilde{\pi} \circ \pi$ maps Ω on to Ω''. Similarly to (2.5) one can verify that

$$E^+ \left[E^+(Z|\omega') \mid \omega'' \right] = E^+(Z|\omega''). \tag{2.6}$$

Moreover, if Y and Z are max-plus integrable and $Y(\omega) = Y'[\pi(\omega)]$ for all ω, then:

$$E^+[Y|\omega'] = Y'(\omega') \tag{2.7}$$

$$E^+[(Y \oplus Z)|\omega'] = Y'(\omega') \oplus E^+(Z|\omega') \tag{2.8}$$

$$E^+[(Y \otimes Z)|\omega'] = Y'(\omega') \otimes E^+(Z|\omega'). \tag{2.9}$$

3 Max-plus martingales.

Consider a time interval $0 \leq t \leq T$ and a sample space $\Omega = \Omega_T$. For each t, let π_t be a mapping from Ω_T onto a space Ω_t, with π_T the identity. Moreover, for $0 \leq \tau < t \leq T$, $\pi_\tau = \tilde{\pi}_{\tau t} \circ \pi_t$, where $\tilde{\pi}_{\tau t}$ maps Ω_t onto Ω_τ. We denote elements of Ω_t by ω^t. The max-plus conditional density and conditional expectation in Section 2 are now denoted by $Q(\omega|\omega^t)$ and $E^+(Z|\omega^t)$ respectively.

For $0 \leq t \leq T$ let M_t be a family of max-plus integrable random variables, such that $M_t = M_t(\omega^t)$. Then M_t is a *max-plus martingale* if, for $0 \leq \tau < t \leq T$

$$E^+ (M_t|\omega^\tau) = M_\tau. \tag{3.1}$$

More precisely, we should say that M_t is a $\{\pi_t\}$, Q max-plus martingale. Similarly, M_t is a *max-plus super-martingale* if $\leq$ holds in (3.1), and a *max-plus submartingale* if $\geq$ holds.

Our definition of max-plus martingale is closely related to the idea of exponential maxingale, as defined in [P1, Appendix A], [P2, Chap. 2].

Example 3.1 If Z is max-plus integrable, then

$$M_t = E^+ \left(Z|\omega^t \right) \tag{3.2}$$

is a max-plus martingale. This follows from (2.6) with $\omega' = \omega^t$, $\omega'' = \omega^\tau$.

Throughout the rest of this paper, we will be concerned with the following special case. Let $q(v)$ be a real-valued function of class C^1 on Euclidean $I\!R^m$ satisfying the following assumptions:

$$(A1) \quad \begin{array}{rl} (i) & q(v) \geq 0, \quad \min_v q(v) = 0 \\ (ii) & q \text{ is strictly convex on } I\!R^m \\ (iii) & |v|^{-1} q(v) \to +\infty \text{ as } |v| \to \infty. \end{array}$$

In particular, (A1) holds if $q(v) = p^{-1}|v|^p,\ p > 1$. Let Ω be the set of absolutely continuous, $I\!R^m$-valued functions $w_{\cdot}$ on $[0,T]$ such that

$$(3.3) \qquad w_0 = 0, \quad \int_0^T q(v_s)ds < \infty, \quad v_s = \dot{w}_s.$$

Here $\dot{w}_s = dw_s/ds$. Let $Q(= Q_T)$ be the max-plus density given by

$$(3.4) \qquad Q(w_{\cdot}) = -\int_0^T q(\dot{w}_s)ds.$$

Let π_t be the map restricting $w_{\cdot}$ to $[0,t]$:

$$w^t = \pi_t(w_{\cdot}) = w_{\cdot}\ |_{[0,t]}\ .$$

The density $Q_t(w^t)$ induced by π_t and the conditional density $Q(w_{\cdot} \mid w^t)$ are

$$(3.5) \qquad \begin{array}{ll} (a) & Q_t(w^t) = -\int_0^t q(\dot{w}_s)ds \\ (b) & Q(w_{\cdot} \mid w^t) = -\int_t^T q(\dot{w}_s)ds. \end{array}$$

Example 3.2 Let $q^*(\theta)$ be the convex dual of $q(v)$:

$$q^*(\theta) = \max_v \left[\theta \cdot v - q(v)\right], \quad \theta \in I\!R^m.$$

Let $\theta_s = \theta_s(w^s)$ be bounded, on $[0,T]$ for each $w_{\cdot}$ and let

$$(3.6) \qquad M_t = \int_0^t \left[\theta_s \cdot dw_s - q^*(\theta_s)ds\right].$$

For $\tau < t$, we have from (3.5b)

$$E^+\left[M_t \mid w^\tau\right] = M_\tau + \sup_{w_{\cdot}}\left[\int_\tau^t \left[\theta_s \cdot \dot{w}_s - q^*(\theta_s) - q(\dot{w}_s)\right]ds\right].$$

The integral is nonpositive, and hence M_t is a max-plus supermartingale. The integrand is 0 if $q_v(\dot{w}_s) = \theta_s$. Thus, M_t is a max-plus martingale if the functional differential equation $\dot{w}_s = q_v^{-1}(\theta_s)$ has a bounded solution on $[\tau, t]$ with initial data w^τ. In deriving a max-plus analogue of the Ito differential rule in the next section, we will encounter a particular case in which M_t is a max-plus martingale. In that context, $v_s = \dot{w}_s$ is a "disturbance input" to the differential equation (4.1$'$) describing the evolution of a system state $x_s \in I\!R^n$. In computing max-plus expectations, or conditional expectations, taking sup over $w_{\cdot}$ is the same as sup over $v_{\cdot}$, since we have fixed $w_0 = 0$.

Other interesting choices for sample space and max-plus density include:

(a) If the initial state x_0 at time $t = 0$ is unknown, we can take $\omega = (x_0, w_\cdot)$ and

$$Q(\omega) = \phi_0(x_0) - \int_0^T q(\dot{w}_s) ds,$$

where $\phi_0(x_0)$ is the likelihood of x_0. The likelihood (max-plus probability density) of x_s satisfying (4.1) evolves with time s according to (5.11) below.

(b) Let $\omega = x_\cdot$ and consider a "large deviations" functional of the form

$$I(x_\cdot) = \int_0^T \Lambda(x_s, \dot{x}_s) ds,$$

where $\Lambda(x, y) \geq 0$, $\Lambda(x, \bar{y}(x)) = 0$. Under suitable further assumptions, $Q(x_\cdot) = -I(x_\cdot)$ is a max-plus probability density. For example, if x_s satisfies (4.1) below and $\sigma(x)$ is an $n \times n$ matrix with $\sigma^{-1}(x)$ bounded, then we can take

$$\Lambda(x, y) = q \left[\sigma^{-1}(x)(y - f(x)) \right].$$

The choice $q(v) = \frac{1}{2}|v|^2$ is of special interest for Wentzell-Freidlin type large deviations of dynamical systems under small Brownian motion-type perturbations [FW][DE].

