Stability in the Stefan problem with surface tension (I)

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Abstract

We develop a high-order energy method to prove asymptotic stability of flat steady surfaces for the Stefan problem with surface tension - also known as the Stefan problem with Gibbs-Thomson correction.

1 Introduction

The Stefan problem is one of the best known parabolic two-phase free boundary problems. It is a simple model of phase transitions in liquid-solid systems.

Let $\Omega \subset \mathbb{R}^n$ denote a domain that contains a liquid and a solid separated by an interface Γ. As the melting or cooling take place the boundary moves and we are naturally led to a free boundary problem. The unknowns are the temperatures of the liquid and the solid denoted respectively by v^+ and v^- and the location of the interface Γ separating the two different phases.

We shall assume that $\Omega = \mathbb{T}^{n-1} \times [-1,1]$ where $\mathbb{T}^{n-1}$ stands for an $(n-1)$-dimensional torus. Let us assume that the moving interface $\Gamma(t)$ is a graph given by $x_n = \rho(t,x')$. Here $\rho : [0,T] \times \mathbb{T}^{n-1} \to \mathbb{R}$ is some smooth function such that $\bigcup_{0 \le t \le T} \Gamma(t) \subset \Omega$ and $T > 0$. Define the liquid/solid phase $\Omega^{\pm}(t)$ by setting $\Omega^{\pm}(t) = \left\{ (x',x_n) \in \Omega \,\middle|\, x_n \gtrless \rho(x',t) \right\}$. We note that $\Omega = \Omega^+(t) \cup \Omega^-(t)$. In order to formulate the problem we first specify the initial conditions. Let $\Gamma_0 = \mathrm{graph}(\rho_0)$ be the initial position of the free boundary and $v_0 : \Omega \to \mathbb{R}$ be the initial temperature. The unknowns are the interface $\Big\{\Gamma(t); t\geq 0\Big\}$ and the temperature function $v:\Omega\times[0,T]\to\mathbb{R}$. We denote the normal velocity of Γ by V and normalize it to be positive if Γ is locally expanding $\Omega^-(t)$. The mean curvature of $\Gamma(t)$ is given by

$$\kappa(t)=\nabla\cdot\big(\frac{\nabla\rho(t)}{\sqrt{1+|\nabla\rho(t)|^2}}\big).$$

The Stefan problem with surface tension is now given by:

$$\partial_t v-\Delta v=0 \qquad \text{in}\,\Omega,\, t>0, \tag{1.1}$$
$$v=\kappa(t) \qquad \text{on}\,\Gamma(t),\, t\geq 0, \tag{1.2}$$
$$\partial_n v=0 \qquad \text{on}\,\mathbb{T}^{n-1}\times\{x_n=\pm 1\} \tag{1.3}$$
$$V=[\partial_\nu v]_-^+ \qquad \text{on}\,\Gamma(t),\, t>0, \tag{1.4}$$
$$v(\cdot,0)=v_0 \qquad \text{in}\,\Omega, \tag{1.5}$$
$$\Gamma(0)=\Gamma_0. \tag{1.6}$$

Given v we write v^+ and v^- for the restriction of v to $\Omega^+(t)$ and $\Omega^-(t)$, respectively. With this notation $[\partial_\nu v]_-^+$ stands for the jump of the normal derivatives across the interface $\Gamma(t)$, namely

$$[\partial_\nu v]_-^+:=\partial_\nu v^+-\partial_\nu v^-,$$

where ν stands for the unit normal on the hypersurface $\Gamma(t)$ with respect to $\Omega^-(t)$. If we replace the boundary condition (1.2) with

$$v=0 \qquad \text{on} \quad \Gamma(t),\;\; t\geq 0, \tag{1.7}$$

then we are referring to the *classical Stefan problem.*

The difficulties in dealing with the existence of solutions of the problem (1.1) - (1.6) arise from the nonlinear coupling between the temperature v and the boundary ρ. This connection is expressed through the boundary conditions (1.2) and (1.4). The equation (1.4) is a Neumann-type boundary condition for v. It is hyperbolic in nature as opposed to the parabolic diffusion process in the regions Ω^+ and Ω^-.

From the technical point of view the first major obstacle for the analysis is the moving boundary. To deal with this issue we shall first transform the problem to the fixed domain by applying the so-called Hanzawa transform.

To this end let us fix a small positive constant $\alpha < \frac{1}{3}$ and choose a cut-off function $\phi \in C^\infty(\mathbb{R})$ with $\mathrm{Im}(\phi) \subset [0,1]$ and

$$\phi(z) = \begin{cases} 1, & |z| \leq \alpha, \\ 0, & |z| > 1-\alpha \end{cases} \qquad ||\phi'||_{L^\infty(\mathbb{R})} < C.$$

Define now a diffeomorphism

$$\Theta(x', x_n, t) = (x', x_n + \phi(x_n)\rho(x', t), t)$$

and the function $u(x', x_n, t) = v(\Theta(x', x_n, t))$. Observe that $u(x', 0, t) = v(x', \rho(x', t), t)$ and at the outer boundaries $\partial\Omega^\pm := \mathbb{T}^{n-1} \times \{x_n = \pm 1\}$, we have $v|_{\partial\Omega^\pm} = u|_{\partial\Omega^\pm}$. This is the version of the transform first introduced by Hanzawa (cf. [11]). In the new coordinates the heat operator $\partial_t - \Delta$ transforms into a more complicated operator whose coefficients depend on the interface function ρ and the cut-off function ϕ. Following the calculations from [4], we find that the Laplace operator Δ in the new coordinates takes the form

$$\Delta_\Theta u = \Delta_{x'} u + a_\rho u_{nn} - B_\rho \cdot \nabla_{x'} u_n - d_\rho u_n,$$

where

$$a_\rho := \frac{1 + |\phi\nabla\rho|^2}{(1+\phi'\rho)^2}, \quad B_\rho := \frac{2\phi}{1+\phi'\rho}\nabla\rho, \tag{1.8}$$

$$d_\rho := \frac{\phi\Delta\rho}{1+\phi'\rho} - \frac{(\phi^2)'|\nabla\rho|^2}{(1+\phi'\rho)^2} + \frac{\phi''\rho(1+|\phi\nabla\rho|^2)}{(1+\phi'\rho)^3}. \tag{1.9}$$

Furthermore, the operator ∂_t in the new coordinates reads

$$(\partial_t)_\Theta u = \partial_t u + e_\rho u_n$$

where

$$e_\rho := -\frac{\phi\rho_t}{1+\phi'\rho}. \tag{1.10}$$

Note that RHS of (1.2) remains unchanged in the new coordinates. In order to transform the boundary condition (1.4) into the new coordinates we first observe that

$$(\partial_i)_\Theta u = \partial_i u - \frac{\phi\partial_i\rho}{1+\phi'\rho}\partial_n u, \ 1 \leq i \leq n-1, \qquad (\partial_n)_\Theta u = \frac{1}{1+\phi'\rho}\partial_n u.$$

We thus conclude that at the boundary $\mathbb{T}^{n-1}\times\{x_n=0\}$:

$$\nabla v|_\Gamma = \nabla_\Theta u|_{x_n=0} = (\nabla_{x'}u,0) - \frac{\partial_n}{1+\phi'\rho}(\phi\nabla\rho,1) = (\nabla_{x'}u,0) - \partial_n u(\nabla\rho,1).$$

The outward unit normal is given by $\nu(x',t) = \frac{(-\nabla\rho(x',t),1)}{\sqrt{1+|\nabla\rho|^2}}$. Thus the normal velocity V takes the form $V = \frac{-\partial_t\rho}{\sqrt{1+|\nabla\rho|^2}}$. Using the above expressions we derive the formula for $[\partial_n v]_-^+|_\Gamma$. Namely on $\mathbb{T}^{n-1}\times\{x_n=0\}$ we have

$$[\partial_n v]_-^+|_\Gamma = [(\nabla_{x'}u,0) - \partial_n u(\nabla\rho,1)]_-^+ \cdot \frac{(-\nabla\rho(x',t),1)}{\sqrt{1+|\nabla\rho|^2}} = [\partial_n u]_-^+\sqrt{1+|\nabla\rho|^2}.$$

It is thus easy to see that the equation (1.4) transforms into

$$\partial_t\rho = (1+|\nabla\rho|^2)[\partial_n u]_+^-.$$

For the sake of notational simplicity we also set

$$\langle\rho\rangle := \sqrt{1+|\nabla\rho|^2}.$$

The Stefan problem (1.1) - (1.6) now takes the following form:

$$u_t - \Delta_{x'}u - a_\rho u_{nn} + B_\rho\cdot\nabla_{x'}u_n + c_\rho u_n = 0 \tag{1.11}$$
$$u = \kappa \qquad \text{on } \mathbb{T}^{n-1}\times\{x_n=0\} \tag{1.12}$$
$$\partial_n u = 0 \qquad \text{on } \mathbb{T}^{n-1}\times\{x_n=\pm 1\} \tag{1.13}$$
$$u(x,0) = u_0(x) \qquad x\in\Omega \tag{1.14}$$
$$\rho(x',0) = \rho_0(x') \qquad x'\in\mathbb{T}^{n-1} \tag{1.15}$$
$$\rho_t = \langle\rho\rangle^2[u_n]_+^- \qquad \text{on } \mathbb{T}^{n-1}\times\{x_n=0\}, \tag{1.16}$$

where we set $c_\rho := d_\rho + e_\rho$ with d_ρ and e_ρ given by (1.9) and (1.10) respectively. Recall that a_ρ and B_ρ are given by (1.8). In order to deal with the hyperbolic equation (1.16) we introduce the regularization of the jump relation (1.16):

$$\rho_t + \epsilon\Delta^2\rho_t = \langle\rho\rangle^2[u_n]_+^- \qquad \text{on } \mathbb{T}^{n-1}\times\{x_n=0\}. \tag{1.17}$$

We shall refer to the problem of finding the solution to (1.11)-(1.15) and (1.17) as to the regularized Stefan problem.

Notation. For notational simplicity we define for any multi-index $\mu = (\mu_1,\dots,\mu_{n-1})$ and $s\in\mathbb{N}$

$$\partial_s^\mu = \partial_t^s\partial_{x_1}^{\mu_1}\dots\partial_{x_{n-1}}^{\mu_{n-1}}. \tag{1.18}$$

Note that the operator ∂_s^μ acts only in directions tangential to the boundary $\mathbb{T}^{n-1}\times\{x_n=0\}$. The Latin letters are always used to refer to the differentiation with respect to the time variable t and Greek letters to refer to the differentiation with respect to the first $n-1$ spatial variables $x_1,\dots,x_{n-1}$. If each component of μ' is not greater then that of μ and $s'\le s$, we write $(\mu',s')\le(\mu,s)$. We write $(\mu',s')<(\mu,s)$ if $(\mu',s')\le(\mu,s)$ and $|\mu'|<|\mu|$ or $s'<s$. We also denote $C_{s'}^{\mu'}=\binom{(\mu,s)}{(\mu',s')}$. For given functions $\omega:\mathbb{T}^{n-1}\to\mathbb{R}$ and $\mathcal{U}:\Omega\to\mathbb{R}$, we denote $\omega_i=\partial_{x_i}\omega$, $i=1,\dots,n-1$ and $\mathcal{U}_n=\partial_{x_n}\mathcal{U}$. With $x=(x_1,\dots,x_n)$ and $x'=(x_1,\dots,x_{n-1})$ we set

$$\nabla_{x'}\mathcal{U}=(\partial_{x_1}\mathcal{U},\dots,\partial_{x_{n-1}}\mathcal{U}),\quad |\nabla_{x'}^2\mathcal{U}|^2=\sum_{i,j=1}^{n-1}(\partial_{x_ix_j}\mathcal{U})^2,\quad \Delta_{x'}\mathcal{U}=\sum_{i=1}^{n-1}\partial_{x_ix_i}\mathcal{U}.$$

The Einstein summation convention is used throughout the paper when dealing with repeated indices. The letter C will stand for a generic constant that may change from line to line.

We define the following high-order energy norms:

$$\begin{aligned}\mathcal{E}(\mathcal{U},\omega;\psi)(t):=&\sum_{|\mu|+2s\le 2k}\int_\Omega\Big\{(\partial_s^\mu\mathcal{U}(t))^2+|\partial_s^\mu\nabla_{x'}\mathcal{U}(t)|^2+a_{\psi(t)}(\partial_s^\mu\mathcal{U}_n(t))^2\Big\}+\\&+\sum_{|\mu|+2s\le 2k}\int_{\mathbb{T}^{n-1}}\Big\{|\partial_s^\mu\nabla\omega(t)|^2\langle\psi(t)\rangle^{-1}+I_{\psi(t)}(\nabla^2\partial_s^\mu\omega,\nabla^2\partial_s^\mu\omega)(t)\Big\},\end{aligned}\tag{1.19}$$

$$\begin{aligned}\mathcal{D}(\mathcal{U},\omega;\psi)(t):=&\sum_{|\mu|+2s\le 2k}\int_\Omega\Big\{(\partial_s^\mu\mathcal{U}_t(t))^2+|\partial_s^\mu\nabla_{x'}\mathcal{U}(t)|^2+a_{\psi(t)}(\partial_s^\mu\mathcal{U}_n(t))^2+\\&|\nabla_{x'}^2\partial_s^\mu\mathcal{U}(t)|^2+2a_{\psi(t)}|\partial_s^\mu\nabla_{x'}\mathcal{U}_n(t)|^2+\big(a_{\psi(t)}\partial_s^\mu\mathcal{U}_{nn}(t)\big)^2\Big\}\\&+2\sum_{|\mu|+2s\le 2k}\int_{\mathbb{T}^{n-1}}|\partial_s^\mu\nabla\omega_t(t)|^2\langle\psi(t)\rangle^{-1},\end{aligned}\tag{1.20}$$

where for given functions $\omega,\psi:\mathbb{T}^{n-1}\to\mathbb{R}$, we define

$$I_\psi(\nabla^2\omega,\nabla^2\omega):=|\nabla^2\omega|^2\langle\psi\rangle^{-1}-\sum_{k=1}^{n-1}\big(\nabla\omega_k\cdot\nabla\psi\big)^2\langle\psi\rangle^{-3}.\tag{1.21}$$

Recall that $a_{\psi(t)}$ is given by (1.8). It is crucial to observe that I_ψ is a positive definite bilinear form. Namely,

$$I_\psi(\nabla^2\omega,\nabla^2\omega) = \int_{\mathbb{T}^{n-1}} \Big\{ |\nabla^2\omega|^2 \langle\psi\rangle^{-3} + \Big(|\nabla^2\omega|^2 |\nabla\psi|^2 - \sum_{k=1}^{n-1} \big(\nabla\omega_k\cdot\nabla\psi\big)^2 \Big) \langle\psi\rangle^{-3} \Big\}$$
$$\geq \int_{\mathbb{T}^{n-1}} |\nabla^2\omega|^2 \langle\psi\rangle^{-3}. \tag{1.22}$$

Note that we have used the Cauchy-Schwarz inequality in the last estimate. The *instant energy* $\mathcal{E}$ and the *dissipation* $\mathcal{D}$ are respectively given by

$$\mathcal{E} \equiv \mathcal{E}(u,\rho) := \mathcal{E}(u,\rho;\rho), \tag{1.23}$$

$$\mathcal{D} \equiv \mathcal{D}(u,\rho) := \mathcal{D}(u,\rho;\rho). \tag{1.24}$$

In the rest of the paper we shall always assume $k \geq n$, where n is the dimension of the space the domain Ω belongs to. Observe that the stationary solutions to the Stefan problem (1.11) - (1.16) are given by $(u,\rho) \equiv (0,\bar{\rho})$, where $\bar{\rho} \in \mathbb{R}$ is a given constant. Note that $\mathcal{E}(u,\rho-\bar{\rho}) = \mathcal{E}(u,\rho)$. The main result of the paper is the following theorem.

