Oct.16, 2000

ABSTRACT

Stochastic Inter-temporal Optimization in Discrete Time

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Takashi Negishi, Rama Ramachandran and Kazuo Mino (ed.) Economic Theory, Dynamics and Markets: Essays in Honor of Ryuzo Sato, Kluwer, 2001

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The standard model of inter-temporal optimization is based upon certainty equivalence and ignores risk and uncertainty. We solve a modification of the standard model of inter-temporal optimization in an environment where the return to capital is stochastic, and we impose the constraint that there be no default on the short-term debt. We derive benchmarks for optimal foreign debt in a world of uncertainty. Insofar as the actual debt exceeds the benchmark, the expected utility of consumption is reduced. Default occurs with probability (1-p) when the debt exceeds the maximum debt $f(2)_{max}$. The main reasons for a deviation between the actual debt and the optimal debt are that the borrower is overly optimistic about the distribution function of the return to investment, does not optimize with the "no default" constraint, and/or there is a moral hazard problem.

We also consider an inter-temporal optimization model involving extreme prudence. The lender, who may be an institutional investor, has infinite risk aversion and will only lend for projects where the profitability of the investment is almost sure. In this case also, we derive the optimal debt, which is our benchmark for risk management.

JEL classification: D61, D81, D9, F34

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1. The Need for a Paradigm of Risk Management of Short-term Foreign Currency Denominated Debt

Data on the credit rating of bonds issued in the first half of the 1990s suggest that investors in emerging market securities paid little attention to credit risk, or that they were comfortable with the high level of credit risk that they were incurring[1]. The compression of the interest rate yield spread prior to[2] and the subsequent turmoil in emerging markets have raised doubts about the ability of investors to appropriately assess and price risk. After the 1997 crises, Moody's credit rating agency wrote that there was a need for a "paradigm shift" that involves greater analytic emphasis on the risks associated with the reliance on short-term debt for otherwise creditworthy borrowers.

The literature in international finance concerning inter-temporal optimization in discrete time makes assumptions that imply certainty equivalence[3]. A major implication is that investment should be increased as long as the expectation of the marginal product of capital exceeds the expected interest rate. If the capital output ratio were constant, investment would then greatly exceed saving, and an extremely high foreign debt would be incurred. The certainty equivalence assumptions ignore the risks inherent in such a high level of investment and foreign debt. The standard approach fails to address the questions of how should one optimize under uncertainty, or how to evaluate what debt is likely to lead to default. We develop a paradigm for inter-temporal optimization under uncertainty in a finite horizon discrete time context, with the <u>constraint</u> that there be no

[1] This section relies on International Monetary Fund, International Capital Markets, Washington DC (1999), and International Monetary Fund, Anticipating Balance of Payments Crises, Occasional Paper #186, (1999).
[2] The market expectations as embodied in interest rates did not widen significantly prior to the Mexican crisis. In the Asian crises, spreads hardly increased in the months prior to the floatation of the Bhat. The credit rating agencies and the market analysts all failed to signal the Asian crises in advance. They downgraded these countries only after the crises.
[3] See the reference to Obstfeld and Rogoff below. Similarly, the use of the Maximum Principle in continuous time assumes perfect certainty. Neither approach is useful in a world of risk and uncertainty. By contrast, Infante and Stein (1973) used dynamic programming to solve for intertemporal optimization in an environment where there is not perfect knowledge. The derived suboptimal feedback control drives the economy to the unknown perfect certainty optimal path.

default on short- term foreign currency denominated debt.

2. A Discrete Time Finite Horizon Model, Risk and Risk Aversion

In an earlier paper[4], we solved the problem of the optimal consumption, capital and foreign debt in continuous time over an infinite horizon, where the productivity of capital and the interest rate have Brownian motion components. By allowing the shocks to be correlated, we relate the external shocks to the vulnerability of the banking sector. The technique of analysis is dynamic programming. Here, we solve a modification of the standard, well-known, model of inter-temporal optimization.

The contribution of our paper is as follows. We show how to solve the inter-temporal optimization problems without making the usual[5] certainty equivalence assumptions. The standard two period model can be solved by calculus, whereas the infinite horizon case discussed in our earlier paper employed the dynamic programming method with the technical mathematical difficulties encountered in the theory of continuous time stochastic control. The effects of different ways of describing the uncertainty upon the optimal consumption, investment, the current account deficit and debt, are explicitly considered. The object is to select consumption and investment - and the resulting short term debt - in the first period to maximize the expected present value of consumption over both periods. The constraint is that, regardless of the state of nature in the second period, there will be no default on the debt.