Example 3.3 Suppose that $V_t = V_t(w^t)$, $\ell_t = \ell_t(w^t)$ are bounded and satisfy for $0 \leq \tau < t \leq T$

$$V_\tau = E^+ \left[\left(\int_\tau^t \ell_s ds + V_t \right) | w^\tau \right]. \tag{3.7}$$

Let

$$M_t = \int_0^t \ell_s ds + V_t.$$

By (2.7), (2.9) and (3.7), M_t is a max-plus martingale.

Example 3.4 We define the max-plus integral of ℓ_s on $[\tau, t]$ as

$$\oplus \int_\tau^t \ell_s ds = \sup_{[\tau, t]} \ell_s. \tag{3.8}$$

Suppose that $\Phi_t = \Phi_t(w^t)$, $\ell_t = \ell_t(w^t)$ are bounded and satisfy for $0 \leq \tau < t \leq T$

$$\Phi_\tau = E^+ \left[\left(\oplus \int_\tau^t \ell_s ds \oplus \Phi_t \right) \Big| w^\tau \right]. \tag{3.9}$$

Let

$$M_t = \oplus \int_0^t \ell_s ds \oplus \Phi_t.$$

Since

$$\oplus \int_0^t \ell_s ds = \left(\oplus \int_0^\tau \ell_s ds\right) \oplus \left(\oplus \int_\tau^t \ell_s ds\right), \tag{3.10}$$

(2.7), (2.8) and (3.9) imply that M_t is a max-plus martingale.

Examples 3.3 and (3.4) will arise in Sections 5 and 7, in connection with max-plus multiplicative and max-plus additive dynamic programming principles.

4 Max-plus stochastic differential equations.

Let w_s be any $I\!R^m$-valued absolutely continuous function on $[0, T]$ satisfying (3.3), and let the max-plus density $Q(w_\cdot)$ be as in (3.4). The formulas (3.5)(a)(b) reflect the property that, with this choice of $Q(w_\cdot)$, w_s is a "max-plus independent increments process" in the terminology of [Qu]. When $q(v) = \frac{1}{2}|v|^2$, w_s is a max-plus analogue of a m-dimensional Brownian motion process.

Let $x_s \in I\!R^n$ satisfy the differential equation

$$dx_s = f(x_s)ds + \sigma(x_s)dw_s. \tag{4.1}$$

Equivalently, (4.1) can be written as

$$\dot{x}_s = f(x_s) + \sigma(x_s)v_s, \tag{4.1$'$}$$

where $v_s = \dot{w}_s$ may be regarded as an unknown disturbance entering linearly into the dynamics of a system state x_s. If w_s in (4.1) were replaced by a Brownian motion, then one would have an Ito-sense stochastic differential equation. In keeping with this analogy, we call (4.1) a *max-plus stochastic differential equation.* We assume:

(A2) f and σ are of class C^1 with f, f_x, σ, σ_x bounded. These assumptions are unnecessarily strong, but are made to avoid various technical complications. In this section and Section 5, we give some basic results which are intended as initial steps toward a max-plus stochastic calculus, analogous to the Ito calculus associated with solutions to Ito-sense stochastic differential equations.

Max-plus differential rule. The following analogue (4.4) of the Ito stochastic differential rule holds. For $x, p \in I\!R^n$ let

$$H(x,p) = f(x) \cdot p + q^*\left(\sigma'(x)p\right). \tag{4.2}$$

Let $G(t, x)$ be any class C^1 function. If x_s satisfies (4.1) on $[0, T]$, let

$$\begin{aligned} &(a) \quad M_t^G = \textstyle\int_0^t \left[\theta_s \cdot dw_s - q^*(\theta_s)ds\right], \\ &(b) \qquad \theta_s = \sigma'(x_s)G_x(s, x_s). \end{aligned} \tag{4.3}$$

An elementary calculation gives

$$dG(s,x_s) = \left[\frac{\partial G}{\partial s}(s,x_s) + H(x_s, G_x(s,x_s))\right] ds + dM_s^G. \tag{4.4}$$

Proposition 4.1 *Let $G(t,x)$ be of class C^1. Then*
(a) M_t^G is a max-plus supermartingale.
(b) M_t^G is a max-plus martingale if G_x is bounded.

Proof. Part (a) is immediate from Example 3.2. Moreover, M_t^G is a max-plus martingale provided that the equation $v_s = \dot{w}_s = q_v^{-1}(\theta_s)$ has a bounded solution. Such a solution is obtained by substituting $v_s = q_v^{-1}(\theta_s)$ in (4.1′) and using (4.3b). ■

Remark 4.1 The assumption that G_x is bounded in Proposition 4.1 (b) can be weakened. Consider, for simplicity, $q(v) = \frac{1}{2}|v|^2$. Then $q^*(\theta) = \frac{1}{2}|\theta|^2$, and $q_v^{-1}(\theta) = \theta$. With $v_s = \sigma'(x_s)G_x(s,x_s)$, (4.1′) becomes

$$\dot{x}_s = f(x_s) + (\sigma\sigma')(x_s)G_x(s,x_s).$$

This has a solution on $[\tau, t]$ with any initial data x_τ provided $|G_x(s,x)| \leq C(1+|x|)$ for some C. On the other hand, without such a growth condition on $|G_x|, M_t^G$ may be only a supermartingale. For example, in dimension $n=1$, take $f(x)=0$, $\sigma(x)=1$, $G(x) = \frac{1}{3}x^3$. Then

$$E^+\left[M_t^G \mid w^\tau\right] - M_\tau^G = \sup_{v.} \frac{1}{2}\int_\tau^t \left(v_s - x_s^2\right)^2 ds$$

where $v_s = \dot{x}_s$ and x_τ is given from (4.1) on the interval $[0,\tau]$ and x_0. The supremum cannot be 0 if $t-\tau$ is less than the blow-up time of the solution to $\dot{x}_s = x_s^2$ with initial data x_τ.