Theorem 1.1 *There exists a sufficiently small constant $M^* > 0$ such that if the initial data satisfy (u_0,ρ_0) satisfy*

$$\mathcal{E}(u_0,\rho_0) + \Big| \int_{\mathbb{T}^{n-1}} \rho_0 - \int_\Omega u_0(1+\phi'\rho_0) \Big| \leq M^*,$$

then there exists a unique global solution to the Stefan problem (1.11) - (1.16) satisfying the global bound

$$\mathcal{E}(u,\rho)(t) + \frac{1}{2}\int_s^t \mathcal{D}(u,\rho)(\tau)\,d\tau \leq \mathcal{E}(u,\rho)(s), \quad t \geq s \geq 0. \tag{1.25}$$

Moreover, given the stationary solution $(u,\rho) \equiv (0,\bar{\rho})$, such that

$$\int_{\mathbb{T}^{n-1}} \bar{\rho} = \int_{\mathbb{T}^{n-1}} \rho_0 - \int_\Omega u_0(1+\phi'\rho_0),$$

then for such small initial datum there exist constants $K_1, K_2 > 0$ such that

$$\mathcal{E}(u,\rho)(t) + ||\rho(t)-\bar{\rho}||_2^2 \leq K_1 e^{-K_2 t} \quad \textit{for all} \quad t \geq 0.$$

The proof of this theorem will strongly rely on careful examination of the regularized Stefan problem (1.11) - (1.15) and (1.17). For this purpose we introduce the appropriate energy norms incorporating the additional viscosity coefficient ϵ.

$$\begin{aligned}\mathcal{E}_\epsilon(\mathcal{U},\omega;\psi) := \mathcal{E}(\mathcal{U},\omega;\psi) + \sum_{|\mu|+2s\le 2k} \epsilon \int_{\mathbb{T}^{n-1}} \Big\{ |\partial_s^\mu \Delta\nabla\omega|^2 \langle\psi\rangle^{-1} \\ + I_\psi(\nabla^2\partial_s^\mu\Delta\omega, \nabla^2\partial_s^\mu\Delta\omega) \Big\}\end{aligned} \tag{1.26}$$

$$\mathcal{D}_\epsilon(\mathcal{U},\omega;\psi) := \mathcal{D}(\mathcal{U},\omega;\psi) + 2\epsilon \sum_{|\mu|+2s\le 2k} \int_{\mathbb{T}^{n-1}} |\nabla^2\partial_s^\mu\Delta\omega|^2 \langle\psi\rangle^{-1}. \tag{1.27}$$

The above norms are the weighted versions of parabolic Sobolev norms given by

$$||(\mathcal{U},\omega)||_{\mathcal{E}_\epsilon} := \sum_{|\mu|+2s\le 2k} \Big\{ ||\partial_s^\mu \mathcal{U}||^2_{H^1(\Omega)} + ||\partial_s^\mu \nabla\omega||^2_{H^1} + \epsilon ||\partial_s^\mu \Delta\nabla\omega||^2_{H^1} \Big\} \tag{1.28}$$

and

$$||(\mathcal{U},\omega)||_{\mathcal{D}_\epsilon} := \sum_{|\mu|+2s\le 2k} \Big\{ ||\partial_s^\mu \mathcal{U}_t||^2_{L^2(\Omega)} + ||\partial_s^\mu \nabla\mathcal{U}||^2_{H^1(\Omega)} + ||\partial_s^\mu \nabla\omega_t||^2_2 + \epsilon ||\partial_s^\mu \Delta\nabla\omega_t||^2_2 \Big\}. \tag{1.29}$$

Given $||\psi||_\infty$ small enough so that $(1+\phi'\psi)^2 \ge \delta > 0$ and $||\nabla\psi||_\infty$ bounded, we conclude that there exists $C>0$ so that

$$\frac{1}{C}||(\mathcal{U},\omega)||_{\mathcal{E}_\epsilon} \le \mathcal{E}_\epsilon \le C||(\mathcal{U},\omega)||_{\mathcal{E}_\epsilon}, \quad \frac{1}{C}||(\mathcal{U},\omega)||_{\mathcal{D}_\epsilon} \le \mathcal{D}_\epsilon \le C||(\mathcal{U},\omega)||_{\mathcal{D}_\epsilon}.$$

In this sense the above norms are equivalent and this observation will be often implicitly used throughout the paper. The major part of the analysis will be concerned with proving the following result, which states that the regularized Stefan problem has unique global solutions with small initial data - independent of ϵ.

Theorem 1.2 *There exists a sufficiently small constant $M>0$ independent of ϵ, such that the following statement holds: if for given initial data $(u_0^\epsilon, \rho_0^\epsilon)$ the inequality*

$$\mathcal{E}_\epsilon(u_0^\epsilon, \rho_0^\epsilon) + \Big| \int_{\mathbb{T}^{n-1}} \rho_0^\epsilon - \int_\Omega u_0^\epsilon (1+\phi'\rho_0^\epsilon) \Big| \le M$$

holds, then there exists a unique global solution $(u^\epsilon, \rho^\epsilon)$ *to the regularized Stefan problem (1.11)- (1.15) and (1.17). Moreover,*

$$\mathcal{E}_\epsilon(u,\rho)(t) + \frac{1}{2}\int_0^t \mathcal{D}_\epsilon(u,\rho)(\tau)\,d\tau \leq \mathcal{E}_\epsilon(u_0,\rho_0), \quad t \geq 0. \tag{1.30}$$

The Stefan problem has been studied in a variety of mathematical literature over the past century (see for instance [20]). It has been known that classical Stefan problem admits unique global classical solutions in $\mathbb{R}^1$ ([6], [7] and [12]). Local classical solutions are established in [11] and [16].

If the diffusion equation (1.1) is replaced by the elliptic equation $\Delta v = 0$, then the resulting problem is called the Hele-Shaw problem (or the quasi-stationary Stefan problem) with surface tension. Global solutions for the Hele-Shaw problem in two dimensions with small initial data have been established in [3]. In [1], stability of the solutions close to the steady state sphere is established. Global stability for the one-phase Hele-Shaw problem is established in [9]. Local-in-time solutions in parabolic Hölder spaces in arbitrary dimensions are established in [2].

As to the Stefan problem with surface tension, global weak existence theory (without uniqueness) is analyzed in [14] and [18]. In [8] the authors consider the Stefan problem with small surface tension i.e. $\sigma \ll 1$ if (1.2) is substituted by $v = \sigma\kappa(t)$. The local existence result for the Stefan problem is studied in [17]. In [4] the authors prove a local existence and uniqueness result in suitable Besov spaces, relying on the L^p-regularity theory.

We establish a *global-in-time* existence, uniqueness and exponential decay of classical solutions to the Stefan problem with surface tension near a flat steady state (Theorem 1.1). The major difficulty consists of proving Theorem 1.2 which establishes the existence and uniqueness result for the regularized Stefan problem with the energy estimate

$$\mathcal{E}_\epsilon(t) + \int_0^t \mathcal{D}_\epsilon(\tau)\,d\tau \leq \mathcal{E}_\epsilon(0) + C\int_0^t \sqrt{\mathcal{E}_\epsilon(\tau)}\mathcal{D}_\epsilon(\tau)\,d\tau, \tag{1.31}$$

where C does not depend on ϵ. Combined with the smallness assumption on the instant energy $\mathcal{E}_\epsilon$ the estimate (1.31) gives (1.30) and the global-in-time existence. For a fixed ϵ we first construct local-in-time solution for the regularized Stefan problem (1.11)- (1.15) and (1.17). The crux of our method is the use of high order energy estimates, for the differential operator ∂_s^μ acts only in tangential directions with respect to the boundary $\mathbb{T}^{n-1} \times \{x_n = 0\}$.

This is very convenient when deriving the energy identities because the Neumann boundary operator commutes with ∂_s^μ. The diffusion equation (1.11) is then used to control high order derivatives of u with respect to the normal direction x_n, as it is presented in Lemmas 3.5 and 3.6. We set up an iteration scheme, which generates a sequence of iterates $\{(u^m,\rho^m)\}_{m\in\mathbb{N}}$. Such iteration is well defined, but it breaks the natural energy setting due to lack of exact cancellations in the presence of the cross-terms. With fixed ϵ, we crucially use the regularization to prove that $\{(u^m,\rho^m)\}_{m\in\mathbb{N}}$ is a Cauchy sequence in the energy space. As $m\to\infty$ the unpleasant cross-terms disappear and we recover (1.31) in the limit. We conclude the proof of Theorem 1.1 by letting $\epsilon\to 0$.

This work is the first step in our program of developing a robust energy method to investigate and characterize morphological stabilities/instabilities arising in numerous free boundary problems in applied PDE. In particular, in a forth-coming paper we are going to establish stability and instability(!) of steady spheres in the Stefan problem with surface tension.

The article is organized as follows: In Chapter 2 we derive general energy identities for a model Stefan problem. In Chapter 3 the iteration scheme for proving the local existence is set up and the actual energy identities are derived, based on Chapter 2. Furthermore, some basic estimates are established, which are then used in Chapter 4 to prove the crucial energy estimates. Chapter 5 is entirely devoted to the proof of the local-in-time existence nd uniqueness. The main results, Theorems 1.1 and 1.2 are proved in Chapter 6.

2 Energy identities

Let $I=[0,q]$ for some $0<q\le\infty$. The derivation of the energy identities crucially depends on the following model problem:

$$\mathcal{U}_t-\Delta_{x'}\mathcal{U}-a_\psi\mathcal{U}_{nn}=f \quad \text{on } \Omega\times I \tag{2.32}$$
$$\mathcal{U}=\Delta\chi\langle\psi\rangle^{-1}-\psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}+G \quad \text{on } \mathbb{T}^{n-1}\times\{x_n=0\}\times I \tag{2.33}$$
$$\partial_n\mathcal{U}=0 \quad \text{on } \mathbb{T}^{n-1}\times\{x_n=\pm1\}\times I \tag{2.34}$$
$$[\mathcal{U}_n]_+^-=\left(\omega_t+\epsilon\Delta^2\omega\right)\langle\psi\rangle^{-2}+h \quad \text{on } \mathbb{T}^{n-1}\times\{x_n=0\}\times I \tag{2.35}$$

We shall denote
$$g:=-\psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}+G.$$

For most of the identities we shall derive, only the leading term $\Delta\chi\langle\psi\rangle^{-1}$ in (2.33) will be relevant. We can thus write the equation (2.33) in the alternative form

$$\mathcal{U} = \Delta\chi\langle\psi\rangle^{-1} + g \quad \text{on } \mathbb{T}^{n-1}\times\{x_n=0\}\times I. \tag{2.36}$$

We define the energies $\bar{\mathcal{E}}_\epsilon$ and $\bar{\mathcal{D}}_\epsilon$ (for the model problem) by setting $k=0$ in the definitions (1.26) and (1.27) of $\mathcal{E}_\epsilon$ and $\mathcal{D}_\epsilon$, respectively.

Lemma 2.1 *Let each of the functions $\mathcal{U}$, ω, χ and ψ be five times continuously differentiable with respect to the space variable and each of its spatial partial derivatives of order ≤ 5 once continuously differentiable with respect to the time variable. The following identity holds:*

$$\frac{d}{dt}\bar{\mathcal{E}}_\epsilon(\mathcal{U},\omega;\psi) + \bar{\mathcal{D}}_\epsilon(\mathcal{U},\omega;\psi) = \int_\Omega \{P+R\} - \int_{\mathbb{T}^{n-1}}\{Q+S+T\}, \tag{2.37}$$

where

$$\begin{aligned}
&P \equiv P(\mathcal{U},\psi,f) := f\mathcal{U} - (a_\psi)_n\mathcal{U}_n\mathcal{U}, \\
&R \equiv R(\mathcal{U},\psi,f) := f^2 - 2f\left(B_\psi\cdot\nabla_{x'}\mathcal{U}_n\right) + \mathcal{U}_n^2(a_\psi)_t - 2\mathcal{U}_t\mathcal{U}_n(a_\psi)_t \\
&-2\nabla_{x'}\mathcal{U}_n\cdot\nabla_{x'}(a_\psi)\mathcal{U}_n + 2\Delta_{x'}\mathcal{U}\mathcal{U}_n(a_\psi)_n,
\end{aligned} \tag{2.38}$$

$$\begin{aligned}
&Q \equiv Q(\chi,\omega,\psi,g,h) = \nabla\omega_t\cdot\left(\nabla\chi - \nabla\omega\right)\langle\psi\rangle^{-1} + \epsilon\Delta\nabla\omega_t\cdot\left(\Delta\nabla\chi - \Delta\nabla\omega\right)\langle\psi\rangle^{-1} \\
&-\frac{1}{2}\left\{|\nabla\omega|^2\langle\psi\rangle_t^{-1} + \epsilon|\Delta\nabla\omega|^2\langle\psi\rangle_t^{-1}\right\} + \omega_t\nabla\chi\cdot\nabla(\langle\psi\rangle^{-1}) \\
&+\epsilon\Delta\nabla\omega_t\cdot\nabla(\langle\psi\rangle^{-1})\Delta\chi - (\omega_t+\epsilon\Delta^2\omega_t)g - \langle\psi\rangle^2 h\mathcal{U}|_{\mathbb{T}^{n-1}},
\end{aligned} \tag{2.39}$$