Part 2.1 is a discussion of a modification of the standard model. Part 2.2 is an intuitive and graphic presentation of some of our results. The general mathematical solution is in part 3. Part 4 considers an extremely prudent approach to inter-temporal optimization by an agent who has infinite risk aversion. This would be the case if the lenders were institutional investors who are infinitely risk averse and will only lend for

[4] Fleming and Stein, CESifo Working paper #204 (1999).
[5] The intertemporal optimization analysis in Obstfeld and Rogoff (1996: 60-87) makes assumptions that imply certainty equivalence. Hence risk is not considered in their resulting optimal consumption, investment and debt. They are aware of this deficiency, and write (p.81) the following: "…consumption is determined according to the certainty equivalent principle. People make decisions under uncertainty by acting as if future stochastic variables were sure to turn out equal to their conditional means. Certainty equivalence is rarely a rational basis for decisions."

projects which are almost sure things. We derive optimal investment and debt in this most prudent case.

2.1 Stochastic Optimal Control: Discrete Time over a Two Period Horizon

We consider a series of repeating two period cycles. In the first period, the GDP denoted by Y(1) = b(1)K(1), where K(1) is the capital stock and b(1) is the current productivity of capital. The country selects consumption and investment. The current account deficit, equal to consumption plus investment less the GDP, is financed through short-term foreign debt. The interest rate r is known. The productivity of capital b(2) in period t =2 is the crucial stochastic variable. The GDP is proportional to capital: Y(t)/K(t) = b(t). The GDP in the second period is Y(2) = b(2)[K(1) + I(1)], where K(1) is the initial capital and I(1) is the rate of investment in the first period. The GDP in the second period Y(2) is stochastic, because it depends upon b(2), the stochastic productivity of the capital or investment. The variables are measured in $US.

The productivity of investment b(2) is stochastic for the following reason. Dollars are borrowed at interest rate r to purchase capital and produce an output, which is sold in the world market. The dollar value of the output depends upon several factors: the terms of trade (export/import prices), the exchange rate of the country and the productivity of the investment, measured in domestic currency. If the terms of trade deteriorate, the investment is ill advised or the exchange rate depreciates, the productivity of the investment b(2) measured in $US declines. Then the repayment of the dollar denominated debt is more costly. Instead of viewing the effect of exchange rate uncertainty upon the interest payments denominated in foreign currency, we view the uncertainty via the productivity of investment[6].

The risk/uncertainty is contained in the net return on investment x = [b(2) - r], the stochastic productivity of capital b(2) less r, the known interest rate. The range of b(2) is r

[6] A good example of the uncertainty, and our use of b(2) to describe it, concerns Mexico and Thailand. The firms and the banks borrowed US dollars on the assumption that the exchange rate would continue to be fixed relative to the US dollar. When the terms of trade declined, the return in domestic currency declined. The firms had difficulty in repaying the banks that, in turn, had more non-performing loans. It then was difficult for the banks to repay the foreign creditors, and the exchange rate depreciated. The depreciation

$\pm$ a/2, a > 0. The values of the net return x = [b(2) - r] are symmetrical around zero with a range a > 0, as described in Box 1, with probabilities (p, 1-p), 1 > p > 0, in the good and bad case respectively. This is a simple and general formulation that makes minimal assumptions about the distribution function.

BOX 1. UNCERTAINTY CONCERNING THE NET RETURN x = [b(2) - r]		
b(2)	Pr(b)	
$b^+(2)$ = r + a/2	1 > p > 0	good case
$b^-(2)$ = r – a/2	(1-p) > 0	bad case

Productivity of capital in the second period is b(2) = Y(2)/K(2). The interest rate is r. The net return is x = b(2) - r. The expected net return E(x) = E[b(2)-r] = a(p - 1/2); range [b(2)-r] = a > 0. Risk concerns the realization of the "bad" case. Var (x) = $a^2p(1-p)$

The optimization concerns the sum of the utility of consumption in the first period plus the expectation of utility of consumption in the second period. The consumption in the second period is equation (1). It is equal to the stochastic GDP less the repayment of debt and interest (1+r)L(2) less investment I(2). Default/rescheduling of debt will occur if consumption in period two falls below a certain minimum tolerable level, which we shall call $C(t)_{min}$ = 0. The "no default" constraint is that C(2) in equation (1) be positive.

(1) C(2) = b(2)[K(1) + I(1)] - (1+r)L(2) - I(2) > 0.

The debt carried into the second period L(2) is consumption plus investment less GDP in the first period, the trade deficit. In a series of repeating cycles K(3) = K(1),or I(1) + I(2) = 0. Then the consumption in the second period is equation (1.1), using the no-default constraint.

(1.1) C(2) = b(2)K(1) + [b(2) - r]I(1) + (1+r)[b(1)K(1) - C(1)] > 0

The net return on investment [b(2) - r] is a/2 in the good case with probability p, and (-a/2) in the bad case with probability (1-p). The consumption in the good case is $C^+(2)$, and is $C^-(2)$ in the bad case.

2.1.1 The Maximal Debt

The maximal debt is defined as the debt associated with zero consumption. If the bad case materializes, and the actual debt exceeds the maximal debt $L(2)_{max}$, there will be

aggravated the decline in b(2) arising from the decline in the terms of trade. With a depreciated currency, default/renegotiation. Consumption $C^-(2)$ will be positive in the bad case where $b^-(2) = r - a/2$, if inequality (1.1) is satisfied. The debt carried into the second period L(2) is investment I(1) less saving S(1) = b(1)K(1) - C(1). Hence the no-default constraint for the maximum debt, $L(2)_{max}$, can be expressed either as (2.1) or (2.2).