Max-plus Markov property. The family of solutions x_s to the differential equation (4.1), with arbitrary initial data, form a deterministic counterpart of the Markov diffusion process obtained when w_s is replaced by a Brownian motion. In the terminology of [BCOQ] x_s is called a decision diffusion process, and in [DMD] x_s is called a Bellman-Maslov process. We will not give a precise definition of max-plus Markov process, nor will we make use of such a definition. Instead, we will consider max-plus versions of such associated concepts as transition densities, backward and forward PDEs. For $0 \leq t < s \leq T$, let $P(t,x;s,\xi)$ denote the max-plus transition density of x_s satisfying (4.1), or equivalently (4.1′), with initial data $x_t = x$. Then

$$P(t,x;s,\xi) = -\inf_{v.}\left\{\int_t^s q(v_r)dr : x_t = x, x_s = \xi\right\}, \tag{4.5}$$

provided the set of such $v_{\cdot}$ with $q(v_r)$ integrable on $[t, s]$ is not empty. Otherwise, $P(t, x; s, \xi) = -\infty$. The function P is the max-plus analogue of the transition density of a Markov diffusion process. The following analogue of the Chapman-Kolmogorov equation holds: for $0 \leq \tau < t < s \leq T$

$$P(\tau, y; s, \xi) = \sup_x \left\{ P(\tau, y; t, x) + P(t, x; s, \xi) \right\}. \tag{4.6}$$

Let us indicate the initial data $x_t = x$ by writing $E^+ = E^+_{tx}$. Also for $t < r < T$, let $x^{r,T}$ denote the restriction of the sample path $x_{\cdot}$ to the time interval $[r, T]$.

Proposition 4.2 *Let $t < r < T$. For any bounded function $F(x^{r,T})$,*

$$E^+_{tx}\left[F(x^{r,T})\right] = E^+_{tx}\left\{E^+_{r,x_r}\left[F(x^{r,T})\right]\right\}.$$

Proof. From (3.5b)

$$\begin{aligned} E^+\left[F(x^{r,T}) \middle| w^r\right] &= \sup_{w_{\cdot}}\left[F(x^{r,T}) - \int_r^T q(\dot{w}_r)dr\right] \\ &= E^+_{r,x_r}\left[F(x^{r,T})\right]. \end{aligned}$$

The proposition the follows from (2.5) with $\omega = w_{\cdot}$ and $\omega' = w^r$. ∎

5 Backward and forward PDEs.

The generator of a Markov diffusion process is a linear, second order, parabolic partial differential operator of elliptic type (possibly degenerate). The corresponding time-dependent backward and forward linear PDEs are of parabolic type. In the max-plus setting, the generator of a Markov diffusion is replaced by the first-order operator H in (4.2) and (4.4). The corresponding backward and forward PDEs (see (5.6)) and (5.11) below are nonlinear with respect to the usual arithmetic operations. However, these PDEs are linear with respect to max-plus addition and scalar multiplication [FM2]. Equations (5.6) and (5.11) are first order PDEs. Solutions of these PDEs are typically not smooth, but should be interpreted as solutions in the viscosity sense.

We begin by describing a solution $V(t, x)$ to the max-plus backward PDE (5.6), in which an inhomogeneous term $\ell(x)$ has been included. In control theory terminology, ℓ is a "running cost" function. Equation (5.6) is considered with data (5.7) at a final time T, where g is a "terminal cost" function. To avoid various technical complications, we assume in addition to $(A2)$:

(A3) $\quad$ ℓ, g are of class C^1 with ℓ, ℓ_x, g, g_x bounded.

Given initial data $x_t = x$, consider the following control problem. Let x_s be the solution to (4.1′) for $t \leq s \leq T$ with control v_s, and let

$$J(t,x;v_\cdot) = \int_t^T [\ell(x_s) - q(v_s)]\, ds + g(x_T). \tag{5.1}$$

Let $V(t,x)$ be the value function:

$$V(t,x) = \sup_{v_\cdot} J(t,x;v_\cdot). \tag{5.2}$$

The value function is rewritten as a max-plus expectation as follows. Let

$$Z_{tT} = \int_t^T \ell(x_s)ds + g(x_T). \tag{5.3}$$

Then

$$V(t,x) = E^+_{tx}(Z_{tT}), \tag{5.2′}$$

where the subscripts tx merely indicate the initial data $x_t = x$.

Proposition 5.1 *A minimizing control $v^0_\cdot$ exists. Moreover, $|v^0_s| \leq M$ for some constant M.*

Proof. The existence of a minimizing $v^0_\cdot$ follows from [FR, p. 68, Cor. 4.1], using assumptions (A1)-(A3). To obtain a bound for v^0_s, we use Pontryagin's principle [FR, Sec 2.11]. The costate p_s satisfies

$$\dot{p}_s = -p_s \cdot \left[f_x(x^0_s) + \sigma_x(x^0_s)v^0_s\right] - \ell_x(x^0_s) \tag{5.4}$$

where x^0_s is the solution to (4.1′) with $x^0_t = x$ when $v_s = v^0_s$. Moreover, $p_T = g_x(x^0_T)$. Since $p_s \cdot \sigma(x^0_s)v - q(v)$ is maximum for $v = v^0_s$, we have

$$v^0_s = q_v^{-1}\left(\sigma'(x^0_s)p_s\right). \tag{5.5}$$

From (5.1)

$$-\left[(T-t)||\ell|| + ||g||\right] \leq J(t,x;0) \leq J\left(t,x;v^0_\cdot\right) \leq (T-t)||\ell|| + ||g||,$$

where $||\ \ ||$ is the sup norm. Thus,

$$\int_t^T q\left(v^0_s\right) ds \leq 2\left[(T-t)||\ell|| + ||g||\right].$$

By (A1)(iii), this gives a bound for the L^1-norm of $v^0_.$ which is uniform with respect to $(t,x) \in [0,T] \times I\!R^n$. From (5.4) there is a constant C_1 such that

$$\left| \frac{d}{ds} \log \left(1 + |p_s|^2 \right) \right| \leq C_1 \left(1 + |v^0_s| \right).$$

Since $p_T = g_x(x^0_T)$ is bounded, this implies that $|p_s| \leq C_2$ for some constant. Since $\sigma(x^0_s)$ is bounded (by (A2)), this implies using (5.5) that $|v^0_s| \leq M$ for some M. ■

Proposition 5.1 implies that the value function $V(t,x)$ is the same if the control cutoff $|v_s| \leq M$ is imposed. One then has:

Theorem 5.1 *The value function V satisfies a Lipschitz condition on $[0,T] \times I\!R^n$. Moreover, $V(t,x)$ is the unique bounded, Lipschitz continuous viscosity solution to the backward PDE*

$$\frac{\partial V}{\partial t} + H(x, V_x) + \ell(x) = 0, \quad 0 \leq t \leq T \tag{5.6}$$

with terminal data

$$V(T,x) = g(x). \tag{5.7}$$

The fact that V is Lipschitz is a special case of [FSon, Lemmas 4.8.1 and 4.8.2]. The remainder of Theorem 5.1 follows from [FSon, Thm. 2.7.1 and Cor. 2.9.1]. For closely related results, see [BCD, Sec. 3.3].

If (5.6)-(5.7) has a smooth (class C^1) solution $G(t,x)$ with G_x bounded, then it follows from the max-plus differential rule (4.4) that $G(t,x) = V(t,x)$. However, there is typically no such smooth solution.