$$\begin{aligned}
&S \equiv S(\chi,\omega,\psi,g,h) := 2\nabla\omega_t\cdot\left(\nabla\chi_t - \nabla\omega_t\right)\langle\psi\rangle^{-1} + 2\epsilon\Delta\nabla\omega_t\cdot\left(\Delta\nabla\chi_t - \Delta\nabla\omega_t\right)\langle\psi\rangle^{-1} \\
&+2\nabla\chi_t\cdot\nabla(\langle\psi\rangle^{-1})\omega_t + 2\epsilon\Delta\nabla\omega_t\cdot\nabla(\langle\psi\rangle^{-1})\Delta\chi_t - 2\left(\omega_t+\epsilon\Delta^2\omega_t\right)\left(\Delta\chi\langle\psi\rangle_t^{-1} + g_t\right) \\
&-2h\langle\psi\rangle^2\mathcal{U}_t|_{\mathbb{T}^{n-1}},
\end{aligned} \tag{2.40}$$

$$\begin{aligned}
&T \equiv T(\chi,\omega,\psi,G,h) = A(\chi,\omega,\psi,g,h) + B(\chi,\omega,\psi,g,h) + 2\Delta G\left(\omega_t+\epsilon\Delta^2\omega_t\right) \\
&\quad +2\Delta\mathcal{U}|_{\mathbb{T}^{n-1}}h\langle\psi\rangle^2.
\end{aligned} \tag{2.41}$$

$$\tag{2.42}$$

Here, the functions A and B are given by:

$$\begin{aligned}
&A := 2\Delta\omega_t\left(\Delta\chi - \Delta\omega\right)\langle\psi\rangle^{-1} - 2\Delta\omega_t\psi_i\psi_j(\chi_{ij}-\omega_{ij})\langle\psi\rangle^{-3} - |\nabla^2\omega|^2\langle\psi\rangle_t^{-1} \\
&+2\omega_{it}\nabla\omega_i\cdot\nabla(\langle\psi\rangle^{-1}) - 2\nabla\omega_t\cdot\nabla(\langle\psi\rangle^{-1})\Delta\omega + \omega_{jk}\omega_{ik}\psi_j\psi_{it}\langle\psi\rangle^{-3} + \left(\nabla\omega_k\cdot\nabla\psi\right)^2\langle\psi\rangle_t^{-3} \\
&-2\omega_{jk}\left\{\left[\psi_i\psi_j\omega_{kt}\langle\psi\rangle^{-3}\right]_i - \psi_i\psi_j\omega_{ikt}\langle\psi\rangle^{-3}\right\} + 2\omega_{kt}\left\{\left[\psi_i\psi_j\omega_{ij}\langle\psi\rangle^{-3}\right]_k - \psi_i\psi_j\omega_{ijk}\langle\psi\rangle^{-3}\right\}
\end{aligned} \tag{2.43}$$

and

$$
\begin{aligned}
&B := 2\epsilon\Delta^2\omega_t\big(\Delta^2\chi-\Delta^2\omega\big)\langle\psi\rangle^{-1}-2\epsilon\Delta^2\omega_t\psi_i\psi_j(\Delta\chi_{ij}-\Delta\omega_{ij})\langle\psi\rangle^{-3}\\
&-\epsilon|\nabla^2\Delta\omega|^2\langle\psi\rangle_t^{-1}+2\epsilon\Delta\omega_{it}\nabla\Delta\omega_i\cdot\nabla(\langle\psi\rangle^{-1})\\
&-2\epsilon\Delta\nabla\omega_t\cdot\nabla(\langle\psi\rangle^{-1})\Delta^2\omega+\epsilon\Delta\omega_{jk}\Delta\omega_{ik}\psi_j\psi_{it}\langle\psi\rangle^{-3}+\epsilon\big(\Delta\nabla\omega_k\cdot\nabla\psi\big)^2\langle\psi\rangle_t^{-3}\\
&-2\epsilon\Delta\omega_{jk}\Big\{\big[\psi_i\psi_j\Delta\omega_{kt}\langle\psi\rangle^{-3}\big]_i-\psi_i\psi_j\Delta\omega_{ikt}\langle\psi\rangle^{-3}\Big\}\\
&+2\epsilon\Delta\omega_{kt}\Big\{\big[\psi_i\psi_j\Delta\omega_{ij}\langle\psi\rangle^{-3}\big]_k-\psi_i\psi_j\Delta\omega_{ijk}\langle\psi\rangle^{-3}\Big\}\\
&+2\epsilon\Big\{\Delta\big(\Delta\chi\langle\psi\rangle^{-1}-\psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}\big)-\big(\Delta^2\chi\langle\psi\rangle^{-1}-\psi_i\psi_j\Delta\chi_{ij}\langle\psi\rangle^{-3}\big)\Big\}\Delta^2\omega_t.
\end{aligned}
\tag{2.44}
$$

Proof. We start by multiplying the equation (2.32) by $\mathcal{U}$ and integrating over Ω. By a direct computation,

$$
\frac{1}{2}\partial_t\int_\Omega\mathcal{U}^2+\int_\Omega\Big\{|\nabla_{x'}\mathcal{U}|^2+a_\psi(\mathcal{U}_n)^2\Big\}-\int_{\mathbb{T}^{n-1}}\langle\psi\rangle^2[\mathcal{U}_n]_+^-\mathcal{U}=\int_\Omega P(\mathcal{U},\psi,f),
\tag{2.45}
$$

where $P(\mathcal{U},\psi,f)$ is given by (2.38). Using the boundary conditions (2.36) and (2.35), we obtain

$$
\begin{aligned}
-\langle\psi\rangle^2[\mathcal{U}_n]_+^-\mathcal{U}&=-\big\{\omega_t\Delta\chi\langle\psi\rangle^{-1}+\epsilon\Delta^2\omega_t\Delta\chi\langle\psi\rangle^{-1}\big\}\\
&-\langle\psi\rangle^2h\big[\Delta\chi\langle\psi\rangle^{-1}+g\big]-(\omega_t+\epsilon\Delta^2\omega_t)g.
\end{aligned}
\tag{2.46}
$$

Integrating by parts in the first term on the RHS of (2.46) above we arrive at

$$
\begin{aligned}
&-\textstyle\int_{\mathbb{T}^{n-1}}\Big\{\omega_t\Delta\chi\langle\psi\rangle^{-1}+\epsilon\Delta^2\omega_t\Delta\chi\langle\psi\rangle^{-1}\Big\}=\\
&\textstyle\int_{\mathbb{T}^{n-1}}\nabla(\omega_t\langle\psi\rangle^{-1})\cdot\nabla\chi+\epsilon\Delta\nabla\omega_t\cdot\nabla(\Delta\chi\langle\psi\rangle^{-1}).
\end{aligned}
\tag{2.47}
$$

By the product rule, the integrand on the RHS of (2.47) can be written as

$$
\nabla\omega_t\langle\psi\rangle^{-1}\cdot\nabla\chi+\epsilon\Delta\nabla\omega_t\cdot\Delta\nabla\chi\langle\psi\rangle^{-1}+\omega_t\nabla(\langle\psi\rangle^{-1})\cdot\nabla\chi+\epsilon\Delta\omega_t\nabla(\langle\psi\rangle^{-1})\cdot\Delta\nabla\chi.
$$

In each of the terms $\nabla\omega\langle\psi\rangle^{-1}\cdot\nabla\chi$ and $\Delta\nabla\omega_t\cdot\Delta\nabla\chi\langle\psi\rangle^{-1}$ we set $\chi=\omega+(\chi-\omega)$ to obtain

$$
\begin{aligned}
&\frac{1}{2}\partial_t\Big\{|\nabla\omega|^2\langle\psi\rangle^{-1}+\epsilon|\Delta\nabla\omega|^2\langle\psi\rangle^{-1}\Big\}+\nabla\omega_t\cdot\big(\nabla\chi-\nabla\omega\big)\langle\psi\rangle^{-1}+\\
&+\epsilon\Delta\nabla\omega_t\cdot\big(\Delta\nabla\chi-\Delta\nabla\omega\big)\langle\psi\rangle^{-1}-\frac{1}{2}\Big\{|\nabla\omega|^2\langle\psi\rangle_t^{-1}+\epsilon|\Delta\nabla\omega|^2\langle\psi\rangle_t^{-1}\Big\}+\\
&+\omega_t\nabla\chi\cdot\nabla(\langle\psi\rangle)^{-1}+\epsilon\Delta\nabla\omega_t\cdot\nabla(\langle\psi\rangle^{-1})\Delta\chi
\end{aligned}
\tag{2.48}
$$

Thus plugging (2.48) into (2.47) yields

$$-\int_{\mathbb{T}^{n-1}}\langle\psi\rangle^2[\mathcal{U}_n]_+^-\mathcal{U}=\frac{1}{2}\partial_t\int_{\mathbb{T}^{n-1}}\Big\{|\nabla\omega|^2\langle\psi\rangle^{-1}+\epsilon|\Delta\nabla\omega|^2\langle\psi\rangle^{-1}\Big\}+\int_{\mathbb{T}^{n-1}}Q(\chi,\omega,\psi,g,h), \tag{2.49}$$

where $Q(\chi,\omega,\psi,g,h)$ is given by (2.40). To complete the derivation of (2.37), we take the square of the equation (2.32) and integrate over Ω:

$$\begin{aligned}&\partial_t\Big\{\int_\Omega|\nabla_{x'}\mathcal{U}|^2+a_\psi\mathcal{U}_n^2\Big\}+\int_\Omega\Big\{\mathcal{U}_t^2+|\nabla_{x'}^2\mathcal{U}|^2+2a_\psi|\nabla_{x'}\mathcal{U}_n|^2+a_\psi^2\mathcal{U}_{nn}^2\Big\}\\&-2\int_{\mathbb{T}^{n-1}}\langle\psi\rangle^2[\mathcal{U}_n]_+^-\mathcal{U}_t+2\int_{\mathbb{T}^{n-1}}\langle\psi\rangle^2[\mathcal{U}_n]_+^-\Delta_{x'}\mathcal{U}=\int_\Omega R(\mathcal{U},\psi,f),\end{aligned} \tag{2.50}$$

where $R(\mathcal{U},\psi,f)$ is given by (2.39). The goal is to evaluate the two integrals over $\mathbb{T}^{n-1}$ on LHS of (2.50) using the boundary conditions (2.36) and (2.35). We first treat the integral $\int_{\mathbb{T}^{n-1}}\langle\psi\rangle^2[\mathcal{U}_n]_+^-\mathcal{U}_t$. Integrating by parts in the leading order term, we obtain

$$\begin{aligned}-2\int_{\mathbb{T}^{n-1}}\langle\psi\rangle^2[\mathcal{U}_n]_+^-\mathcal{U}_t=&\,2\int_{\mathbb{T}^{n-1}}\Big\{|\nabla\omega_t|^2\langle\psi\rangle^{-1}+\epsilon|\Delta\nabla\omega_t|^2\langle\psi\rangle^{-1}\Big\}\\&+\int_{\mathbb{T}^{n-1}}S(\chi,\omega,\psi,g,h),\end{aligned} \tag{2.51}$$

where $S(\chi,\omega,\psi,g,h)$ is given by (2.41). Note that the expression (2.41) is obtained similarly to (2.48) by setting $\chi=\omega+(\chi-\omega)$ in the leading order terms.

The second integral over $\mathbb{T}^{n-1}$ in the identity (2.50) is more delicate. We shall make use of the boundary conditions (2.33) and (2.35) to evaluate it. The relation (2.33) is used is to exploit the full algebraic structure of the curvature-type term $\Delta\chi\langle\psi\rangle^{-1}-\psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}$, which is important in the energy estimates later on. We have:

$$\begin{aligned}&\langle\psi\rangle^2[\mathcal{U}_n]_+^-\Delta_{x'}\mathcal{U}=\Delta\big(\Delta\chi\langle\psi\rangle^{-1}-\psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}\big)(\omega_t+\epsilon\Delta^2\omega_t)\\&+\Delta G\big(\omega_t+\epsilon\Delta^2\omega_t\big)+\langle\psi\rangle^2h\Delta_{x'}\mathcal{U}.\end{aligned} \tag{2.52}$$

Observe that

$$
\begin{aligned}
&2\int_{\mathbb{T}^{n-1}}\Delta\big(\Delta\chi\langle\psi\rangle^{-1}-\psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}\big)\omega_t = 2\int_{\mathbb{T}^{n-1}}\big(\Delta\chi\langle\psi\rangle^{-1}-\psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}\big)\Delta\omega_t =\\
&\partial_t\int_{\mathbb{T}^{n-1}}|\nabla^2\omega|^2\langle\psi\rangle^{-1}-\int_{\mathbb{T}^{n-1}}|\nabla^2\omega|^2\langle\psi\rangle_t^{-1}+2\int_{\mathbb{T}^{n-1}}\omega_{it}\nabla\omega_i\cdot\nabla(\langle\psi\rangle^{-1})-\\
&2\int_{\mathbb{T}^{n-1}}\nabla\omega_t\cdot\nabla(\langle\psi\rangle^{-1})\Delta\omega+2\int_{\mathbb{T}^{n-1}}\Delta\omega_t\big(\Delta\chi-\Delta\omega\big)\langle\psi\rangle^{-1}-\\
&2\int_{\mathbb{T}^{n-1}}\omega_{kkt}\psi_i\psi_j\omega_{ij}\langle\psi\rangle^{-3}-2\int_{\mathbb{T}^{n-1}}\Delta\omega_t\big(\psi_i\psi_j(\chi_{ij}-\omega_{ij})\langle\psi\rangle^{-3}\big).
\end{aligned}
\tag{2.53}
$$

Here, just like in (2.48) we substituted $\chi=\omega+(\chi-\omega)$ in the leading order terms. Note that we have repeatedly used integration by parts. Integrating by parts twice, we obtain

$$
\begin{aligned}
&-2\int_{\mathbb{T}^{n-1}}\omega_{kkt}\psi_i\psi_j\omega_{ij}\langle\psi\rangle^{-3} = 2\int_{\mathbb{T}^{n-1}}\omega_{kt}\psi_i\psi_j\omega_{ijk}\langle\psi\rangle^{-3}+\\
&2\int_{\mathbb{T}^{n-1}}\omega_{kt}\Big\{\big[\psi_i\psi_j\omega_{ij}\langle\psi\rangle^{-3}\big]_k-\psi_i\psi_j\omega_{ijk}\langle\psi\rangle^{-3}\Big\} =\\
&-2\int_{\mathbb{T}^{n-1}}\omega_{ikt}\psi_i\psi_j\omega_{jk}\langle\psi\rangle^{-3}-2\int_{\mathbb{T}^{n-1}}\omega_{jk}\Big\{\big[\psi_i\psi_j\omega_{kt}\langle\psi\rangle^{-3}\big]_i-\psi_i\psi_j\omega_{ikt}\langle\psi\rangle^{-3}\Big\}\\
&+2\int_{\mathbb{T}^{n-1}}\omega_{kt}\Big\{\big[\psi_i\psi_j\omega_{ij}\langle\psi\rangle^{-3}\big]_k-\psi_i\psi_j\omega_{ijk}\langle\psi\rangle^{-3}\Big\}.
\end{aligned}
\tag{2.54}
$$

We now single out the t-derivative in the first term on RHS of (2.54) to obtain

$$
\begin{aligned}
-2\omega_{ikt}\psi_i\psi_j\omega_{jk}\langle\psi\rangle^{-3} = -\partial_t\Big\{\sum_{k=1}^{n-1}\big(\nabla\omega_k\cdot\nabla\psi\big)^2\langle\psi\rangle^{-3}\Big\}+\omega_{jk}\omega_{ik}\psi_j\psi_{it}\langle\psi\rangle^{-3}\\
+\big(\nabla\omega_k\cdot\nabla\psi\big)^2\langle\psi\rangle_t^{-3}.
\end{aligned}
\tag{2.55}
$$

We combine the identities (2.53), (2.54) and (2.55) to conclude

$$
\begin{aligned}
&2\int_{\mathbb{T}^{n-1}}\Delta\big(\Delta\chi\langle\psi\rangle^{-1}-\psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}\big)\omega_t =\\
&\partial_t\int_{\mathbb{T}^{n-1}}|\nabla^2\omega|^2\langle\psi\rangle^{-1}-\partial_t\Big\{\sum_{k=1}^{n-1}\int_{\mathbb{T}^{n-1}}\big(\nabla\omega_k\cdot\nabla\psi\big)^2\langle\psi\rangle^{-3}\Big\}+\int_{\mathbb{T}^{n-1}}A\\
&=\partial_t\int_{\mathbb{T}^{n-1}}I_\psi(\nabla^2\omega,\nabla^2\omega)+\int_{\mathbb{T}^{n-1}}A,
\end{aligned}
$$

where I_ψ and A are given by (1.21) and (2.43) respectively.