(2.1) $L(2)_{max} = [b^-(2)K(1) + (1 + b^-(2))S(1)] / (a/2)$

(2.2) $L(2)_{max} = b^-(2)K(1) + (1 + b^-(2))I(1)] / (1 + r)$

If the actual debt exceeds the maximal, then with probability (1-p) > 0 there will be a default. This equation is independent of any optimization, and just depends upon the chosen values for saving S(1) and investment I(1) in period t = 1, and the bad value $b^-(2)$ realized in period t = 2..

2.1.2 Optimization

The criterion function is equation (3): the maximization, over the set of controls and the constraints, of the expectation [E] of the present value [J] of the utility of consumption. The utility is a HARA function $U(C(t)) = (1/\gamma)C(t)^\gamma$, where $\gamma < 1$. Risk aversion is $(1-\gamma) > 0$. The most interesting and useful cases are when $\gamma \leq 0$. The $\gamma = 0$ case is the logarithmic utility function. Then the utility of zero consumption is minus infinity; and the optimization will avoid that situation. Controls C(1) and I(1) are selected in period t=1 to maximize the sum of the expected utility of consumption over the two periods, subject to the no-default constraint. This is equation (3), using (1.1) for C(2).

(3) $\max E[J] = \max (1/\gamma)C(1)^\gamma + (1/\gamma)[pC^+(2)^\gamma + (1-p)C^-(2)^\gamma]$

Three cases are considered. The first two assume that $(1-\gamma) > 0$ is finite. The third case, discussed in part 4, is very important and less well known. It involves a very conservative approach to risk management. It is called the Large Deviations [LD] approach to risk. The meaning of the very conservative LD approach is discussed below.

2.2. Intuitive and Graphic Description of the Solution

We describe intuitively the solution for the constrained optimal (C-O) investment, debt and consumption in the case where the utility function is logarithmic, $\gamma = 0$. The C-O

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the net value added of the economy commanded fewer dollars to repay the loans.

debt, investment and saving are described in equations (4)-(6). An asterisk denotes constrained optimal quantities. All are measured as fractions of current GDP, which is Y(1) = b(1)K(1). Optimal saving/GDP in period one equation (4) is independent of the expected net return.

(4) $S(1)/Y(1) = h^*(1)$, independent of E(x)

In the standard literature that makes the certainty equivalence assumptions, the optimal stock of capital is adjusted until the expected marginal productivity is equal to the interest rate. If the expected productivity of capital Y(t)/K(t) = b(t) is constant and exceeds the interest rate E(x) > 0, then investment would be maximal. A maximal amount would be borrowed at rate r to finance the maximal investment, and a maximal amount of risk is assumed.

In our model where risk is explicitly taken into account and there is a no-default/rescheduling constraint, we obtain a very different result. Even though there are no diminishing returns to capital, optimal investment/GDP denoted i*(1) is described by equation (5). Optimal investment is zero, for values of the expected net return E(x) = E[b(2) - r] less than ρ, where ρ > 0 depends upon $(a/2)^2$ the square of downside risk. When E(x) > ρ, the rate of optimal investment is proportional to the <u>positive</u> difference between the expected net return and a function of the downside risk ρ, until we reach the maximum $i(1)_{max}$, defined below.

(5) $i^*(1) = \zeta B(E(x) - \rho) < i(1)_{max}$; $B > 0$, $\rho \sim (a/2)^2 > 0$.

$\zeta = 1$ if $E(x) > \rho$; $\zeta = 0$ if $E(x) \leq \rho$

The C-O debt/GDP carried into period two, denoted as f*(2) = L*(2)/Y(1) is equation (6),the difference between C-O investment i*(1) less the C-O saving h*(1) = S(1)/Y(1).

(6) $f^*(2) = i^*(1) - h^*(1)$.

The maximum debt/GDP = $L(2)_{max}/Y(1) = L(2)_{max}/b(1)K(1) = f(2)_{max}$ is derived from equation (2.1) above. It is equation (6.1). Since investment equals the debt plus saving, we obtain the maximal investment, equation (6.2).

(6.1) $f(2)_{max} = [b^-(2)/b(1) + (1+b^-(2))h(1)]/(a/2)$.

(6.2) $i(1)_{max} = h(1) + f(2)_{max}$

Our conclusion can be described in figure 1, based upon (4)-(6). The expected net return denoted by E(x) = E[b(2) - r] = a(p - 1/2), is plotted on the abscissa and the optimal debt/GDP is plotted on the ordinate. The country should be a net creditor as long as the expected net return on investment E(x) = E[b(2) - r] = a(p - 1/2) is below e > 0. It should be a debtor when E(x) exceeds e > 0. The maximum C-O debt/GDP is $f(2)_{max}$ the debt/GDP that produces a zero consumption when the bad case occurs. Then default/debt rescheduling will occur with probability is (1-p).