Remark 5.1 Assumptions (A2)(A3) can be weakened in various ways. For instance if σ is constant, the bound $|v^0_s| \leq M$ in Proposition 5.1 follows from bounds on $f_x,\ \ell_x,\ g_x$ only. The value function V is continuous and satisfies a uniform Lipschitz condition in x. In Theorem 5.1, a uniqueness result for viscosity solutions in [M, Sec. 4] can be used.

<u>Max-plus probabilistic interpretation.</u> Let x_s satisfy (4.1$'$) for $0 \leq s \leq T$, with arbitrary initial data x_0 at time 0. For $0 \leq t < T$, $V(t, x_t)$ is the sup in (5.2) with x_t as initial data for (4.1$'$) on the time interval $[t,T]$. The dynamic programming principle implies that, for $\tau < t$,

$$V(\tau, x_\tau) = \sup_{v.} \left[\int_\tau^t [\ell(x_s) - q(v_s)]\, ds + V(t, x_t) \right]. \tag{5.8}$$

Equivalently (since $v_s = \dot{w}_s$)

$$V(\tau, x_\tau) = E^+ \left[\left(\int_\tau^t \ell(x_s) ds + V(t, x_t) \right) \mid w^\tau \right]. \tag{5.8$'$}$$

Let

$$M_t = \int_0^t \ell(x_s)ds + V(t, x_t). \tag{5.9}$$

Then from Example 3.3, with $V_t = V(t, x_t)$:

Proposition 5.2 *M_t is a max-plus martingale.*

The property that M_t is a max-plus martingale is equivalent to the dynamic programming principle. It holds under much weaker assumptions than (A2) and (A3). Since max-plus multiplication is the same as ordinary addition, we may call (5.8) a max-plus multiplicative dynamic programming principle. A corresponding max-plus additive dynamic programming principle will be considered in Section 7.

Forward PDE. Suppose that the state x_0 at time $t = 0$ is unknown, and let $\phi(x_0)$ be the likelihood (max-plus density) of x_0. The likelihood of x_s satisfying (4.1) is $\phi(s, x_s)$, where

$$\phi(s, \xi) = \sup_{x_0} \{\phi_0(x_0) + P(0, x_0; s, \xi)\} \tag{5.10}$$

with P the max-plus transition density in (4.5). The max-plus forward PDE is (see [FM2][HJ])

$$\frac{\partial \phi}{\partial s} = \tilde{H}(\xi, \phi_\xi), \quad s \geq 0 \tag{5.11}$$

where ϕ_ξ is the gradient and

$$\tilde{H}(\xi, p) = -f(\xi) \cdot p + q^* \left(-\sigma'(\xi)p\right). \tag{5.12}$$

The initial data for (5.11) are $\phi(0, \xi) = \phi_0(\xi)$. If ϕ_0 is locally Lipschitz, it can be shown that ϕ is locally Lipschitz and satisfies (5.11) in the viscosity sense. We omit the proof, which is essentially the same as for the case $q(v) = \frac{1}{2}|v|^2$ considered in [FM2].

Remark 5.2 A function ψ on $I\!R^n$ is called *semiconvex* if: for every $R > 0$ there exists C_R such that $\psi(x) + C_R|x|^2$ is convex on the ball $\{|x| \leq R\}$. If f, σ, ℓ are assumed to be class C^2 in addition to (A2), (A3), then ϕ_0 semiconvex implies $\phi(s, \cdot)$ semiconvex for $s > 0$. Moreover, in the "nondegenerate" case with $\sigma(x)$ a $n \times n$ matrix having bounded inverse $\sigma^{-1}(x)$, semiconvexity of $\phi(x, \cdot)$ holds without assuming semiconvexity of ϕ_0. See [FM2, Sec. 4]. Similarly, for the value function, $V(t, \cdot)$ is semiconvex if g is semiconvex. The class of semiconvex functions has a natural relationship to max-plus stochastic processes governed by (4.1), analogous to the relationship between class C^2 functions on $I\!R^n$ and Ito sense stochastic differential equations.

6 H_∞-norm of a nonlinear system.

In this section, we consider the idea of H_∞-norm of a nonlinear system from a max-plus probability perspective. We begin with a brief review of basic ideas concerning H_∞-norms and dissipation inequalities. In this section we take $q(v) = \frac{1}{2}|v|^2$. Let x_s satisfy $(4.1')$ for $s \geq 0$ with unknown initial state x_0 and disturbance $v_\cdot \in L^2([0,T]; \mathbb{R}^m)$ for each $T > 0$. Assume that the unperturbed system (with $v_s = 0$) is globally asymptotically stable to 0. Let L be continuous on $\mathbb{R}^n$, with $L(0) = 0$, $L(x) > 0$ for all $x \neq 0$. The system is said to have L^2-gain $\leq \gamma$ if there exists $W(x) \geq 0$ with $W(0) = 0$ such that: for every $T, x_0, v_\cdot$

$$\int_0^T L(x_s)ds \leq \gamma^2 \left[\int_0^T |v_s|^2 ds + 2W(x_0) \right]. \tag{6.1}$$

We recall below sufficient conditions that (6.1) holds for some $\gamma < \infty$. The least such γ is called the H_∞-norm. See [VdS][BCD][Sor].

Let $\mu = (2\gamma^2)^{-1}$. Then (6.1) is equivalent to

$$E^+_{x_0} \left(\mu \int_0^T L(x_s)ds \right) \leq W(x_0) \tag{6.1'}$$

for all x_0 and all $T > 0$. A sufficient condition that $(6.1')$ hold is that the following dissipation inequality hold with $W(x) \geq 0$, $W(0) = 0$ [VdS]. For $0 \leq \tau < t$ and any $v_\cdot$

$$W(x_t) + \mu \int_\tau^t L(x_s)ds \leq \frac{1}{2} \int_\tau^t |v_s|^2 ds + W(x_\tau). \tag{6.2}$$

Such a function W is called *a storage function.* The dissipation inequality (6.2) is equivalent to the statement that

$$M_t = W(x_t) + \mu \int_0^t L(x_s)ds \tag{6.3}$$

is a max-plus super martingale. If W is of class C^1, then (6.2) is equivalent to the partial differential inequality

$$H(x, W_x(x)) + \mu L(x) \leq 0, \quad x \in \mathbb{R}^n, \tag{6.4}$$

$$H(x,p) = f(x) \cdot p + \frac{1}{2}|\sigma'(x)p|^2.$$

Note that H is the same as in (4.2) with $q^*(\theta) = \frac{1}{2}|\theta|^2$. The following is a sufficient condition that the system $(4.1')$ has finite H_∞-norm. Assume that there exist positive constants c, C, such that for all $x, y \in \mathbb{R}^n$

$$(x - y) \cdot (f(x) - f(y)) \leq -c|x - y|^2. \tag{6.5}$$

Also assume that $f(0)=0$ and $L(x)\le C|x|^2$. Then there exists $K>0$ such that $W(x)=K|x|^2$ satisfies (6.4) for sufficiently small $\mu>0$, and hence for γ sufficiently large.