In the ϵ-dependent part on RHS of (2.52) we set $\omega^* = \Delta\omega$ and $\chi^* = \Delta\chi$. We can write

$$\Delta\big(\Delta\chi\langle\psi\rangle^{-1} - \psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}\big)\Delta^2\omega_t = \big(\Delta\chi^*\langle\psi\rangle^{-1} - \psi_i\psi_j\chi^*_{ij}\psi^{-3}\big)\Delta\omega^*_t$$
$$+\Big\{\Delta\big(\chi^*\langle\psi\rangle^{-1} - \psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}\big) - \big(\Delta\chi^*\langle\psi\rangle^{-1} - \psi_i\psi_j\chi^*_{ij}\langle\psi\rangle^{-3}\big)\Big\}\Delta\omega^*_t.$$

We may now apply the same computation as in (2.53) to conclude

$$2\epsilon\int_{\mathbb{T}^{n-1}}\Delta\big(\Delta\chi\langle\psi\rangle^{-1} - \psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}\big)\Delta^2\omega_t = \partial_t\int_{\mathbb{T}^{n-1}}\epsilon I_\psi(\nabla^2\Delta\omega,\nabla^2\Delta\omega) + \int_{\mathbb{T}^{n-1}}B,$$

where B is given by (2.44). We combine the above identities to write the final form of the second integral over $\mathbb{T}^{n-1}$ in the identity (2.50):

$$2\int_{\mathbb{T}^{n-1}}\langle\psi\rangle^2[\mathcal{U}_n]^-_+\Delta_{x'}\mathcal{U} = \partial_t\Big\{\int_{\mathbb{T}^{n-1}}I_\psi(\nabla^2\omega,\nabla^2\omega) + \epsilon I_\psi(\nabla^2\Delta\omega,\nabla^2\Delta\omega)\Big\}$$
$$+\int_{\mathbb{T}^{n-1}}T(\chi,\omega,\psi,G,h), \tag{2.56}$$

where T is given by (2.42). By summing the identities (2.45) and (2.50), plugging (2.49) in (2.45) and (2.51) and (2.56) into (2.50) and collecting terms, we conclude the proof of the lemma. □

3 Iteration scheme and the basic estimates

We shall set up an iterative scheme in order to solve the regularized Stefan problem locally-in-time. For given ρ^m and Cauchy data $u_0^\epsilon \in C^\infty(\Omega)$, $\rho_0^\epsilon \in C^\infty(\mathbb{T}^{n-1})$, we solve the following problem:

$$u_t^{m+1} - \Delta_{x'}u^{m+1} - a_{\rho^m}u_{nn}^{m+1} + B_{\rho^m}\cdot\nabla_{x'}u_n^{m+1} + c_{\rho^m}u_n^{m+1} = 0 \tag{3.57}$$
$$u^{m+1} = \kappa^m \qquad \text{on } \mathbb{T}^{n-1}\times\{x_n = 0\} \tag{3.58}$$
$$\partial_n u^{m+1} = 0 \qquad \text{on } \mathbb{T}^{n-1}\times\{x_n = \pm 1\} \tag{3.59}$$
$$u^{m+1}(x,0) = u_0^\epsilon(x), \quad \rho^{m+1}(x',0) = \rho_0^\epsilon(x'). \tag{3.60}$$

Here

$$\kappa^m := \nabla\cdot\big(\frac{\nabla\rho^m}{\langle\rho^m\rangle}\big). \tag{3.61}$$

The solution to the problem (3.57) - (3.60) exists and is smooth (see Chapter 4 of [13]). Having obtained u^{m+1}, we solve the equation

$$\rho_t^{m+1} + \epsilon\Delta^2\rho_t^{m+1} = \langle\rho^m\rangle^2 [u_n^{m+1}]_+^- \qquad \text{on } \mathbb{T}^{n-1}\times\{x_n=0\} \tag{3.62}$$

for ρ^{m+1}. We aim for proving the convergence of the sequence (u^m, ρ^m) to the solution of the regularized Stefan problem in the energy space. Applying the tangential differential operator ∂_s^μ (recall (1.18)) to the equations (3.57), (3.58) and (3.62), we obtain

$$\partial_s^\mu u_t^{m+1} - \partial_s^\mu \Delta_{x'} u^{m+1} - a_{\rho^m}\partial_s^\mu u_{nn}^{m+1} = f_{\mu,s}^m$$

$$\begin{aligned}\partial_s^\mu u^{m+1} &= \partial_s^\mu \kappa^m = \Delta\partial_s^\mu\rho^m \langle\rho^m\rangle^{-1} + g_{\mu,s}^m \\ &= \Delta\partial_s^\mu\rho^m \langle\rho^m\rangle^{-1} - \rho_i^m\rho_j^m\Delta\partial_s^\mu\rho_{ij}^m \langle\rho^m\rangle^{-1} + G_{\mu,s}^m\end{aligned}$$

$$[\partial_s^\mu u_n^{m+1}]_+^- \langle\rho^m\rangle^2 = \partial_s^\mu\rho_t^{m+1} + \epsilon\Delta^2\rho_t^{m+1} + h_{\mu,s}^m\langle\rho^m\rangle^2,$$

where

$$f_{\mu,s}^m = \Big\{\partial_s^\mu\Big(a_{\rho^m}u_{nn}^{m+1}\Big) - a_{\rho^m}\partial_s^\mu u_{nn}^{m+1}\Big\} - \partial_s^\mu\Big(B_{\rho^m}\cdot\nabla_{x'}u_n^{m+1}\Big) - \partial_s^\mu\Big(c_{\rho^m}u_n^{m+1}\Big), \tag{3.63}$$

$$g_{\mu,s}^m = \sum_{\substack{|\mu'|+s'\\<|\mu|+s}} C_{s'}^{\mu'}\partial_{s'}^{\mu'}\Delta\rho^m\partial_{s-s'}^{\mu-\mu'}(\langle\rho^m\rangle^{-1}) + \partial_s^\mu(\nabla\rho^m\cdot\nabla(\langle\rho^m\rangle^{-1})), \tag{3.64}$$

$$\begin{aligned}G_{\mu,s}^m = &\sum_{\substack{|\mu'|+s'\\<|\mu|+s}} C_{s'}^{\mu'}\partial_{s'}^{\mu'}\Delta\rho^m\partial_{s-s'}^{\mu-\mu'}(\langle\rho^m\rangle^{-1}) - \\ &-\big\{\partial_s^\mu\big(\rho_i^m\rho_j^m\rho_{ij}^m\langle\rho^m\rangle^{-1}\big) - \rho_i^m\rho_j^m\partial_s^\mu\rho_{ij}^m\langle\rho^m\rangle^{-1}\big\},\end{aligned} \tag{3.65}$$

$$h_{\mu,s}^m = \sum_{\substack{|\mu'|+s'\\<|\mu|+s}} C_{s'}^{\mu'}\partial_{s'}^{\mu'}(\rho_t^{m+1} + \epsilon\Delta^2\rho_t^{m+1})\partial_{s-s'}^{\mu-\mu'}(\langle\rho^m\rangle^{-2}). \tag{3.66}$$

For any $l\in\mathbb{N}$ let us define

$$\mathcal{E}^l := \mathcal{E}_\epsilon(u^l,\rho^l;\rho^{l-1}), \quad \mathcal{D}^l := \mathcal{D}_\epsilon(u^l,\rho^l;\rho^{l-1}), \tag{3.67}$$

where $\mathcal{E}_\epsilon$ and $\mathcal{D}_\epsilon$ are defined by (1.26) and (1.27) respectively. Setting $\mathcal{U} = \partial_s^\mu u^{m+1}$, $\omega = \partial_s^\mu\rho^{m+1}$, $\chi = \partial_s^\mu\rho^m$, $\psi = \rho^m$, $f = f_{\mu,s}^m$, $g = g_{\mu,s}^m$, $G = G_{\mu,s}^m$ and $h = h_{\mu,s}^m$, the identity (2.37) implies

$$\frac{d}{dt}\mathcal{E}^{m+1}(t) + \mathcal{D}^{m+1}(t) = \int_\Omega\big\{P^m + R^m\big\} - \int_{\mathbb{T}^{n-1}}\big\{Q^m + S^m + T^m\big\}. \tag{3.68}$$

Here $P^m = \sum_{|\mu|+2s\le 2k} P^m_{\mu,s}$ and R^m, Q^m, S^m and T^m are defined analogously, whereby

$$P^m_{\mu,s} = P(\partial^\mu_s u^{m+1}, \rho^m, f^m_{\mu,s}), \quad Q^m_{\mu,s} = Q(\partial^\mu_s \rho^m, \partial^\mu_s \rho^{m+1}, \rho^m, g^m_{\mu,s}, h^m_{\mu,s}) \tag{3.69}$$

and

$$\begin{aligned} &R^m_{\mu,s} := R(\partial^\mu_s u^{m+1}, \rho^m, f^m_{\mu,s}), \quad S^m_{\mu,s} := S(\partial^\mu_s \rho^m, \partial^\mu_s \rho^{m+1}, \rho^m, g^m_{\mu,s}, h^m_{\mu,s}), \\ &T^m_{\mu,s} := T(\partial^\mu_s \rho^m, \partial^\mu_s \rho^{m+1}, \rho^m, g^m_{\mu,s}, h^m_{\mu,s}). \end{aligned} \tag{3.70}$$

The inequality (1.22) implies that the instant energy $\mathcal{E}^l$ is positive definite. In order to estimate P^m, R^m, Q^m, S^m and T^m we first need to establish some basic auxiliary estimates.

Lemma 3.1 *The following identity holds*

$$\partial_t \Big\{ \int_\Omega u^{m+1}(1+\phi'\rho^m) \Big\} = \partial_t \Big\{ \int_{\mathbb{T}^{n-1}} \rho^{m+1} \Big\}. \tag{3.71}$$

Proof. We multiply the equation (3.57) with $(1+\phi'\rho^m)$ and integrate over Ω. We thus obtain

$$\begin{aligned} &\int_\Omega (1+\phi'\rho^m)(u^{m+1}_t - \Delta_{x'} u^{m+1}) - \int_\Omega \frac{1+|\phi\nabla\rho^m|^2}{1+\phi'\rho^m} u^{m+1}_{nn} + 2\int_\Omega \phi\nabla\rho^m \cdot \nabla_{x'} u^{m+1}_n \\ &+ \int_\Omega \phi\Delta\rho^m u^{m+1}_n - \int_\Omega \Big(\frac{(\phi^2)'|\nabla\rho^m|^2}{1+\phi'\rho^m} - \frac{\phi''\rho^m(1+|\phi\nabla\rho^m|^2)}{(1+\phi'\rho^m)^2} \Big) u^{m+1}_n \\ &- \int_\Omega \phi\rho^m_t u^{m+1}_n = 0. \end{aligned}$$

Integrating by parts we have $\int_\Omega -\phi\rho^m_t u^{m+1}_n = \int_\Omega \phi'\rho^m_t u^{m+1}$. Using this identity, we obtain

$$\int_\Omega (1+\phi'\rho^m) u^{m+1}_t + \int_\Omega (1+\phi'\rho^m) e_\rho u^{m+1}_n = \partial_t \Big\{ \int_\Omega (1+\phi'\rho^m) u^{m+1} \Big\}.$$

Observe that the integration by parts implies

$$\int_\Omega -(1+\phi'\rho^m)\Delta_{x'} u^{m+1} = \int_\Omega \phi'\nabla_{x'}\rho^m \cdot \nabla_{x'} u^{m+1}$$

and $\int_\Omega \phi\Delta\rho^m u^{m+1}_n = -\int_\Omega \phi\nabla\rho^m \cdot \nabla_{x'} u^{m+1}_n$. Thus

$$-\int_\Omega (1+\phi'\rho^m)\Delta_{x'} u^{m+1} + \int_\Omega 2\phi\nabla\rho^m \cdot \nabla_{x'} u^{m+1}_n + \int_\Omega \phi\Delta\rho^m u^{m+1}_n = 0$$

Note further that

$$\partial_n\Big(\frac{1+|\phi\nabla\rho^m|^2}{1+\phi'\rho^m}\Big)=\frac{(\phi^2)'|\nabla\rho^m|^2}{1+\phi'\rho^m}-\frac{\phi''\rho^m(1+|\phi\nabla\rho^m|^2)}{(1+\phi'\rho^m)^2}$$

Using integration by parts again we have

$$-\int_\Omega\frac{1+|\phi\nabla\rho^m|^2}{1+\phi'\rho^m}u^{m+1}_{nn}-\int_\Omega\Big(\frac{(\phi^2)'|\nabla\rho^m|^2}{1+\phi'\rho^m}-\frac{\phi''\rho^m(1+|\phi\nabla\rho^m|^2)}{(1+\phi'\rho^m)^2}\Big)u^{m+1}_n=$$
$$-\int_{\mathbb{T}^{n-1}}\langle\rho^m\rangle^2\,[u^{m+1}_n]^-_+=-\partial_t\Big\{\int_{\mathbb{T}^{n-1}}\rho^{m+1}\Big\}.$$