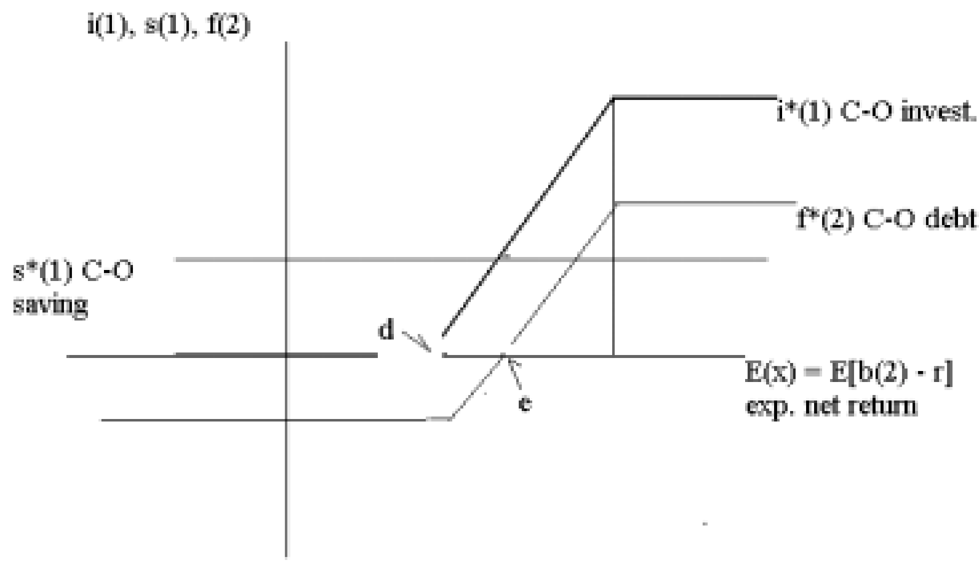

Figure 1. Constrained optimal saving, investment and debt - per Y(1)

3. Mathematical Technique and Solution

We solve our modification of the standard model by taking explicit account of the uncertainty, rather than by using the certainty-equivalence approach in the literature, and by using the "no default" constraint, discussed below. Consumption C(2) in equation (1.1) is a stochastic variable. When the productivity of capital takes on the good value $b^+(2) = r + a/2$, with probability p, then consumption $C^+(2)$ is equation (7.1); and when the productivity of capital takes on the bad value $b^-(2) = r - a/2$, with probability (1-p), consumption is $C^-(2)$ in equation (7.2).

(7.1) $C^+(2) = (1+r)[b(1)K(1) - C(1)] + (a/2)I(1) + (r + a/2)K(1)$

(7.2) $C^-(2) = (1+r)[b(1)K(1) - C(1)] - (a/2)I(1) + (r - a/2)K(1)$

If there is optimal risk management in period t=1, the country would select the controls $C(1) > 0$, $I(1) \geq 0$ to maximize the expectation of the present value of utility of consumption, E[J] in equation (3), which is strictly concave, subject to the no-default constraint **Γ**. Using (7.2), the constraint[7] that $C^-(2)/K(1) = c^-(2) > 0$, and $i(1) = I(1)/K(1) \geq 0$ implies that the values of c(1) and i(1) must lie within a triangle bounded by the coordinate axes and the negatively sloped line $C^-(2) = 0$ graphed in figure 2.

Γ: For $c(1) > 0$ and $i(1) \geq 0$: $(1+r)c(1) + (a/2)i(1) \leq (r - a/2) + (1+ r)b(1)$,

The crucial partial derivatives, to be used in the solution, are equations (8.1) - (8.2) in the good case, and (9.1)-(9.2) in the bad case.

Probability p > 0	Probability (1-p) > 0
(8.1) $dC^+(2)/dC(1) = -s = -(1+r)$;	(9.1) $dC^-(2)/dC(1 = -s = -(1+r)$
(8.2) $dC^+(2)/dI(1) = + a/2$;	(9.2) $dC^-(2)/dI(1) = - a/2$

Since E[J] is strictly concave over **Γ**, the maximum is at a unique C*(1), I*(1), which is either interior to Γ or on the boundary I(1) = 0.When the maximization is interior to Γ, it is found by setting the partial derivatives of E[J] with respect to C(1) and I(1) equal to zero. This gives equations (10) and (11). There will be an interior maximum for C(1), when $\gamma \leq 0$, because a zero consumption implies a utility of minus infinity. The condition for an interior maximum for I(1) is discussed below.

maximization

(10) $C^{\gamma-1}(1) = s\{p[C^+(2)]^{\gamma-1} + (1-p)[C^-(2)]^{\gamma-1}\}$

(11) $\{ p[C^+(2)]^{\gamma-1} (a/2) - (1-p)[C^-(2)]^{\gamma-1} (a/2)\} = 0$

These are two equations in three unknowns: C(1), $C^+(2)$ and $C^-(2)$. Variables $C^+(2)$ and $C^-(2)$ are defined in (7.1) and (7.2). The expression $C^{\gamma-1}(t)$ is the marginal utility of consumption in period t. Define variables A^+ and A^- as the ratio of marginal utilities of consumption in period t = 2, in the good and bad cases respectively, relative to the marginal utility of consumption in period t=1. These are equations (12.1) and (12.2) respectively. The consumption in the second period is a proportion of that in the first period.