The minimal storage function $\tilde W(x)$ is found by the following procedure. Let $\tilde V(T,x_0)$ denote the left side of (6.1′). If in Section 5, we write $V(t,x)$ as $V(t,T;x)$ to exhibit dependence on T, then $V(t,T;x)=\tilde V(T-t;x)$. From dynamic programming and the fact that $L\ge 0$, $\tilde V$ is a nondecreasing function of T. Moreover $\tilde V\ge 0$. Let

$$\tilde W(x_0)=\lim_{T\to\infty}\tilde V(T,x_0). \tag{6.6}$$

Proposition 6.1 *If $\tilde W(x_0)<\infty$ for all x_0, then*

$$M_t=\tilde W(x_t)+\mu\int_0^t L(x_s)ds \tag{6.7}$$

is a max-plus martingale.

Proof. By dynamic programming, for $0\le\tau<t\le T$

$$\tilde V(T-\tau,x_\tau)=E^+\left[\left(\mu\int_\tau^t L(x_s)ds+\tilde V(T-t),x_t)\right)\mid w^\tau\right].$$

Since $\tilde V(T-\tau,x_\tau)$ and $\tilde V(T-t,x_t)$ increase to $\tilde W(x_\tau)$ and $\tilde W(x_t)$ as $T\to\infty$,

$$\tilde W(x_\tau)=E^+\left[\left(\mu\int_\tau^t L(x_s)ds+\tilde W(x_t)\right)\mid w^\tau\right].$$

Thus, M_t is a max-plus martingale. ■

From Proposition 6.1, $\tilde W$ satisfies the dissipation inequality (6.2). Moreover, $\tilde W\ge 0$. To show that $\tilde W$ is a storage function it remains to verify that $\tilde W(0)=0$. Given $\delta>0$, choose x^δ such that

$$\tilde W(x^\delta)<\inf_x\tilde W(x)+\delta.$$

The dissipation inequality (6.2) with $\tau=0$, $v_s=0$ implies that for all $t>0$

$$\int_0^t\mu L(\bar x_s)ds\le\delta$$

where $\dot{\bar x}_s=f(\bar x_s)$ and $\bar x_0=x^\delta$. Since $L(x)>0$ for all $x\ne 0$, $x^\delta\to 0$ as $\delta\to 0$. Thus, $\tilde W(x)$ has a strict minimum at $x=0$. If we take $x_0=0$, then for all $T,v_\cdot$ the dissipation inequality implies

$$0\le\int_0^T\mu L(x_s)ds\le\frac{1}{2}\int_0^T|v_s|^2ds.$$

Hence, $V(T,0)=0$ for all T which implies $\tilde W(0)=0$.

From the definition of $\tilde{W}$ and (6.2), $\tilde{W}$ is the minimal storage function: $\tilde{W} \le W$ for any W satisfying (6.2). If $\tilde{W}$ is locally Lipschitz, then $\tilde{W}$ is a viscosity solution of the PDE

$$H(x, \tilde{W}(x)) + \mu L(x) = 0, \quad x \in I\!R^n \tag{6.8}$$

In [FM1, Sec. 4] the following inequality is proved. In addition to (A2) and (6.5) assume that $\sigma(x) = \sigma$ is constant and the Lipschitz condition

$$|L(x) - L(y)| \le K|x - y|, \quad x, y \in I\!R^n.$$

Then

$$|\tilde{V}(T, x) - \tilde{V}(T, y)| \le c^{-1} K \mu |x - y|. \tag{6.9}$$

Since $c^{-1}\mu K$ is a uniform Lipschitz constant for $\tilde{V}(T, \cdot)$, $\tilde{W}$ is also Lipschitz with constant $c^{-1}\mu K$ provided $0 < \mu \le \bar{\mu}$ where $\bar{\gamma} = (2\bar{\mu})^{-\frac{1}{2}}$ is the H_∞-norm.

Remark 6.1 In contrast to the uniform Lipschitz condition in (6.9), which holds when σ is constant, the Lipschitz constant for $V(t, x)$ in Theorem 5.1 may depend on the time difference $T - t$. If $\sigma(x)$ is not constant, then it is more difficult to obtain a Lipschitz estimate for $\tilde{V}(T, \cdot)$ independent of T. If $|x|\,|\sigma_x(x)\sigma'(x)|$ is bounded for large x, then such estimate can be found by the method of [MI].

7 Max-plus additive dynamic programming.

In this section we consider a max-plus analogue Z^+_{tT} of the criterion Z_{tT} in (5.3). To simplify the exposition, the terminal cost term $g(x_T)$ in (5.3) will now be omitted. As in Sections 4 and 5, assume that f, σ satisfy (A2) and that ℓ satisfies (A3). Let

$$Z^+_{tT} = \oplus \int_t^T \ell(x_s) ds, \tag{7.1}$$

where as before x_s is the solution to (4.1) for $t \le s \le T$ with $x_t = x$ and as in (3.8)

$$\oplus \int_t^T \ell(x_s) ds = \max_{[t,T]} \ell(x_s). \tag{7.2}$$

Corresponding to (5.2$'$) let

$$V^+(t, x) = E^+_{tx}(Z^+_{tT}). \tag{7.3}$$

In Section 9, we obtain $V^+(t, x)$ as a large deviations limit of L_λ norms as $\lambda \to \infty$, if in (3.3) $q(v) = \frac{1}{2}|v|^2$.

Let $0 \le t < r < T$. Since the max-plus time integral in (7.2) is additive under subdivision of $[t,T]$ into $[t,r]$ and $[r,T]$, we have by (2.8) and Proposition 4.2:

$$V^+(t,x) = E^+_{tx}\left[\left(\oplus\int_t^r \ell(x_s)ds\right) \oplus \left(\oplus\int_r^T \ell(x_s)ds\right)\right]$$

$$= E^+_{tx}\left[\left(\oplus\int_t^r \ell(x_s)ds\right) \oplus \left(E^+_{rx_r} \oplus \int_r^T \ell(x_s)ds\right)\right].$$

Hence V^+ satisfies the max-plus additive dynamic programming principle:

(7.4) $$V^+(t,x) = E^+_{tx}\left[\left(\oplus\int_t^r \ell(x_s)ds\right) \oplus V^+(r,x_r)\right].$$

In Example 3.4, let $\Phi_t = V^+(t,x_t)$ and $\ell_s = \ell(x_s)$. Then from (7.4) with $x = x_t$ we obtain:

Proposition 7.1 *$M_t = \left(\oplus\int\limits_0^t \ell(x_s)ds\right) \oplus V^+(t,x_t)$ is a max-plus martingale.*

The function $V(t,x)$ satisfies the backward PDE (5.6) in the viscosity sense (Theorem 5.1). In this section we will show that $V^+(t,x)$ satisfies a corresponding variational inequality (7.8). Let us begin by establishing some properties of V^+.