This finishes the proof of the lemma. □

The importance of this identity is reflected in the fact that it allows to control terms with purely temporal derivatives of ρ^{m+1}:

Lemma 3.2 *There exist positive constants K and $\theta_0<1$ such that for any $\theta\le\theta_0$ such that if*

$$\Big|\int_{\mathbb{T}^{n-1}}\rho_0^\epsilon-\int_\Omega u_0^\epsilon(1+\phi'\rho_0^\epsilon)\Big|<\theta,$$

$$\mathcal{E}^m\le\theta,\quad ||\nabla\rho^{m-1}||_\infty\le1\quad\textit{and}\quad\sum_{p=0}^k||\partial_p\rho^m||_2\le K\sqrt{\mathcal{E}^m}+\theta,$$

then $||\nabla\rho^m||_\infty\le1$ *and* $\sum_{p=0}^k||\partial_p\rho^{m+1}||_2\le K\sqrt{\mathcal{E}^{m+1}}+\theta.$

Proof. Observe first that the assumption on ρ^{m-1} implies that $\langle\rho^{m-1}\rangle\le\sqrt{2}$. Using the Sobolev inequality, we obtain

$$||\nabla\rho^m||_\infty\le C_*||\nabla\rho^m||_{H^{(n+1)/2}}\le C_*\sqrt{2}\mathcal{E}^m\le C_*\sqrt{2}\theta.$$

Thus, choosing $\theta_0\le\frac{1}{C_*\sqrt{2}}$ guarantees $||\nabla\rho^m||_\infty\le1$. By (3.71) we have

$$\int_{\mathbb{T}^{n-1}}\partial_s\rho^{m+1}_t=\int_\Omega\partial_s u^{m+1}_t+\int_\Omega\phi'\sum_{p=0}^{s+1}\binom{s+1}{p}\partial_p u^{m+1}\partial_{s+1-p}\rho^m.$$

Also, $\int_{\mathbb{T}^{n-1}}\rho^{m+1}=\int_\Omega(u^{m+1}+\phi'u^{m+1}\rho^m)+\int_{\mathbb{T}^{n-1}}\rho_0^\epsilon-\int_\Omega u_0^\epsilon(1+\phi'\rho_0^\epsilon)$. Thus, for $0\le s\le k-1$, using the Cauchy-Schwarz inequality, definition (3.67) of $\mathcal{E}^{m+1}$

and the main assumption in the statement of the lemma, we obtain

$$\Big|\int_{\mathbb{T}^{n-1}} \partial_s \rho_t^{m+1}\Big| \leq ||\partial_{s+1} u^{m+1}||_2 + C\sum_{p=1}^{s+1} ||\partial_p u^{m+1}||_{L^2(\Omega)} ||\partial_{s+1-p}\rho^m||_2$$
$$\leq \sqrt{\mathcal{E}^{m+1}} + C\sqrt{\mathcal{E}^{m+1}} \sum_{p=0}^{k} ||\partial_p \rho^m||_2.$$

Furthermore,

$$\Big|\int_{\mathbb{T}^{n-1}} \rho^{m+1}\Big| \leq C||u^{m+1}||_2 + C||u^{m+1}||_{L^2(\Omega)}||\rho^m||_2 + \Big|\int_{\mathbb{T}^{n-1}} \rho_0^\epsilon - \int_\Omega u_0^\epsilon(1+\phi'\rho_0^\epsilon)\Big|$$
$$\leq C\sqrt{\mathcal{E}^{m+1}} + C\sqrt{\mathcal{E}^{m+1}} \sum_{p=0}^{k} ||\partial_p \rho^m||_2 + \theta.$$

By the Poincaré inequality we get

$$\sum_{p=0}^{k} ||\partial_p \rho^{m+1}||_2 \leq C\sum_{p=0}^{k} ||\partial_p \nabla \rho^{m+1}||_2 + C\sum_{p=0}^{k} ||\int_{\mathbb{T}^{n-1}} \partial_p \rho^{m+1}||_2. \tag{3.72}$$

The first term on the right-hand side is estimated by $C\sqrt{\mathcal{E}^{m+1}}$, by the definition (3.67) of $\mathcal{E}^{m+1}$. By the previous two inequalities and the assumptions of the lemma, we can estimate the second sum on RHS of (3.72) by $C\sqrt{\mathcal{E}^{m+1}} + C\sqrt{\mathcal{E}^{m+1}}\big(K\sqrt{\mathcal{E}^m} + \theta\big) + \theta$. Keeping in mind that $\mathcal{E}^m \leq \theta$, we choose $1 \leq K$ large enough and $\theta < \frac{1}{C_*\sqrt{2}}$ small enough so that

$$\sum_{p=0}^{k} ||\partial_p \rho^{m+1}||_2 \leq K\sqrt{\mathcal{E}^{m+1}} + \theta.$$

□

In the following we shall work under the standing assumption

$$\mathcal{E}^m \leq \theta, \quad ||\nabla \rho^{m-1}||_\infty \leq 1 \quad \sum_{p=0}^{k} ||\partial_p \rho^m||_2 \leq K\sqrt{\mathcal{E}^m} + \theta, \tag{3.73}$$

with $\theta \leq \theta_0$ where θ_0 is given as in Lemma 3.2.

Lemma 3.3 *Let $\xi \in C^\infty(J,\mathbb{R})$ and $J \subseteq \mathbb{R}$ an interval such that every derivative of ξ is uniformly bounded on J.*

(a) *Let (μ,r) be a pair of indices such that $|\mu|+2r \le 2k$ and $(|\mu|,r) \ne (0,0)$. Then there exists a positive constant C such that*

$$||\partial_r^\mu \left[\xi(|\nabla\rho^m|^2)\right]||_{H^1} \le C\sqrt{\mathcal{E}^m} \tag{3.74}$$

and

$$||\partial_r^\mu \left[\xi(|\nabla\rho^m|^2)\right]||_{H^3} \le \frac{C}{\sqrt{\epsilon}}\sqrt{\mathcal{E}^m}. \tag{3.75}$$

(b) *There exists a positive constant C such that*

$$||\partial_s^\mu \nabla\rho^m||_{H^2} \le C\sqrt{\mathcal{D}^{m+1}}, \tag{3.76}$$

where $|\mu|+2s \le 2k$. Furthermore,

$$||\partial_{s+1}^\mu \xi(|\nabla\rho^m|^2)||_2 \le C\sqrt{\mathcal{D}^m} \tag{3.77}$$

for all (μ,s) satisfying $|\mu|+2s \le 2k$.

(c) *For any pair of indices (μ,s) such that $|\mu|+2s \le 2k$ there exists a positive constant C and a small parameter λ such that*

$$||\partial_s^\mu \rho_t^{m+1}||_2^2 + 2\epsilon||\partial_s^\mu \Delta\rho_t^{m+1}||_2^2 + \epsilon^2||\partial_s^\mu \Delta^2\rho_t^{m+1}||_2^2 \le \frac{C}{\lambda}\mathcal{E}^{m+1} + C\lambda\mathcal{D}^{m+1}. \tag{3.78}$$

Proof. *Part (a):* Let $\alpha = \mu+\tau$ for any given multi-index of length $n-1$ satisfying $|\tau| \le 1$. Let first $r=0$. By assumption $|\alpha| \ge 1$. Using Moser's inequality (cf. [10], Lemma 5) and Leibniz' rule (cf. [10], Lemma 4), we have

$$||\partial^\alpha \left[\xi(|\nabla\rho^m|^2)\right]||_2 \le C \max_{1\le d\le|\alpha|}\left(||\xi^{(d)}(|\nabla\rho^m|^2))||_\infty ||\nabla\rho^m||_\infty^{|\alpha|-1}\right)||\partial^\alpha(|\nabla\rho^m|^2)||_2$$
$$\le C||\partial^\alpha(|\nabla\rho^m|^2)||_2 \le C||\partial^\alpha\nabla\rho^m||_2||\nabla\rho^m||_\infty \le C\sqrt{\mathcal{E}^m}.$$

Let $r \ge 1$.

$$\partial_r^\alpha\left[\xi(|\nabla\rho^m|^2)\right] = \partial^\alpha\left\{\sum_{d=1}^r \sum_{\substack{s_1+\dots+s_d=r\\ s_i>0}} C_d\xi^{(d)}(|\nabla\rho^m|^2)\partial_{s_1}(|\nabla\rho^m|^2)\dots\partial_{s_d}(|\nabla\rho^m|^2)\right\}$$

$$= \sum_{d=1}^r \sum_{\substack{s_1+\dots+s_d=r\\ s_i>0}} \sum_{\substack{\gamma_1+\dots+\gamma_{d+1}\\ =\alpha}} C_d C_{\gamma_1\dots\gamma_{d+1}} \partial_{s_1}^{\gamma_1}(|\nabla\rho^m|^2)\dots\partial_{s_d}^{\gamma_d}(|\nabla\rho^m|^2)\partial^{\gamma_{d+1}}\left[\xi^{(d)}(|\nabla\rho^m|^2)\right].$$

For any $i=1,\ldots,d$, we have $\partial_{s_i}^{\gamma_i}(|\nabla\rho^m|^2)=\sum_{l=0}^{s_i}\sum_{\delta\le\gamma_i}C_{l,\delta}\partial_l^\delta\nabla\rho^m\partial_{s_i-l}^{\gamma_i-\delta}\nabla\rho^m$. Thus if $|\gamma_i|+2s_i\le k+1$, then by the Sobolev inequality

$$||\partial_l^\delta\nabla\rho^m||_\infty\le||\partial_l^\delta\nabla\rho^m||_{H^{\frac{n+1}{2}}}\le C\sqrt{\mathcal{E}^m},$$

and analogously $||\partial_{s_i-l}^{\gamma_i-\delta}\nabla\rho^m||_\infty\le C\sqrt{\mathcal{E}^m}$, implying $||\partial_{s_i}^{\gamma_i}(|\nabla\rho^m|^2)||_\infty\le C\mathcal{E}^m$. If $|\gamma_{d+1}|\le k+1$, we use the Sobolev and Moser's inequality to conclude

$$||\partial^{\gamma_{d+1}}\big[\xi^{(d)}(|\nabla\rho^m|^2)\big]||_\infty\le C\sqrt{\mathcal{E}^m}.$$

If there exists $1\le j\le d$ such that $|\gamma_j|+2s_j>k+1$, then $|\gamma_i|+2s_i\le k+1$ for $1\le i\le d$, $i\neq j$ and additionally, $|\gamma_{d+1}|\le k+1$. Thus we can estimate the term containing γ_j in superscript in L^2-norm and the remaining terms in L^∞-norm. If on the other hand $|\gamma_i|+2s_i\le k+1$ for every $1\le i\le d$, we estimate the term $\partial^{\gamma_{d+1}}\big[\xi^{(d)}(|\nabla\rho^m|^2)\big]$ in L^2-norm and the remaining terms in L^∞-norm. We conclude that

$$||\partial_r^\alpha\big[\xi(|\nabla\rho^m|^2)\big]||_2\le C\sqrt{\mathcal{E}^m},$$

for the specified range of α-s and r-s. The inequality (3.75)is proved similarly.

Part (b): By (3.58), $\Delta\rho^m=u^{m+1}\langle\rho^m\rangle+\rho_i^m\rho_j^m\rho_{ij}^m\langle\rho^m\rangle^{-2}$. Let $\gamma=\mu+\tau$ where $|\tau|=1$. Applying ∂_s^γ to the above identity, we get

$$\begin{aligned}\partial_s^\gamma\Delta\rho^m=&\sum_{\gamma',s'}C_{s'}^{\gamma'}\partial_{s'}^{\gamma'}(u^{m+1})\partial_{s-s'}^{\gamma-\gamma'}(\langle\rho^m\rangle)+\\&\sum_{\substack{\sum(\gamma_l+s_l)\\=\gamma+s}}C_{s_1,s_2,s_3,s_4}^{\gamma_1,\gamma_2,\gamma_3,\gamma_4}\partial_{s_1}^{\gamma_1}\rho_i^m\partial_{s_2}^{\gamma_2}\rho_j^m\partial_{s_3}^{\gamma_3}\rho_{ij}^m\partial_{s_4}^{\gamma_4}(\langle\rho^m\rangle^{-2})\end{aligned}$$

Observe that $\int_{\mathbb{T}^{n-1}}u^{m+1}=0$ since $u^{m+1}=\kappa^m=\nabla\cdot\big(\nabla\rho^m\langle\rho^m\rangle^{-1}\big)$ on $\mathbb{T}^{n-1}$. Let us fix $(\gamma',s')\le(\gamma,s)$. If $|\gamma'|>0$ note that $\int_\Omega\partial_{s'}^{\gamma'}u^{m+1}=0$. We use the trace inequality and then the Poincaré inequality on Ω to deduce

$$||\partial_{s'}^{\gamma'}u^{m+1}||_{L^2(\mathbb{T}^{n-1})}\le C||\partial_{s'}^{\gamma'}u^{m+1}||_{H^1(\Omega)}\le C||\partial_{s'}^{\gamma'}\nabla u^{m+1}||_{L^2(\Omega)}\le C\sqrt{\mathcal{D}^{m+1}}.$$

If $|\gamma'|=0$ by the Poincaré inequality and the trace inequality:

$$\begin{aligned}||\partial_{s'}u^{m+1}||_{L^2(\mathbb{T}^{n-1})}&\le C||\partial_{s'}\nabla_{x'}u^{m+1}||_{L^2(\mathbb{T}^{n-1})}\\&\le C||\partial_{s'}\nabla_{x'}u^{m+1}||_{H^1(\Omega)}\le C\sqrt{\mathcal{D}^{m+1}}.\end{aligned}$$

By part (a), $||\partial_{s-s'}^{\gamma-\gamma'}(\langle\rho^m\rangle)||_2 \le C + C\sqrt{\mathcal{E}^m}$. Furthermore, if for some $1 \le l \le 4$ we have $|\gamma_l| + 2s_l \ge k$, we estimate the term containing γ_l in superscript in L^2-norm and the remaining terms in L^∞-norm. Using part (a) we deduce

$$||\partial_{s_1}^{\gamma_1}\rho_i^m \partial_{s_2}^{\gamma_2}\rho_j^m \partial_{s_3}^{\gamma_3}\rho_{ij}^m \partial_{s_4}^{\gamma_4}(\langle\rho^m\rangle^{-2})||_2 \le C(\mathcal{E}^m)^{3/2} \sum_{|\mu|+2s\le 2k} ||\partial_s^\mu \nabla\rho^m||_{H^2}.$$