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[7] If $c(2) > 0$ in the bad case, then it must be positive in the good case.

(12.1) $[C^{+}(2)]^{\gamma-1} = A^{+} [C(1)]^{\gamma-1}$

(12.2) $[C^{-}(2)]^{\gamma-1} = A^{-} [C(1)]^{\gamma-1}$

We determine the values of A^{+} and A^{-} by substituting (12.1) and (12.2) into the maximizing relations (10) and (11), and obtain (13) and (14). The latter two equations concern the maximization with respect to control variables C(1) > 0 and I(1) > 0, and we solve for the values of A^{+} and A^{-}. Consider the case where I(1) > 0, and equation (14) is an equality. We now have two equations in two unknowns: the ratios of the marginal utility of $C^{+}(2)$, and of $C^{-}(2)$, relative to the marginal utility of C(1).

(13) $pA^{+} + (1-p)A^{-} = 1/s$

(14) $pA^{+} - (1-p)A^{-} = 0$.

The resulting values of the ratios of marginal utility A^{+} and A^{-}, are equations (15.1)-(15.2). These imply the ratios of consumption $C^{+}(2)/C(1)$ and $C^{-}(2)/C(1)$. They depend upon: the probability (1> p > 0, 1-p) in the good and bad cases and s = 1+ r, one plus the interest rate r.

(15.1) $A^{+} = 1/2ps$

(15.2) $A^{-} = 1/2(1-p)s$

Using (15.1) and (15.2) in (12.1) and (12.2), we obtain the values of consumption C(2) in period two relative to the optimal control C(1), equations (16.1) and (16.2).

(16.1) $C^{+}(2) = (2ps)^{1/1-\gamma}\, C(1)$

(16.2) $C^{-}(2) = (2(1-p)s)^{1/1-\gamma}\, C(1)$

Let us summarize what we have so far. Originally we had four unknowns: C(1),I(1),$C^{+}(2)$, $C^{-}(2)$. Equation (14) implies that the expected relative marginal utility $pU'[C^{+}(2)]/U'[C(1)]$ in the good case should equal the expected relative marginal utility $(1-p)U'[C^{-}(2)]/U'[C(1)]$ in the bad case. Equation (13) states that the sum of the expected relative marginal utilities should equal 1/s = 1/(1+r). Equations (16.1) and (16.2), derived from the maximization, give us $C^{+}(2)$ and $C^{-}(2)$as a proportion of C(1). Since we know $C^{+}(2)$ and $C^{-}(2)$as proportions of C(1), we now need only solve for C(1) and I(1). We do this as follows.

Substitute equation (16.1), the relative consumption that results in the good case, in equation (7.1) to obtain equation (17). Similarly substitute equation (16.2), the relative consumption that results in the bad case, in equation (7.2), to obtain equation (18). These two equations permit us to solve for the optimal controls, $c(1) = C(1)/K(1) > 0$ and $i(1) = I(1)/K(1) \geq 0$, as a fraction of the initial capital K(1), when there is uncertainty about the future productivity of capital. These equations, and constraint Γ, are graphed in figure 2. The crucial parameters S_1, S_2, and N are defined in table 1; and the solution for c(1), i(1) and f(2) is in table 2, below.

TABLE 1. Definitions of Crucial Terms: three cases

	CASE [A] $-\infty < \gamma < 1$	CASE [B] $\gamma = 0$	CASE [LD] Large Deviations $(1-p) = e^{-\alpha(1-\gamma)}$, $\alpha > 0$, $\gamma \Rightarrow -\infty$, $(1-p)^{1/1-\gamma} = e^{-\alpha} = B$ 1>B> 0 weight on bad case
S_1 weight on good case	$(2ps)^{1/1-\gamma} + s$	$s(2p + 1)$	$1 + s$
S_2 weight on bad case	$(2(1-p)s)^{1/1-\gamma} + s$	$s(2(1-p)+1)$	$B + s$
$S_1 - S_2$	$(2ps)^{1/1-\gamma} - (2(1-p)s)^{1/1-\gamma}$	$2s(2p - 1)$	$1 - B > 0$
$S_1 + S_2$	$2s + (2s)^{1/1-\gamma} [p^{1/1-\gamma} + (1-p)^{1/1-\gamma}]$	$4s$	$(1+B +2 s)$
S_1/S_2		$(2p + 1)/(2(1-p)+1)$	$(1+s)/(B+s) > 1$
N	$(1+ b(1))s + (a/2) - 1 = sY^* + (a/2)$	$sY^* + (a/2)$	$sY^* + (a/2)$
(N – a)	$(1+ b(1))s - (a/2) - 1 = sY^* - (a/2)$	$sY^* - (a/2)$	$sY^* - (a/2)$

Note: The values of the net return b(2) - r are symmetrical around zero with a range $a > 0$, and probabilities (p, 1-p). Expected net return $x = E[b(2)-r] = a(p - 1/2)$. The interest rate is r and define $s = 1 + r$. "Safe wealth" is defined as $Y^* = [(1+ b(1))s - 1]/s$, and the risk premium in case [B] is defined as $\rho = (a/2)^2/[(1/2)sY^*]$.