Lemma 7.1 *(a) $\|V^+\| \le \|\ell\|$ where $\|\ \|$ is the sup norm.*
(b) $V^+(T,x) = \ell(x)$.
(c) $V^+(t,x)$ is a nonincreasing function of t.

Proof. Parts (a) and (b) are immediate from the definition (7.3). Write $V^+(t,x)$ as $V^+(t,T;x)$ to indicate dependence on the final time T. By (7.2) V^+ is a nondecreasing function of T. For $0 < \alpha < T - t$,

$$V^+(t+\alpha,T;x) = V^+(t,T-\alpha;x) \le V^+(t,T;x).$$

This is (c). ■

For $0 \le t \le s \le T$, let

(7.5) $$U(t,s;x) = E^+_{tx}[\ell(x_s)].$$

Lemma 7.2 *(a)* $V^+(t,x) = \oplus \int_t^T U(t,s;x)ds$.
(b) V^+ *satisfies a Lipschitz condition on* $[0,T] \times \mathbb{R}^n$.
(c) *There exists* M *such that*

$$V^+(t,x) = \sup_{|v_s| \le M} \left[Z_{tT}^+ - \int_t^T q(v_s) ds \right].$$

Proof. By [FSon, Lemmas 4.8.1 and 4.8.2] U satisfies a Lipschitz condition, jointly in the variables t, s, x. Lemma 7.2 (a) is equivalent to the statement

$$E_{tx}^+ \left[\oplus \int_t^T \ell(x_s) ds \right] = \oplus \int_t^T \left[E_{tx}^+ \ell(x_s) \right] ds. \tag{7.6}$$

To prove (7.6) consider any partition Π of $[t,T]$ into intervals $[s_i, s_{i+1}]$, $i = 1, \cdots, m-1$, with $s_1 = t$ $s_m = T$, and $|\Pi| = \max_i (s_{i+1} - s_i)$. Then

$$E_{tx}^+ \left(\bigoplus_{i=1}^{m-1} \ell(x_{s_i}) \right) = \bigoplus_{i=1}^{m-1} E_{tx}^+ [\ell(x_{s_i})]. \tag{7.7}$$

Next, take a sequence of such partitions $\Pi_j, j = 1, 2, \cdots$ such that Π_{j+1} is a refinement of Π_j and $|\Pi_j| \to 0$ as $j \to \infty$. Since $U(t,s;x) = E_{tx}^+[\ell(x_s)]$ is a continuous function of s, the right side of (7.7) tends to the right side of (7.6). Moreover, $\bigoplus_{i=1}^{m-1} \ell(x_{s_i})$ is a nondecreasing function of j and tends to $\oplus \int_t^T \ell(x_s) ds$ as $j \to \infty$. Hence the left side of (7.7) tends to the left side of (7.6). This proves (a).

Part (b) follows from (a) and Lipschitz continuity of U. Part (c) states that

$$V^+(t,x) = \bar{E}_{tx}^+ (Z_{tT}^+)$$

where $\bar{E}_{tx}^+$ is the max-plus expectation with $Q(w_\cdot)$ replaced by $\bar{Q}(w_\cdot)$, where

$$\bar{Q}(w_\cdot) = \begin{cases} Q(w_\cdot) & \text{if } |\dot{w}_s| \le M \text{ } a.e. \text{ in } [t,T] \\ \\ -\infty & \text{otherwise.} \end{cases}$$

The same proof as for Proposition 5.1 shows that there exists M such that

$$E_{tx}^+[\ell(x_s)] = \bar{E}_{tx}^+[\ell(x_s)].$$

Moreover, the same proof as for (7.6) shows that

$$\bar{E}_{tx}^+ \left[\oplus \int\limits_t^T \ell(x_s) ds \right] = \oplus \int\limits_t^T \bar{E}_{tx}^+ [\ell(x_s)] ds.$$

Then (c) follows from (a). ■

Since V^+ is Lipschitz continuous, V^+ is differentiable at almost every $(t,x) \in [0,T] \times I\!R^n$. Let us next show that V^+ satisfies the dynamic programming variational inequality (7.8) at each such point.

Theorem 7.1 *If V^+ is differentiable at (t,x) and $t < T$, then*

$$0 = \max \left[\ell(x) - V^+(t,x),\ V_t^+(t,x) + H(x, V_x^+(t,x)) \right] \tag{7.8}$$

where H is as in (4.2).

Proof. By Lemma 7.1 (b) and (c), $\ell(x) - V^+(t,x) \leq 0$. By (7.4), for $t < r \leq T$

$$V^+(t,x) \geq E^+_{tx} V^+(r, x_r) \geq V^+(r, x_r) - \int_t^r q(v_s) ds,$$

where $v_s = \dot{w}_s$. In particular, let $v_s = v$ be any constant for $t \leq s \leq r$. By differentiability of V^+ at (t,x)

$$0 \geq V_t^+(t,x) + (f(x) + \sigma(x)v) \cdot V_x^+(t,x) - q(v)$$

for all v. Therefore

$$0 \geq V_t^+(t,x) + H(x, V_x^+(t,x)). \tag{7.9}$$

To complete the proof, it suffices to show that equality holds in (7.9) when $\ell(x) < V^+(t,x)$. By Lemma 7.2 (c) we can take $|v_s| \leq M$. If $r - t$ is small enough and $|v_s| \leq M$ for $t \leq r \leq s$, then

$$\oplus \int_t^r \ell(x_s) ds < V^+(r, x_r).$$

By (7.4)

$$V^+(t,x) = E^+_{tx}[V^+(r, x_r)]. \tag{7.10}$$

By [FSon, Thm. 1.6.1], equality holds in (7.9) when $\ell(x) < V^+(t,x)$. ■

As in [FSon p. 20] we call V^+ a generalized solution to the variational inequality if (7.8) holds for almost all (t,x). Unfortunately, besides the value function V^+ there are typically infinitely many other generalized solutions to (7.8) with the same terminal data $\ell(x)$ at time T. This difficulty is resolved by considering viscosity solutions rather than generalized solutions. The following analogue of Theorem 5.2 holds.

Theorem 7.2 *$V^+(t,x)$ is the unique bounded, Lipschitz continuous viscosity solution to the variational inequality* (7.8) *with the terminal data.*

$$V^+(T,x) = \ell(x). \tag{7.11}$$

The proof of Theorem 7.2 involves small changes in standard arguments to prove the corresponding results for viscosity solutions of first order PDEs. We sketch them in the Appendix.