Thus, summing over all pairs (μ, s) yields

$$\sum_{\substack{|\mu|+2s\le 2k \\ \gamma=\mu+\tau, |\tau|=1}} ||\partial_s^\gamma \Delta\rho^m||_2 \le (C + C\sqrt{\mathcal{E}^m})\sqrt{\mathcal{D}^{m+1}} + C(\mathcal{E}^m)^{3/2} \sum_{|\mu|+2s\le 2k} ||\partial_s^\mu \nabla\rho^m||_{H^2}. \tag{3.79}$$

Since $\int_{\mathbb{T}^{n-1}} \nabla\rho^m = 0$ and $|\gamma| \ge 1$ elliptic regularity implies

$$\sum_{|\mu|+2s\le 2k} ||\partial_s^\mu \nabla\rho^m||_{H^2} \le C \sum_{\substack{|\mu|+2s\le 2k \\ \gamma=\mu+\tau, |\tau|=1}} ||\partial_s^\gamma \Delta\rho^m||_2. \tag{3.80}$$

Combining (3.79), (3.80) and choosing $\mathcal{E}^m$ sufficiently small we deduce the claim.
Part (c): We apply the differential operator ∂_s^μ to the 'jump relation' (3.62) and take squares on both sides to obtain

$$||\partial_s^\mu \rho_t^{m+1}||_2^2 + 2\epsilon||\partial_s^\mu \Delta\rho_t^{m+1}||_2^2 + \epsilon^2||\partial_s^\mu \Delta^2\rho_t^{m+1}||_2^2 = ||\partial_s^\mu\big(\langle\rho^m\rangle^2 [u_n^{m+1}]_+^-\big)||_2^2.$$

Next

$$\begin{aligned}
||\partial_s^\mu\Big(\langle\rho^m\rangle^2 [u_n^{m+1}]_+^-\Big)||_2 &= ||\sum_{\mu', s'} C_{s'}^{\mu'} \partial_{s'}^{\mu'}(\langle\rho^m\rangle^2)\partial_{s-s'}^{\mu-\mu'}[u_n^{m+1}]_+^-||_2 \\
&\le C \sum_{\substack{|\mu'|+2s' \\ \le k}} ||\partial_{s'}^{\mu'}(\langle\rho^m\rangle^2)||_\infty ||\partial_{s-s'}^{\mu-\mu'}[u_n^{m+1}]_+^-||_2 + \\
&\quad + C \sum_{\substack{|\mu'|+2s' \\ > k}} ||\partial_{s'}^{\mu'}(\langle\rho^m\rangle^2)||_2 ||\partial_{s-s'}^{\mu-\mu'}[u_n^{m+1}]_+^-||_{L^\infty(\mathbb{T}^{n-1})} \\
&\le (C\sqrt{\mathcal{E}^m} + 1)\Big(\frac{C}{\lambda}\mathcal{E}^{m+1} + C\lambda\mathcal{D}^{m+1}\Big) \le \frac{C}{\lambda}\mathcal{E}^{m+1} + C\lambda\mathcal{D}^{m+1}.
\end{aligned}$$

Here we assume $\mathcal{E}^m \le 1$. Terms involving L^∞-norm are first estimated by the Sobolev inequality. Then we use the standard trace inequality $||v||_{L^2(\partial\Omega)} \le$

$\frac{C}{\lambda}||v||_{L^2(\Omega)}+\lambda||\nabla v||_{L^2(\Omega)}$ to bound the terms involving $[u_n^{m+1}]_+^-$. Observe that the same proof is easily adapted to yield the bound

$$||\partial_s\rho_t^{m+1}||_2\leq C\sqrt{\mathcal{D}^{m+1}}, \qquad \text{for} \quad s\leq k. \tag{3.81}$$

□

Remark. The estimate (3.78) will play a crucial role in the energy estimates for the problem of local existence. For any pair (μ,s), $|\mu|+2s\leq 2k$ and $|\mu|\geq 1$, the elliptic regularity and the estimate (3.78) imply

$$||\nabla^4\partial_s^{\mu-1}\nabla\rho_t^{m+1}||_2\leq \frac{C}{\lambda\epsilon^2}\sqrt{\mathcal{E}^{m+1}}+\lambda\sqrt{\mathcal{D}^{m+1}}. \tag{3.82}$$

Lemma 3.4 *There exists a positive constant C such that*

(a) *For $r\geq 0$, $|\mu|+2s\leq 2k$ and $(|\mu|,s)\neq(0,0)$*

$$||\partial_s^\mu\partial_{n^r}(a_{\rho^m})||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^m} \tag{3.83}$$

(b) *For $r\geq 0$ and $|\mu|+2s\leq 2k$*

$$||\partial_s^\mu\partial_{n^r}(B_{\rho^m})||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^m} \tag{3.84}$$

(c) *For $r\geq 0$ and $|\mu|+2s\leq 2k$*

$$||\partial_s^\mu\partial_{n^r}(c_{\rho^m})||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^m}+C\sqrt{\mathcal{D}^m}. \tag{3.85}$$

Furthermore, for $r\geq 0$, $|\mu|+2s\leq 2k-1$

$$||\partial_s^\mu\partial_{n^r}(c_{\rho^m})||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^m}. \tag{3.86}$$

Proof. *Part (a):* We note:

$$\begin{aligned}\partial_s^\mu\partial_{n^r}(a_{\rho^m}) &= \partial_s^\mu\partial_{n^r}\big((1+|\phi\nabla\rho^m|^2)(1+\phi'\rho^m)^{-2}\big)\\ &= \partial_s^\mu\Big(\sum_{p=0}^{r}\partial_{n^p}F_1(|\phi\nabla\rho^m|)\partial_{n^{r-p}}F_2(\phi'\rho^m)\Big),\end{aligned}$$

where $F_1(x):=1+x^2$ and $F_2(x):=(1+x)^{-2}$. Observe that

$$\partial_{n^{r-p}}F_2(\phi'\rho^m)=\sum_{d=1}^{r-p}\sum_{\substack{s_1+\dots s_d\\=r-p}}C_dF_2^{(d)}(\phi'\rho^m)(\rho^m)^d\phi^{(s_1)}\dots\phi^{(s_d)},$$

and thus

$$||\partial_s^\mu \partial_{n^{r-p}} F_2(\phi'\rho^m)||_{L^2(\Omega)} \le C\sum_{d=1}^{r-p} ||\partial_s^\mu \Big[F_2^{(d)}(\phi'\rho^m)(\rho^m)^d\Big]||_{L^2(\Omega)}$$
$$\le C\sum_{d=1}^{r-p}\sum_{\mu',s'} C_{s'}^{\mu'} ||\partial_{s'}^{\mu'}\big[[F_2^{(d)}(\phi'\rho^m)\big]\partial_{s-s'}^{\mu-\mu'}\big[G_d(\rho^m)\big]||_{L^2(\Omega)},$$

where $G_d(x) := x^d$. By Moser's inequality, for any pair of indices (ν,q) such that $|\nu|+2q\le 2k$, $(|\nu|,q)\ne(0,0)$ we have $||\partial_q^\nu\big[[F_2^{(d)}(\phi'\rho^m)\big]||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^m}$ if $|\nu|>0$, and $||\partial_q^\nu\big[[F_2^{(d)}(\phi'\rho^m)\big]||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^m}+\theta$ if $|\nu|=0$. We have used Lemma 3.2. Similarly, if $|\nu|+2q\le 2k$, we have $||\partial_q^\nu\big[G_d(\rho^m)\big]||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^m}$ when $|\nu|>0$, and $||\partial_q^\nu\big[G_d(\rho^m)\big]||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^m}+\theta$ when $|\nu|=0$. Thus

$$||\partial_{s'}^{\mu'}\big[[F_2^{(d)}(\phi'\rho^m)\big]\partial_{s-s'}^{\mu-\mu'}\big[G_d(\rho^m)\big]||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^m}, \tag{3.87}$$

where we hit the terms with lower order derivatives with L^∞-norms, depending on whether $|\mu'|+2s'\le k$ or $|\mu'|+2s'\ge k$. Additionally, we use the assumption that $\theta<1$ and $\mathcal{E}^m<1$. This implies $||\partial_s^\mu\partial_{n^{r-p}}F_2(\phi'\rho^m)||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^m}$ for $0\le p\le r$ and $|\mu|+2s\le 2k$. In the same way we prove $||\partial_s^\mu\partial_{n^p}F_1(|\phi\nabla\rho^m|)||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^m}$ for $0\le p\le r$, $|\mu|+2s\le 2k$ and $(|\mu|,s)\ne(0,0)$. Using the same idea of estimating lower order terms with L^∞-norms as in the proof of (3.87), we conclude the proof of part (a). The proof of part (b) follows in a completely analogous way.

Part (c): To prove part (c), recall that $c_{\rho^m}=d_{\rho^m}+e_{\rho^m}$, where d_{ρ^m} and e_{ρ^m} are given by

$$d_{\rho^m} := \frac{\phi\Delta\rho^m}{1+\phi'\rho^m}-\frac{(\phi^2)'|\nabla\rho^m|^2}{(1+\phi'\rho^m)^2}+\frac{\phi''\rho^m(1+|\phi\nabla\rho^m|^2)}{(1+\phi'\rho^m)^3},\quad e_{\rho^m} := \frac{\phi\rho_t^m}{1+\phi'\rho^m}.$$

The analogous proof as in the part (a) implies that $||\partial_s^\mu\partial_{n^r}(d_{\rho^m})||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^m}$. Furthermore, since $||\partial_s^\mu\rho_t^m||\le C\sqrt{\mathcal{D}^m}$ if $|\mu|+2s\le 2k$, we use the same method as in part (a) to prove $||\partial_s^\mu\partial_{n^r}(e_{\rho^m})||_{L^2(\Omega)}\le C\sqrt{\mathcal{D}^m}$. We thus conclude (3.85). From the definition of $\mathcal{E}^m$ (cf. (3.67)) and the assumption of Lemma 3.2, we have $||\partial_s^\mu\rho_t^m||\le C\sqrt{\mathcal{E}^m}$ for $|\mu|+2s\le 2k-1$, $|\mu|>0$; also $||\partial_s\rho_t^m||\le C\sqrt{\mathcal{E}^m}+\theta$ for all $s\le k-1$. Now we use the same method as in the proof of part (a) to deduce (3.86). □

Lemma 3.5 *For all pairs of indices (μ,s) such that $|\mu|+2s+r\le k+\frac{n}{2}+3$, the inequality $||\partial_s^\mu\partial_{n^r}u^{m+1}||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^{m+1}}$ holds.*

Proof. We prove the claim by induction in r. In case $r=1$ the claim is obvious from the definition of $\mathcal{E}^m$. Let the claim be true for all $r\le\vartheta$ for some $1<\vartheta<k+n/2+3$. We have to prove the claim for $r=\vartheta+1$, i.e.

$$||\partial_s^\mu\partial_{n^{\vartheta+1}}u^{m+1}||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^{m+1}}, \tag{3.88}$$

if $|\mu|+2s\le k+\frac{n}{2}-\vartheta+2$. Let (μ,s) be such a pair of indices. Then

$$\begin{aligned}
&\partial_s^\mu\partial_{n^{\vartheta+1}}u^{m+1}=\partial_s^\mu\partial_{n^{\vartheta-1}}u_{nn}^{m+1}\\
&=\partial_s^\mu\partial_{n^{\vartheta-1}}\Big\{\big(u_t^{m+1}-\Delta_{x'}u^{m+1}+B_{\rho^m}\cdot\nabla_{x'}u_n^{m+1}+c_{\rho^m}u_n^{m+1}\big)a_{\rho^m}^{-1}\Big\}\\
&=\sum_{\mu',s'}\sum_{w=0}^{\vartheta-1}C(\mu',s',w)\Big\{\partial_{s'}^{\mu'}\partial_{n^w}u_t^{m+1}-\partial_{s'}^{\mu'}\partial_{n^w}\Delta_{x'}u^{m+1}+\\
&+\sum_{\mu'',s''}\sum_{p=0}^{w}C(\mu'',s'',p)\partial_{s''}^{\mu''}\partial_{n^p}\big(B_{\rho^m}\big)\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}\nabla_{x'}u^{m+1}+\\
&+\sum_{\mu'',s''}\sum_{p=0}^{\vartheta-1}C(\mu'',s'',p)\partial_{s''}^{\mu''}\partial_{n^p}\big(c_{\rho^m}\big)\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}u^{m+1}\Big\}\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big).
\end{aligned} \tag{3.89}$$

Observe that $||\partial_{s'}^{\mu'}\partial_{n^w}u_t^{m+1}||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^{m+1}}$ and $||\partial_{s'}^{\mu'}\partial_{n^w}\Delta_{x'}u^{m+1}||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^{m+1}}$ for any triple of indices $(\mu',s',w)\le(\mu,s,\vartheta-1)$ by inductive hypothesis. Further, by Lemma 3.4 and the Sobolev inequality $||\partial_{s''}^{\mu''}\partial_{n^p}B_{\rho^m}||_{L^\infty(\Omega)}\le C\sqrt{\mathcal{E}^m}$, $||\partial_{s''}^{\mu''}\partial_{n^p}c_{\rho^m}||_{L^\infty(\Omega)}\le C\sqrt{\mathcal{E}^m}$ and $||\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big)||_{L^\infty(\Omega)}\le C\sqrt{\mathcal{E}^m}+C$, for $(\mu'',s'',p)\le(\mu',s',w)\le(\mu,s,\vartheta-1)$. Applying these estimates to the above identity and using the inductive assumption, we obtain

$$||\partial_s^\mu\partial_{n^{\vartheta+1}}u^{m+1}||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{E}^{m+1}}+C\mathcal{E}^m\sqrt{\mathcal{E}^{m+1}}\le C\sqrt{\mathcal{E}^{m+1}},$$

where we recall the smallness assumption on $\mathcal{E}^m$, specially $\mathcal{E}^m\le 1$. This finishes the proof of the lemma. □

Lemma 3.6 *If $|\mu|+r+2s\le 2k+1$,*

$$||\partial_s^\mu\partial_{n^r}u^{m+1}||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^{m+1}}. \tag{3.90}$$

Proof. We prove the claim by induction in r. In case $r=1$ the claim is obvious from the definition of $\mathcal{E}^m$. Let the claim be true for all $r\leq\vartheta$ for some $1<\vartheta<2k+1$. We have to prove the claim for $r=\vartheta+1$, i.e.