The condition for <u>an interior maximum</u> is that the solution to (17), (18) satisfy c(1) > 0, i(1) > 0. This is equivalent to $S_1(N-a) > S_2N$. See table 2 below. When $S_1(N-a) \le S_2N$, the maximum occurs on the boundary i(1) = 0. These two equations are quite different from the equations in the literature, which assume certainty equivalence, because we explicitly consider the nature of the uncertainty.

(17) $S_1\, c(1) – (a/2)\, i(1) = N$

(18) $S_2\, c(1) + (a/2)\, i(1) = N - a$

Table 2

Optimal controls: consumption/capital, c(1) saving/capital h(1), investment/capital i(1), and the debt/capital f(2).

(19) $c^*(1) = (2N – a)/ (S_1 + S2)$

(20) $h^*(1) = b(1) - c(1)$

(21) $i^*(1) = \zeta\, (a/2)[(N-a)S_1 – NS_2]/[(S_1+S_2)] < i(1)_{max}$

(22) $f^*(2) = [C(1) + I(1) – Y(1)]/K(1) = i(1) - h(1) = c(1) + i(1) – b(1)$

See table 1 for definitions of N, S_1, S_2.

Indicator function: $\zeta = 1$ when $[(N-a)S_1 – NS_2] > 0$, and $\zeta =0$ otherwise. Equation (5.3) gives the value of $i(1)_{max}$

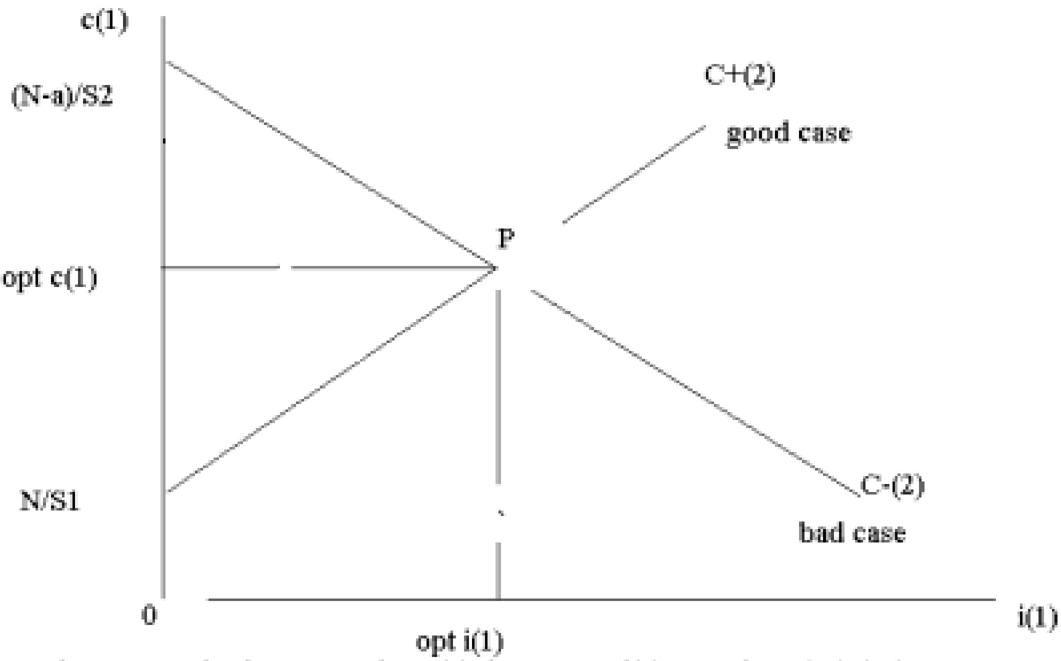

Figure 2. Optimal consumption c(1), investment i(1): equations (17)-(18)

3. The Logarithmic case : $\gamma = 0$.

When $\gamma = 0$, the optimal consumption/capital equation (19) can be written as (23), and optimal saving as equation (24).