8 Infinite horizon sup norm bounds.

In this section, we consider bounds for max-plus expectations on the infinite time interval $t \geq 0$. From these bounds we will obtain sup norm bounds which are max-plus additive analogues of the H_∞ bounds considered in Section 6. Let us again take $q(v) = \frac{1}{2}|v|^2$. Let ℓ be continuous on $I\!R^n$ with $\ell(x) \geq 0$. We are interested in bounds of the following type. We seek a function W^+ such that: for every T, x_0, v.

$$\oplus \int_0^T \ell(x_s)ds \leq \frac{1}{2}\int_0^T |v_s|^2 ds + W^+(x_0). \tag{8.1}$$

At the end of the section (Remark 8.2) we mention an extension of (8.1) considered in [DM].

Since the left side of (8.1) is the maximum of $\ell(x_s)$ over $[0,T]$, this inequality bounds the sup norm of $\ell(x_s)$ in terms of the L^2-norm of v_s plus a function depending only on the initial data x_0. Equivalently,

$$E^+_{x_0}\left(\oplus \int_0^T \ell(x_s)ds\right) \leq W^+(x_0) \tag{8.1$'$}$$

for all x_0 and T. According to (7.6) this is also equivalent to

$$E^+_{x_0}[\ell(x_s)] \leq W^+(x_0) \tag{8.1$''$}$$

for all x_0 and $s \geq 0$.

Proposition 8.1 *If $W^+(x)$ is a C^1 function such that $\ell(x) \leq W^+(x)$ and $H(x, W^+_x(x)) \leq 0$ for all x, then* (8.1$'$) *holds.*

Proof. By Proposition 4.1(a)

$$\ell(x_t) \leq W^+(x_t) = W^+(x_0) + \int_0^t H(x_s, W^+_x(x_s))ds + M_t$$

where M_t is a max-plus supermartingale and $M_0 = 0$. Since $H(x_s, W_x^+(x_s)) \leq 0$ and $E_{x_0}^+(M_t) \leq 0$,

$$E_{x_0}^+[\ell(x_t)] \leq E_{x_0}^+ W(x_t) \leq W^+(x_0).$$

We then obtain (8.1′) from (7.6). ■

Example 8.1 Assume (6.5) and also $f(0) = 0,\ 0 \leq \ell(x) \leq C|x|^2$. Let $W^+(x) = K|x|^2$. Then

$$\begin{aligned} H(x, W_x^+(x)) &= f(x) \cdot W_x^+(x) + \frac{1}{2}|\sigma'(x) W_x^+(x)|^2. \\ &\leq 2\left[-cK + \|\sigma'\|^2 K^2\right]|x|^2, \end{aligned}$$

which is nonpositive if $\|\sigma'\|^2 K \leq c$. If $C\|\sigma'\|^2 \leq c$, then we can choose K such that $C \leq K$ and $\|\sigma'\|^2 K \leq c$. Then (8.1′) holds.

The assumptions of Proposition 8.1 are equivalent to the statement that $W^+(x)$ is a classical supersolution of the variational inequality

$$0 = \max[\ell(x) - W(x), H(x, W_x(x))]. \tag{8.2}$$

We next find by the same procedure as in Section 6 the minimal choice $\tilde{W}^+(x_0)$ for $W^+(x_0)$ in (8.1). Let us denote the left side of (8.1′) by $\tilde{V}^+(T, x_0)$. Then $\tilde{V}^+$ is a nondecreasing function of T. As in (6.6), let

$$\tilde{W}^+(x_0) = \lim_{T \to \infty} \tilde{V}^+(T, x_0). \tag{8.3}$$

If $\tilde{W}^+(x_0) < \infty$, then $\tilde{W}^+$ is the desired minimal choice. If in Section 7, we write $V^+(t,x)$ as $V^+(t,T;x)$, then $V^+(t,T;x) = \tilde{V}^+(T-t,x)$. By using the max-plus additive dynamic programming principle (7.4) and Example 3.4, we obtain in the same way as for Proposition 6.1 that

$$M_t^+ = \tilde{W}^+(x_t) \oplus \left(\oplus \int_0^t \ell(x_s) ds \right) \tag{8.4}$$

is a max-plus martingale, provided $\tilde{W}^+(x) < \infty$ for all x.

If we assume in addition to (A2) and (A3) that $\sigma(x) = \sigma$ is constant and that (6.5) holds, then in the notation of Section 7

$$|U(t,s;x) - U(t,s;y)| \leq c^{-1} K |x - y|$$

where K is a Lipschitz constant for ℓ. This is obtained in the same way as (6.9). By Lemma 7.2(a)

$$|\tilde{V}^+(T,x) - \tilde{V}^+(T,y)| \leq c^{-1} K |x - y|.$$

Let $T \to \infty$ to conclude that $\tilde{W}^+$ is Lipschitz with constant $c^{-1}K$. Moreover $\tilde{W}^+$ is a viscosity solution of the variational inequality (8.2).

Remark 8.2 Dower and McEneaney [DM] consider the following more general formulation. Let $\ell_1,\ \ell_2$ be nonnegative functions. One seeks $\bar{W}(x)$ such that : for all T, x_0, v.

$$\int_0^T \ell_1(x_s)ds + \ell_2(x_T) \le \frac{1}{2}\int_0^T |v_s|^2 ds + \bar{W}(x_0). \tag{8.6}$$

This formulation is reduced to the one considered above (with $\ell_1(x) = 0$) as follows. Consider an augmented state (x_s, ζ_s) where x_s satisfies (4.1) and $\zeta_s \ge 0$ satisfies

$$\dot{\zeta}_s = \ell_1(x_s). \tag{8.7}$$

Let $\ell(x,\zeta) = \zeta + \ell_2(x)$. Then (8.1′) becomes

$$E_{x_0\zeta_0}\left(\oplus\int_0^T \ell(x_s,\zeta_s)ds\right) \le W^+(x_0,\zeta_0). \tag{8.6′}$$

The left side of (8.6′) is the form $\zeta_0 + \bar{V}(T,x_0)$ where $\bar{V}(T,x_0)$ is obtained by putting $\zeta_0 = 0$. Thus, in (8.6′) we seek W^+ of the form $W^+(x,\zeta) = \zeta + \bar{W}(x)$, and from (8.6′) obtain the desired in equality (8.6). The variational inequality (8.2) becomes

$$0 = \max[\zeta + \ell_2(x) - W(x,\zeta), \quad H(x, W_x) + \ell_1(x)W_\zeta]. \tag{8.8}$$

For W of the form $W(x,\zeta) = \zeta + \bar{W}(x)$, this becomes when $\zeta = 0$

$$0 = \max[\ell_2(x) - \bar{W}(x), \quad H(x, \bar{W}_x(x)) + \ell_1(x)], \tag{8.9}$$

which is the dynamic programming variational inequality obtained in [DM].