$$||\partial_s^\mu\partial_{n^{\vartheta+1}}u^{m+1}||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^{m+1}},$$

if $|\mu|+2s+\vartheta\leq 2k$. Let (μ,s) be such a pair of indices. Then

$$\begin{aligned}
&\partial_s^\mu\partial_{n^{\vartheta+1}}u^{m+1}=\partial_s^\mu\partial_{n^{\vartheta-1}}u_{nn}^{m+1}\\
&=\partial_s^\mu\partial_{n^{\vartheta-1}}\Big\{\big(u_t^{m+1}-\Delta_{x'}u^{m+1}+B_{\rho^m}\cdot\nabla_{x'}u_n^{m+1}+c_{\rho^m}u_n^{m+1}\big)a_{\rho^m}^{-1}\Big\}\\
&=\sum_{\mu',s'}\sum_{w=0}^{\vartheta-1}C(\mu',s',w)\Big\{\partial_{s'}^{\mu'}\partial_{n^w}u_t^{m+1}-\partial_{s'}^{\mu'}\partial_{n^w}\Delta_{x'}u^{m+1}+\\
&+\sum_{\mu'',s''}\sum_{p=0}^{w}C(\mu'',s'',p)\partial_{s''}^{\mu''}\partial_{n^p}\big(B_{\rho^m}\big)\cdot\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}\nabla_{x'}u^{m+1}+\\
&+\sum_{\mu'',s''}\sum_{p=0}^{w}C(\mu'',s'',p)\partial_{s''}^{\mu''}\partial_{n^p}\big(c_{\rho^m}\big)\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}u^{m+1}\Big\}\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big).
\end{aligned}\tag{3.91}$$

We analyze separately the case when $|\mu'|+w+2s'\leq k$ and $|\mu'|+w+2s'\geq k$.
Case 1. In the case $|\mu'|+w+2s'\geq k$ note that

$$|\mu-\mu'|+2(s-s')+\vartheta-w\leq 2k-k=k.$$

By the Sobolev inequality and Lemma 3.4, we have

$$||\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big)||_{L^\infty(\Omega)}\leq C\sqrt{\mathcal{E}^m}.$$

If $(\mu'',s'',p)\leq(\mu',s',w)$, $|\mu''|+p+2s''\leq k$ and

$$||\partial_{s''}^{\mu''}\partial_{n^p}\big(c_{\rho^m}\big)||_{L^\infty(\Omega)}\leq||\partial_{s''}^{\mu''}\partial_{n^p}\big(c_{\rho^m}\big)||_{H^{n/2+1}(\Omega)}\leq C\sqrt{\mathcal{E}^m},$$

where we have used the Sobolev inequality and Lemma 3.4 respectively. Similarly $||\partial_{s''}^{\mu''}\partial_{n^p}\big(B_{\rho^m}\big)||_{L^\infty(\Omega)}\leq C\sqrt{\mathcal{E}^m}$. This implies

$$\begin{aligned}
&\int_\Omega|\partial_{s''}^{\mu''}\partial_{n^p}\big(B_{\rho^m}\big)|^2|(\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}\nabla_{x'}u^{m+1})|^2(\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big))^2\\
&\leq C||\partial_{s''}^{\mu''}\partial_{n^p}\big(B_{\rho^m}\big)||_{L^\infty(\Omega)}^2||\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}\nabla_{x'}u^{m+1}||_{L^2(\Omega)}^2\times\\
&\quad\times||\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big)||_{L^\infty(\Omega)}^2\leq C(\mathcal{E}^m)^2\mathcal{E}^{m+1},
\end{aligned}$$

where we have used the inductive hypothesis to deduce

$$||\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}\nabla_{x'}u^{m+1}||_{L^2(\Omega)}^2 \leq C\mathcal{E}^{m+1}.$$

Analogously

$$\int_\Omega (\partial_{s''}^{\mu''}\partial_{n^p}\big(c_{\rho^m}\big))^2(\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}u^{m+1})^2(\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big))^2$$
$$\leq C(\mathcal{E}^m)^2\mathcal{E}^{m+1}$$

If on the other hand $|\mu''|+p+2s''>k$, we use the Sobolev inequality and Lemma 3.5 to get

$$||\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}u^{m+1}||_{L^\infty(\Omega)} \leq ||\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}u^{m+1}||_{H^{n/2+1}(\Omega)} \leq C\sqrt{\mathcal{E}^{m+1}},$$

where we note that

$$|\mu'-\mu''|+(w-p+1)+n/2+1+2(s'-s'')\leq k+n/2+1,$$

so that Lemma 3.5 is applicable. In analogous fashion it follows

$$||\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}\nabla_{x'}u^{m+1}||_{L^\infty(\Omega)} \leq C\sqrt{\mathcal{E}^{m+1}}.$$

We also note that

$$\int_\Omega \Big(\partial_{s'}^{\mu'}\partial_{n^w}\partial u_t^{m+1}-\partial_{s'}^{\mu'}\partial_{n^w}\Delta_{x'}u^{m+1}\Big)^2(\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big))^2$$
$$\leq ||\partial_{s'}^{\mu'}\partial_{n^w}u_t^{m+1}-\partial_{s'}^{\mu'}\partial_{n^w}\Delta_{x'}u^{m+1}||_{L^2(\Omega)}^2||\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big)||_{L^\infty(\Omega)}^2$$
$$\leq C\mathcal{E}^{m+1}\mathcal{E}^m \leq C\mathcal{E}^{m+1}.$$

Observe that we have used the inductive hypothesis in the last inequality above. This completes the first case.

Case 2. In the case $|\mu'|+w+2s'\leq k$, by the Sobolev inequality and Lemmas 3.4 and 3.5,

$$||\partial_{s''}^{\mu''}\partial_{n^p}\big(B_{\rho^m}\big)\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}\nabla_{x'}u^{m+1}||_{L^\infty(\Omega)} \leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{E}^{m+1}},$$

and $||\partial_{s''}^{\mu''}\partial_{n^p}\big(c_{\rho^m}\big)\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}u^{m+1}||_{L^\infty(\Omega)} \leq C\sqrt{\mathcal{E}^{m+1}}$ for $(\mu'',s'',p)\leq(\mu',s',w)$. By Lemma 3.4, $||\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big)||_{L^2(\Omega)} \leq C\sqrt{\mathcal{E}^m}+C$. We also note

$$\int_\Omega \Big(\partial_{s'}^{\mu'}\partial_{n^w}u_t^{m+1}-\partial_{s'}^{\mu'}\partial_{n^w}\Delta_{x'}u^{m+1}\Big)^2(\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big))^2$$
$$\leq ||\partial_{s'}^{\mu'}\partial_{n^w}u_t^{m+1}-\partial_{s'}^{\mu'}\partial_{n^w}\Delta_{x'}u^{m+1}||_{L^\infty(\Omega)}^2||\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big)||_{L^2(\Omega)}^2$$
$$\leq C\mathcal{E}^{m+1}\mathcal{E}^m+C\mathcal{E}^{m+1} \leq C\mathcal{E}^{m+1}.$$

By the Sobolev inequality and Lemma 3.5,

$$||\partial_{s'}^{\mu'}\partial_{n^w}u_t^{m+1}-\partial_{s'}^{\mu'}\partial_{n^w}\Delta_{x'}u^{m+1}||_{L^\infty(\Omega)}^2\leq C\mathcal{E}^{m+1}.$$

We combine the above estimates to conclude $||\partial_s^\mu\partial_{n^{\vartheta+1}}u^{m+1}||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^{m+1}}$ and this completes the second case and finishes the proof of the lemma. □

4 Energy estimates

Lemma 4.1 *Let K and $\theta\leq\theta_0$ be given as in Lemma 3.2. There exists $0<L\leq\theta$ and $\mathcal{T}^\epsilon$ such that if*

$$\mathcal{E}_\epsilon(u_0^\epsilon,\rho_0^\epsilon)+\Big|\int_{\mathbb{T}^{n-1}}\rho_0^\epsilon-\int_\Omega u_0^\epsilon(1+\phi'\rho_0^\epsilon)\Big|\leq\frac{L}{2}$$

and for some $m\in\mathbb{N}$

$$\sup_{0\leq t\leq\mathcal{T}^\epsilon}\mathcal{E}^m(t)+\int_0^{\mathcal{T}^\epsilon}\mathcal{D}^m(\tau)\,d\tau\leq L,$$

$$||\nabla\rho^{m-1}||_\infty\leq 1,\quad \sum_{p=0}^k||\partial_p\rho^m||_2\leq K\sqrt{\mathcal{E}^m}+\theta,$$

then

$$\sup_{0\leq t\leq\mathcal{T}^\epsilon}\mathcal{E}^{m+1}(t)+\int_0^{\mathcal{T}^\epsilon}\mathcal{D}^{m+1}(\tau)\,d\tau\leq L.$$

Proof. With the preparation from Chapter 2, we are ready to estimate RHS of (3.68) term by term. Note that the assumptions of Lemma 3.2 are fulfilled and we are thus able to use Lemmas 3.3 - 3.6 in the forthcoming estimates. Let (μ,s) be an arbitrary, but fixed pair of indices satisfying $|\mu|+2s\leq 2k$.

Term $\int_\Omega P_{\mu,s}^m$: Recall that $P_{\mu,s}^m=P(\partial_s^\mu u^{m+1},\rho^m,f^m)$ is given by (3.69), where P is given by (2.38) and f^m by (3.63). Thus, combining (2.38) and (3.63) we can estimate the first term on RHS of $\int_\Omega P_{\mu,s}^m$:

$$\begin{aligned}&\Big|\int_\Omega\Big\{\partial_s^\mu(a_{\rho^m}u_{nn}^{m+1})-a_{\rho^m}\partial_s^\mu u_{nn}^{m+1}\Big\}\partial_s^\mu u^{m+1}\Big|\\&\leq C\sum_{(|\mu'|,s')\neq(0,0)}\Big|\int_\Omega\partial_{s'}^{\mu'}a_{\rho^m}\partial_{s-s'}^{\mu-\mu'}u_{nn}^{m+1}\partial_s^\mu u^{m+1}\Big|=C\sum_{\substack{|\mu'|+2s'\leq k\\(|\mu'|,s')\neq(0,0)}}+C\sum_{|\mu'|+2s'>k}\\&\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{D}^{m+1}}\leq C\sqrt{\mathcal{E}^m}\mathcal{D}^{m+1}.\end{aligned}\tag{4.92}$$

In the first sum, observe that $||\partial_{s'}^{\mu'} a_{\rho^m}||_{L^\infty(\Omega)} \le C\sqrt{\mathcal{E}^m}$ for $|\mu'|+2s' \le k$, by the Sobolev inequality and Lemma 3.3. In the second sum observe that $||\partial_{s-s'}^{\mu-\mu'} u_{nn}^{m+1}||_{L^\infty(\Omega)} \le C\sqrt{\mathcal{D}^{m+1}}$, by the Sobolev inequality and Lemma 3.6. By glancing at (2.38) and (3.63) the second term in the expression $\int_\Omega P_{\mu,s}^m$ is given by $\int_\Omega \partial_s^\mu (B_{\rho^m} \cdot \nabla_{x'} u_n^{m+1}) \partial_s^\mu u^{m+1}$. It is estimated in a completely analogous way and is bounded by $C\sqrt{\mathcal{E}^m}\mathcal{D}^{m+1}$ again. By (3.69), the third term on RHS (2.38) renders the third term in $\int_\Omega P_{\mu,s}^m$. We have

$$\begin{aligned} &\Big|\int_\Omega \partial_s^\mu \left(c_{\rho^m} u_n^{m+1}\right) \partial_s^\mu u^{m+1}\Big| \le C \sum_{\mu',s'} \Big|\int_\Omega \partial_{s'}^{\mu'} c_{\rho^m} \partial_{s-s'}^{\mu-\mu'} u_n^{m+1} \partial_s^\mu u^{m+1}\Big| \\ &= C \sum_{|\mu'|+2s' \le k} + C \sum_{|\mu'|+2s' > k} \\ &\le C\sqrt{\mathcal{E}^m}\mathcal{D}^{m+1} + C\sqrt{\mathcal{D}^m}\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^{m+1}}. \end{aligned} \tag{4.93}$$

In the first sum we estimate $||\partial_s^\mu c_{\rho^m}||_\infty$ like above (since $|\mu'|+2s' \le k$). In the second sum, for $|\mu'|+2s' \ge k$, $||\partial_{s-s'}^{\mu-\mu'} u_n^{m+1}||_{L^\infty(\Omega)} \le C\sqrt{\mathcal{E}^{m+1}}$ by the Sobolev inequality and Lemma 3.6. By Lemma 3.3, $||\partial_{s'}^{\mu'} c_{\rho^m}||_{L^2} \le C\sqrt{\mathcal{D}^m}$. Finally, the fourth term of $\int_\Omega P_{\mu,s}^m$ (again use (2.38) to identify the fourth term and the equations (3.69) and (3.63) to plug in the appropriate values), is estimated by using the Cauchy-Schwarz inequality

$$\begin{aligned} \Big|\int_\Omega (a_{\rho^m})_n \partial_s^\mu u_n^{m+1} \partial_s^\mu u^{m+1}\Big| &\le ||(a_{\rho^m})_n||_{L^\infty(\Omega)} ||\partial_s^\mu u_n^{m+1}||_{L^2(\Omega)} ||\partial_s^\mu u^{m+1}||_{L^2(\Omega)} \\ &\le C\sqrt{D^m}\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^{m+1}}. \end{aligned} \tag{4.94}$$

Term $\int_{\mathbb{T}^{n-1}} Q_{\mu,s}^m$: We now proceed with the estimates for the expression $\int_{\mathbb{T}^{n-1}} Q_{\mu,s}^m$, where $Q_{\mu,s}^m$ is given by (3.69), where Q is defined by (2.40), $g_{\mu,s}^m$ is given by (3.64) and $h_{\mu,s}^m$ is given by (3.66). The first two terms of $\int_{\mathbb{T}^{n-1}} Q_{\mu,s}^m$ are the cross-terms and they deserve special attention. For any $\eta > 0$,

$$\begin{aligned} &\Big|\int_{\mathbb{T}^{n-1}} \partial_s^\mu \nabla \rho_t^{m+1} \left\{\partial_s^\mu \nabla \rho^m \langle \rho^m \rangle^{-1} - \partial_s^\mu \nabla \rho^{m+1} \langle \rho^m \rangle^{-1}\right\}\Big| \\ &\le \eta ||\partial_s^\mu \nabla \rho^{m+1}||_2^2 + \frac{C}{\eta}\Big(||\partial_s^\mu \nabla \rho^m||_2^2 + ||\partial_s^\mu \nabla \rho^{m+1}||_2^2\Big) \le \eta \mathcal{D}^{m+1} + \frac{C}{\eta}(\mathcal{E}^m + \mathcal{E}^{m+1}). \end{aligned} \tag{4.95}$$

$$\Big|\epsilon\int_{\mathbb{T}^{n-1}}\partial_s^\mu\nabla\Delta\rho_t^{m+1}\cdot\Big\{\Delta\partial_s^\mu\nabla\rho^m\langle\rho^m\rangle^{-1}-\Delta\partial_s^\mu\nabla\rho^{m+1}\langle\rho^m\rangle^{-1}\Big\}\Big|$$
$$\le\eta\epsilon||\partial_s^\mu\Delta\nabla\rho_t^{m+1}||_2^2+\frac{\epsilon C}{\eta}\Big(||\partial_s^\mu\Delta\nabla\rho^m||_2^2+||\partial_s^\mu\Delta\nabla\rho^{m+1}||_2^2\Big)\qquad(4.96)$$
$$\le\eta\mathcal{D}^{m+1}+\frac{C}{\eta}(\mathcal{E}^m+\mathcal{E}^{m+1}).$$