(23) $c^*(1) = Y^*/2$

(24) $h^*(1) = Y(1)/K(1) - C(1)/K(1) = b(1) - Y^*/2$

(25) $Y^* = [s(1 + b(1)) - 1]/s, \qquad s = (1+r).$

The Y* term can be expressed in terms of consumption smoothing. Suppose that current capital and its income K(1) + Y(1) = (1 + b(1))K(1) were loaned out at the known short term interest rate r. The assumption that K(3) = K(1) means that I(2) = -I(1). With certainty, the resources available in period t=2 would be [s(1 + b(1))- 1]K(1). Hence the present value of the net resources per unit of capital K(1) that would be available from a riskless position is Y*, defined in equation (24). Since this quantity is known with certainty, refer to Y* as "safe wealth". Therefore, optimal consumption/capital, in the logarithmic $\gamma = 0$ case, is a multiple of "safe wealth". Since the future utility is given the same weight as present utility, optimal consumption is 1/2 of "safe wealth". Equations (23)-(24) involve inter-temporal consumption smoothing. Saving will be positive if the current output/capital b(1) exceeds one-half of "safe wealth" Y*. All of these magnitudes are known with certainty at the time decisions are made.

Equation (21) for investment/capital i(1) = I(1)/K(1) is expressed as equation (26) in the case where $\gamma = 0$. We use the definitions in table 1. The expected net return x = E[b(2)-r] = a(p-1/2) is a crucial variable. Figure 2 indicates that investment will only be positive if (N-a)/S2 > N/S1; that is, the solution lies inside the inner triangle. We use this condition in the indicator function ζ. Term ρ defined in (27) corresponds to a risk premium. It is positively related to the square of the downside risk that b(2) - r = - a/2, and negatively related to safe wealth. The <u>indicator function</u> ζ is: $\zeta = 1$ when $x > \rho$ and $\zeta = 0$ otherwise[8].

(26) $i^*(1) = \zeta\{sY^*/2\ (x - \rho)\} < i(1)_{max}$.

(27) $\rho = (a/2)^2\ /\ (sY^*/2) > 0$.

This equation for optimal investment/capital is i(1) = 0 for $x \leq \rho$, and linear in x until it reaches a maximum $i(1)_{max}$, defined in equation (5.1) above. This is very different from that implied by the certainty equivalence literature.

We now address the question: How should we evaluate the risk involved with holding or issuing short-term debt? The optimal foreign debt per unit of capital f*(2) = L(2)/K(1) incurred during the first period is simply the trade deficit. It is equal to optimal investment i*(1) in equation (26) less optimal saving h*(1) in equation (24), all per unit of capital.

(28) $f^*(2) = i^*(1) - h^*(1) < f(2)_{max}$.

The country should incur short-term debt if the expected net return exceeds quantity 0e > ρ > 0 in figure 1, and should be a short- term lender if the expected net return is less than 0e. Default will occur with probability (1-p) if the debt per unit of capital f(2) exceeds the maximal debt per unit of capital $f^*(2) = L(2)_{max}/K(1)$ in equation (6.1) above.

4. Extreme Prudence: Large Deviations Model

Borrowers with low risk aversion select higher investment and debt ratios than what are selected by those who are more risk averse. A question is whether there are lenders who are willing to lend to borrowers with very low risk aversion $(1-\gamma)$? If the lenders are institutional investors that manage pension funds, they may be infinitely risk-averse: $(1-\gamma) \Rightarrow \infty$. On the basis of sad experience, the lenders may not share the optimism of borrowers that default will not occur[9]. Institutional investors may only be willing to lend for investments where the probability is minimal that the productivity of capital is below the interest rate on short term debt. We develop an alternative approach to optimization, called the Large Deviations [LD] model. The lenders may only be willing to lend at short-term rate r if the borrower optimizes according to the lenders' criteria.

4.1 Optimal investment

There are three main features of the LD model. First: There is infinite risk aversion, the good case is almost a sure thing, and it is most unlikely that the bad event will occur. The probability of the good event $p \sim 1$, and the probability of the bad event $(1-p) = e^{-\alpha(1-\gamma)}$ is almost zero, since risk aversion coefficient $(1-\gamma) \Rightarrow \infty$ and $\alpha > 0$. Second: The values of the net return $[b(2) - r]$ are either $(a/2) > 0$ or $-(a/2) < 0$. Since $p \sim 1$, the expected net return $x = E[b(2) - r] = (a/2) > 0$. Therefore the expected net return is equal to the symmetrical upside and downside risk (a/2, albeit with different probabilities. Denote by x the expected net return $x = E[b(2) - r]$. Third: The crucial parameter $B = (1-p)^{1/1-\gamma} = e^{-\alpha}$. Since $\alpha > 0$, then $1 > B > 0$. We refer to B as the weight placed upon the bad event. It combines infinite risk aversion with minimal risk.

The solution for the optimal value of investment is equation (29). It is derived from equation (21) using the values in column 3 of table 1. A condition that investment is positive is that inequality (30) is satisfied[10].

(29 $i^*(1) = [x(\rho^* - x)]$

(30) $\rho^* > x > 0$, where:

(31) $\rho^* = sY^*(1-B)/(1+2s+B)$, and $x = a/2 = E[b(2)-r]$.

Figure 3 plots investment $i(1) = I(1)/K(1)$ against x which, in the LD case, is both the expected net return $E[b(2)-r]$ and the downside risk a/2. Optimal investment is a

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[8] This is the condition that indicator function: $\zeta = 1$ when $[(N-a)S_1 - NS_2] > 0$, and $\zeta = 0$ otherwise..