9 Large-deviations limits.

In this section we again take $q(v) = \frac{1}{2}|v|^2$. The max-plus SDE (4.1) then arises naturally from the Freidlin-Wentzell theory of large deviations for small random perturbations [FW]. For $\varepsilon > 0$ let x_s^ε satisfy the Ito sense SDE

$$dx_s^\varepsilon = f(x_s^\varepsilon)ds + \varepsilon^{\frac{1}{2}}\sigma(x_s^\varepsilon)dB_s, \quad t \le s \le T \tag{9.1}$$

with $x_t^\varepsilon = x$, where B_s is an m-dimensional Brownian motion. Corresponding to (5.3), let

$$Z_{tT}^\varepsilon = \int_t^T \ell(x_s^\varepsilon)ds + g(x_T^\varepsilon). \tag{9.2}$$

Then under our previous assumptions

$$E^+_{tx}(Z_{tT}) = \lim_{\varepsilon\to 0} \varepsilon \log E_{tx}\left[\exp(\varepsilon^{-1} Z^\varepsilon_{tT})\right]. \tag{9.3}$$

The max-plus additive counterpart in Section 7 arises from another kind of limit. It is a large deviations version of a L_λ-norm as $\lambda = \varepsilon^{-1}$ tends to infinity. Let $F(x)$ be a positive function such that $\ell(x) = \log F(x)$ satisfies our previous assumptions. Let

$$\mathcal{J}^\varepsilon_{tT} = \int_t^T F(x^\varepsilon_s)^{\frac{1}{\varepsilon}} ds = \int_t^T \exp(\varepsilon^{-1}\ell(x^\varepsilon_s))ds.$$

From large deviations theory

$$\lim_{\varepsilon\to 0} \varepsilon \log E_{tx}(\mathcal{J}^\varepsilon_{tT}) = \oplus \int_t^T E^+_{tx}[\ell(x_s)]ds = E^+_{tx}(Z^+_{tT}), \tag{9.4}$$

with Z^+_{tT} as in (7.1).

Appendix

In this Appendix we sketch a proof of Theorem 7.2. We first recall the concept of viscosity solution of the variational inequality (7.8). See [FSon, Chap. 2][BCD, Chap. 3.3]. A continuous function $G(t,x)$ is a viscosity supersolution of (7.8) if the following holds. If ϕ is a function such that $G(t,x)-\phi(t,x)$ has a local minimum at (t_0,x_0) with $0\le t_0<T$, then

$$(A.1) \qquad \max[\ell(x_0)-G(t_0,x_0),\phi_t(t_0,x_0)+H(x_0,\phi_x(t_0,x_0))]\le 0.$$

By adding a constant to ϕ, we can assume that $G(t_0,x_0)=\phi(t_0,x_0)$. Similarly, $G(t,x)$ is a viscosity subsolution if the inequality is reversed in (A.1) whenever $G(t,x)-\phi(t,x)$ has a local maximum at (t_0,x_0). If G is both a viscosity super and sub solution, then G is a viscosity solution.

Proof of Theorem 7.2. By Lemmas 7.1 and 7.2, V^+ is bounded and Lipschitz continuous. We first use the max-plus dynamic programming principle (7.4) to show that V^+ is a viscosity solution of (7.8). Suppose that $V^+-\phi$ has a local minimum at (t_0,x_0), with $V^+(t_0,x_0)=\phi(t_0,x_0)$. Since $\ell(x_0)-V^+(t_0,x_0)\le 0$ by Lemma 7.1(b) and (c), we need to show that

$$(A.2) \qquad \phi_t(t_0,x_0)+H(x_0,\phi_x(t_0,x_0))\le 0.$$

For $t<r$ and $r-t$ small enough

$$0=V^+(t_0,x_0)-\phi(t_0,x_0)\le V^+(r,x_r)-\phi(r,x_r).$$

By (7.4)

$$V^+(t_0,x_0)\ge E^+_{t_0x_0}[V^+(r,x_r)],$$

and hence

$$\phi(t_0,x_0)\ge E^+_{t_0x_0}[\phi(r,x_r)].$$

The same argument used to obtain (7.9) gives (A.2).

Next, suppose that $V^+-\phi$ has a local maximum at (t_0,x_0) with $V^+(t_0,x_0)=\phi(t_0,x_0)$. If $\ell(x_0)=V^+(t_0,x_0)$, then the max in (A.1) is nonnegative. If $\ell(x_0)<V^+(t_0,x_0)$, then the last part of the proof of Theorem 7.1 gives (7.10) for $t<r$ and $r-t$ small enough. This implies (see [FSon p. 76] or [BCD p. 150])

$$(A.2') \qquad 0\le \phi_t(t_0,x_0)+H(x_0,\phi_x(t_0,x_0)).$$

This shows that V^+ is a viscosity solution.

To complete the proof of Theorem 7.2, we need to show that V^+ is unique in the class of bounded, Lipschitz continuous viscosity solutions G of (7.8) with $\ell(x)=G(T,x)$. This is implied by the following comparison principle. Let G_1 be a viscosity subsolution and G_2 a viscosity supersolution to (7.8), both of which are bounded and Lipschitz continuous. If $G_1(x,T) \leq G_2(x,T)$ for all $x \in I\!R^n$, then $G_1(t,x) \leq G_2(t,x)$ for all $(t,x) \in [0,T] \times I\!R^n$. To obtain this comparison principle, only small changes are needed in the standard proof of the corresponding comparison principle for first order PDEs [BCD, Thm. 3.7]. The proof there proceeds by contradiction, assuming that $G_1(t,x) - G_2(t,x)$ has a positive supremum. On [BCD, p. 153] test functions $\phi(t,x)$ and $\psi(s,y)$ are constructed such that $G_1 - \phi$ has a max at $(\bar{t},\bar{x})$ and $G_2 - \psi$ has a min at $(\bar{s},\bar{y})$, where $(\bar{t},\bar{x}),(\bar{s},\bar{y})$ are in a small neighborhood N of a point $(\tilde{t},\tilde{x})$ such that

$$G_1(t,x) - G_2(t,x) > a > 0 \quad \text{for all } (t,x) \in N.$$

Then ψ satisfies (A.2) at $(\tilde{s},\tilde{y})$ and $\ell(\bar{y}) \leq G_2(\bar{s},\bar{y})$. The neighborhood N can be chosen small enough that $\ell(\bar{x}) < G_1(\bar{t},\bar{x})$. Then ϕ satisfies (A.2′) at $(\bar{t},\bar{x})$. By continuing both the proof in [CD, pp. 153–154] a contradiction is obtained. ■

Note. In the definition above of supersolution, the inequality in (A.1) is opposite that used in [FSon] and [BCD]. Similarly, the inequality is opposite for subsolutions. What we call $H(x,p)$ in (4.2) would be denoted in [FSon] and [BCD] by $-H(x,p)$.

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