[9] See the introductory part of this paper.

parabola, with i(1) = 0 at x = 0 and at x = ρ*. It reaches a maximum at ρ*/2. The logic of the parabola is that: when x = E[b(2)-r] = 0, there is a zero net return so it does not pay to invest in risky assets. As x rises, the expected net return rises and investment is induced. Since x = a/2 is also equal to the downside risk, as x rises above ρ*/2, the risk element dominates and decreases investment. At x = ρ*, the risk has total domination and optimal investment returns to zero.

Two investment functions are plotted in figure 3, for a high and a low value of B. As parameter B the weight on the bad event rises, the value of ρ* declines. The parabola of investment/capital declines, with a smaller range of x for which there is positive investment.

4.2 Optimal Debt.

Optimal debt per unit of initial capital f*(2) is optimal investment less optimal saving. Optimal investment i*(1) is equation (29) the parabola. Optimal saving h*(1) is equation (32), derived equation (20), using table 1 column 3. Saving h*(1), is independent of the expected net return x. It is positively related to the weight B placed upon the bad event and to the current productivity of capital b(1), and is negatively related to safe wealth Y*. As the value of B rises, the saving function graphed in figure 3 rises.

(32) h*(1) = [b(1) - 2sY*/(1+B+2s)]

The optimal debt/capital f*(2) is the vertical distance between the investment and saving curves in figure 3. The maximum debt $f(2)_{max}$ is given in equation (6.1). The crucial variable here is the fraction $B = (1-p)^{1/1-\gamma} = e^{-\alpha}$, the weight on the bad event. This is a quantity that the lender/borrower must select. Weight B affects both saving and investment. As B rises to B', the saving function rises from h(1) to h'(1). The rise in B lowers ρ*, and the investment function declines from i(1) to i'(1) in figure 3. For any

[10] This is the condition $[(N-a)S_1-NS_2] > 0$ for positive investment.

level of x = E[b(2)-r] = a/2, as B rises, the optimal foreign debt declines.

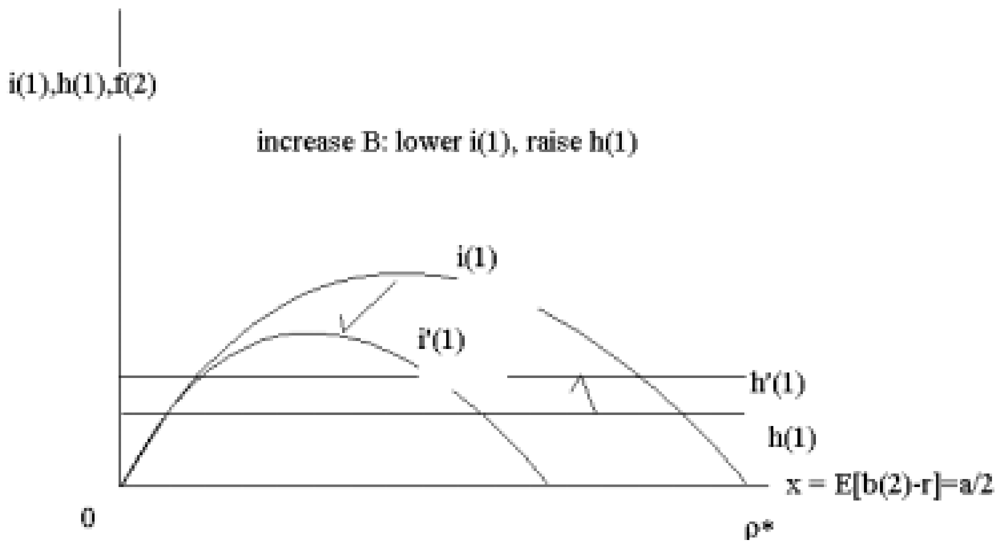

Figure 3. Large deviations. Investment i(1), saving h(1)
debt f(2) = i(1) - h(1)

5. Conclusion

The standard model of inter-temporal optimization is based upon certainty equivalence and ignores risk and uncertainty. We solve a modification of the standard model of inter-temporal optimization in an environment where the return to capital is stochastic, and we impose the <u>constraint</u> that there be no default on the short-term debt. We derive benchmarks for optimal foreign debt in a world of uncertainty. Insofar as the actual debt exceeds the benchmark, the expected utility of consumption is reduced. Default occurs with probability (1-p) when the debt exceeds the maximum debt $f(2)_{max}$. The main reasons for a deviation between the actual debt and the optimal debt are that the borrower is overly optimistic about the distribution function of the return to investment, does not optimize with the "no default" constraint, and/or there is a moral hazard problem.

Stein and Paladino (2000) applied this framework to explain which countries have been forced to reschedule their debts, relative to another set of comparable/control countries that have serviced their debts regularly. In neither was the debt optimal. However, the mean debt/GDP among the defaulters was significantly higher than f_{max}, whereas in the comparable/control countries the mean debt/GDP ratio was approximately f_{max} .

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