Observe that the constant C does not depend on ϵ. The third term in $\int_{\mathbb{T}^{n-1}}Q_{\mu,s}^m$ is given by (2.40) and (3.69). Note that $\partial_t\big(\langle\rho^m\rangle^{-1}\big)=\frac{2\nabla\rho^m\nabla\rho_t^m}{\langle\rho^m\rangle^3}$. By Lemma 3.3, we conclude $||\langle\rho^m\rangle_t^{-1}||_\infty\le C\mathcal{D}^m$. We then obtain

$$\Big|\int_{\mathbb{T}^{n-1}}\Big\{|\partial_s^\mu\nabla\rho^{m+1}|^2\langle\rho^m\rangle_t^{-1}+\epsilon|\partial_s^\mu\Delta\nabla\rho^{m+1}|^2\langle\rho^m\rangle_t^{-1}\Big\}\Big|$$
$$\le C\mathcal{D}^m\big(||\partial_s^\mu\nabla\rho_t^{m+1}||_2^2+\epsilon||\partial_s^\mu\Delta\nabla\rho^{m+1}||_2^2\big)\le C\mathcal{E}^{m+1}\mathcal{D}^m.\qquad(4.97)$$

To estimate the fourth term of $\int_{\mathbb{T}^{n-1}}Q_{\mu,s}^m$ (which is obtained as the fourth term of (2.40) together with the definition (3.69)), we use the Cauchy-Schwarz inequality to get

$$\Big|\int_{\mathbb{T}^{n-1}}\partial_s^\mu\rho_t^{m+1}\partial_s^\mu\nabla\rho^m\cdot\nabla(\langle\rho^m\rangle^{-1})\Big|\le||\partial_s^\mu\rho_t^{m+1}||_2||\partial_s^\mu\nabla\rho^m||_2||\nabla(\langle\rho^m\rangle^{-1})||_\infty\le$$
$$C\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}\le C\sqrt{\mathcal{E}^m}(\mathcal{D}^m+\mathcal{D}^{m+1}).\qquad(4.98)$$

Analogously, the fifth term in $\int_{\mathbb{T}^{n-1}}Q_{\mu,s}^m$ is estimated as follows:

$$\epsilon\Big|\int_{\mathbb{T}^{n-1}}\Delta\nabla\partial_s^\mu\rho_t^{m+1}\cdot\nabla(\langle\rho^m\rangle^{-1})\Delta\partial_s^\mu\rho^m\Big|\le$$
$$||\sqrt{\epsilon}\Delta\nabla\partial_s^\mu\rho_t^{m+1}||_2||\nabla(\langle\rho^m\rangle^{-1})||_\infty||\sqrt{\epsilon}\Delta\partial_s^\mu\rho^m||_2\le C\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{D}^m}\sqrt{\mathcal{E}^m}$$
$$\le C\sqrt{\mathcal{E}^m}(\mathcal{D}^m+\mathcal{D}^{m+1}).\qquad(4.99)$$

We first note that the sixth term of (2.40) contains g. Note that in $\int_{\mathbb{T}^{n-1}}Q_{\mu,s}^m$ $g=g_{\mu,s}^m$, where $g_{\mu,s}^m$ is defined by (3.64). We shall first estimate $||g_{\mu,s}^m||_2$ and $||\sqrt{\epsilon}\nabla g_{\mu,s}^m||_2$ and then use the Cauchy-Schwarz inequality. For any $|\mu'|+s'<|\mu|+s$, we have

$$||\partial_{s'}^{\mu'}\Delta\rho^m\partial_{s-s'}^{\mu-\mu'}(\langle\rho^m\rangle^{-1})||_2+||\partial_s^\mu(\nabla\rho^m\cdot\nabla(\langle\rho^m\rangle^{-1}))||_2\le C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}.$$

The inequality follows by estimating the term with smaller order space-derivatives in L^∞-norm, which can then be estimated by the Sobolev inequality and Lemma 3.3. Similarly, recalling (3.67):

$$||\sqrt{\epsilon}\nabla\big(\partial_{s'}^{\mu'}\Delta\rho^m\partial_{s-s'}^{\mu-\mu'}(\langle\rho^m\rangle^{-1})\big)||_2+||\sqrt{\epsilon}\nabla\big(\partial_s^\mu(\nabla\rho^m\cdot\nabla(\langle\rho^m\rangle^{-1}))\big)||_2\le C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}.$$

Therefore $||g^m_{\mu,s}||_2 + ||\sqrt{\epsilon}\nabla g^m_{\mu,s}||_2 \le C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}$ and we can bound the sixth term in $\int_{\mathbb{T}^{n-1}} Q^m_{\mu,s}$ by

$$\begin{aligned}&\Big|\int_{\mathbb{T}^{n-1}} \big(\partial^\mu_s \rho^{m+1}_t + \epsilon \Delta^2 \rho^{m+1}_t\big) g^m_{\mu,s}\Big| \le ||\partial^\mu_s \rho^{m+1}_t||_2 ||g^m_{\mu,s}||_2 \\ &+||\sqrt{\epsilon}\Delta\nabla \rho^{m+1}_t||_2 ||\sqrt{\epsilon}\nabla g^m_{\mu,s}||_2 \le C\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m} \le C\sqrt{\mathcal{E}^m}(\mathcal{D}^m + \mathcal{D}^{m+1}).\end{aligned} \tag{4.100}$$

The last term in $\int_{\mathbb{T}^{n-1}} Q^m_{\mu,s}$ is extracted from the last term of (2.40), which contains h. By (3.69), $h = h^m_{\mu,s}$ where $h^m_{\mu,s}$ is given by (3.66). For the notational simplicity, we set $h^m_{\mu,s} =: h_1 + \epsilon h_2$, where

$$h_1 := \sum_{\substack{|\mu'|+s' \\ <|\mu|+s}} C^{\mu'}_{s'} \partial^{\mu'}_{s'} \rho^{m+1}_t \partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-2}),\ h_2 := \sum_{\substack{|\mu'|+s' \\ <|\mu|+s}} C^{\mu'}_{s'} \partial^{\mu'}_{s'} \Delta^2 \rho^{m+1}_t \partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-2}). \tag{4.101}$$

Note that for $|\mu'| + s' < |\mu| + s$, we have

$$\begin{aligned}&||\langle\rho^m\rangle^2 \partial^{\mu'}_{s'} \rho^{m+1}_t \partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-2})||_2 \\ &\le ||\langle\rho^m\rangle^2||_\infty ||\partial^{\mu'}_{s'} \rho^{m+1}_t \partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-2})||_2 \le C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}.\end{aligned} \tag{4.102}$$

Here, if $|\mu'| + 2s' \le k$ then $||\partial^{\mu'}_{s'} \rho^{m+1}_t||_\infty \le C\sqrt{\mathcal{E}^{m+1}}$ and if $|\mu'| + 2s' \ge k$ then $||\partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-2})||_\infty \le C\sqrt{\mathcal{E}^m}$. We use the Sobolev inequality and Lemma 3.3 to conclude the estimate. This implies $||h_1||_2 \le C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}$. Note that in similar fashion, for $|\mu'| + s' < |\mu| + s$,

$$||\sqrt{\epsilon}\partial^{\mu'}_{s'} \Delta^2 \rho^{m+1}_t \partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-2})||_2 \le C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}.$$

In other words $||\sqrt{\epsilon}h_2||_2 \le C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}$. From the proof of part (b) of Lemma 3.3, we deduce $||\partial^\mu_s \kappa^m||_2 \le C\sqrt{\mathcal{D}^{m+1}}$ and also $||\sqrt{\epsilon}\partial^\mu_s \kappa^m||_2 \le C\sqrt{\mathcal{D}^{m+1}}$ (recall here (3.61)). Thus the last term of $\int_{\mathbb{T}^{n-1}} Q^m_{\mu,s}$ is bounded by

$$\begin{aligned}&\Big|\int_{\mathbb{T}^{n-1}} \langle\rho^m\rangle^2 h^m_{\mu,s} \partial^\mu_s \kappa^m\Big| \le C||h_1||_2 ||\partial^\mu_s \kappa^m||_2 + C||\sqrt{\epsilon}h_2||_2 ||\sqrt{\epsilon}\partial^\mu_s \kappa^m||_2 \\ &\le C\sqrt{\mathcal{E}^m}\mathcal{D}^{m+1}.\end{aligned} \tag{4.103}$$

Term $\int_\Omega R^m_{\mu,s}$: Note that $R^m_{\mu,s}$ is given by (3.70) where $f^m_{\mu,s}$, is defined by (3.63). Our first task is to estimate the first term of $\int_\Omega R^m_{\mu,s}$, namely

$\int_\Omega (f^m_{\mu,s})^2$. Observe that

$$\int_\Omega (f^m_{\mu,s})^2 \le C \sum_{\substack{(\mu',s')\\ \neq(0,0)}} \int_\Omega (\partial^{\mu'}_{s'} a_{\rho^m})^2 (\partial^{\mu-\mu'}_{s-s'} u^{m+1}_{nn})^2$$
$$+C \sum_{(\mu',s')} \int_\Omega |\partial^{\mu'}_{s'} B_{\rho^m}|^2 |\partial^{\mu-\mu'}_{s-s'} \nabla_{x'} u^{m+1}_n|^2 + C \sum_{\mu',s'} \int_\Omega (\partial^{\mu'}_{s'} c_{\rho^m})^2 (\partial^{\mu-\mu'}_{s-s'} u^{m+1}_n)^2 \tag{4.104}$$

If $|\mu'|+2s'\le k$, by Lemma 3.4

$$||\partial^{\mu'}_{s'} a_{\rho^m}||_{L^\infty(\Omega)} + ||\partial^{\mu'}_{s'} B_{\rho^m}||_{L^\infty(\Omega)} + ||\partial^{\mu'}_{s'} c_{\rho^m}||_{L^\infty(\Omega)} \le C\sqrt{\mathcal{E}^m}.$$

Thus for $|\mu'|+2s'\le k$ and $(\mu',s')\neq(0,0)$, RHS of (4.104) is bounded by $C\mathcal{E}^m\mathcal{D}^{m+1}$. If $|\mu'|+2s'>k$, then $|\mu-\mu'|+2(s-s')\le k-1$ and from Lemma 3.6

$$||\partial^{\mu-\mu'}_{s-s'} u^{m+1}_{nn}||_{L^\infty(\Omega)} + ||\partial^{\mu-\mu'}_{s-s'} \nabla_{x'} u^{m+1}_n||_{L^\infty(\Omega)} + ||\partial^{\mu-\mu'}_{s-s'} u^{m+1}_n||_{L^\infty(\Omega)} \le C\sqrt{\mathcal{E}^{m+1}}.$$

In addition to this, for such (μ',s'), we use Lemma 3.4 to conclude

$$||\partial^{\mu'}_{s'} a_{\rho^m}||_{L^2(\Omega)} + ||\partial^{\mu'}_{s'} B_{\rho^m}||_{L^2(\Omega)} + ||\partial^{\mu'}_{s'} c_{\rho^m}||_{L^2(\Omega)} \le C\sqrt{\mathcal{D}^m}.$$

Thus, for every $\lambda>0$

$$\int_\Omega (\partial^{\mu'}_{s'} a_{\rho^m})^2 (\partial^{\mu-\mu'}_{s-s'} u^{m+1}_{nn})^2 \le ||\partial^{\mu'}_{s'} a_{\rho^m}||^2_{L^2(\Omega)} ||\partial^{\mu-\mu'}_{s-s'} u^{m+1}_{nn}||^2_{L^\infty(\Omega)} \le CD^m\mathcal{E}^{m+1}.$$

Analogously,

$$\int_\Omega \left\{ |\partial^{\mu'}_{s'} B_{\rho^m}|^2 |\partial^{\mu-\mu'}_{s-s'} \nabla_{x'} u^{m+1}_n|^2 + (\partial^{\mu'}_{s'} c_{\rho^m})^2 (\partial^{\mu-\mu'}_{s-s'} u^{m+1}_n)^2 \right\} \le CD^m\mathcal{E}^{m+1},$$

for all $(\mu',s')\le(\mu,s)$ satisfying $|\mu'|+2s'>k$. Combining the estimates for $|\mu'|+2s'\le k$ and $|\mu'|+2s'>k$, we obtain

$$\int_\Omega (f^m_{\mu,s})^2 \le C\mathcal{E}^m\mathcal{D}^{m+1} + CD^m\mathcal{E}^{m+1} \tag{4.105}$$

The second term of the expression $\int_\Omega R^m_{\mu,s}$ (the second term on RHS of (2.39) and (3.70)) is bounded by:

$$\left| \int_\Omega (\partial^\mu_s u^{m+1}_n)^2 (a_{\rho^m})_t \right| \le ||(a_{\rho^m})_t||_{L^\infty(\Omega)} ||\partial^\mu_s u^{m+1}_n||^2_{L^2(\Omega)} \le C\sqrt{\mathcal{E}^m}\mathcal{D}^{m+1} \tag{4.106}$$

and similarly the third term $\left|\int_\Omega \partial_s^\mu u_t^{m+1}\partial_s^\mu u_n^{m+1}(a_{\rho^m})_n\right| \leq C\sqrt{\mathcal{E}^m}\mathcal{D}^{m+1}$. Note further that for the fourth term in $\int_\Omega R_{\mu,s}^m$ (use (2.39) and (3.70)), by Lemma 3.3,

$$\begin{aligned}&\Big|\int_\Omega \nabla_{x'}\partial_s^\mu u_n^{m+1}\cdot\nabla_{x'}a_{\rho^m}\partial_s^\mu u_n^{m+1}\Big|\\ &\leq C||\nabla_{x'}\partial_s^\mu u_n^{m+1}||_{L^2(\Omega)}||\nabla_{x'}a_{\rho^m}||_{L^\infty(\Omega)}||\partial_s^\mu u_n^{m+1}||_{L^2(\Omega)}\leq C\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}.\end{aligned} \tag{4.107}$$

By Lemma 3.3, the last term in $\int_\Omega R_{\mu,s}^m$ (last term on RHS of (2.39) and (3.70)) is bounded by:

$$\begin{aligned}&\Big|\int_\Omega \Delta_{x'}\partial_s^\mu u^{m+1}(a_{\rho^m})_n\partial_s^\mu u_n^{m+1}\Big|\\ &\leq ||\Delta_{x'}\partial_s^\mu u^{m+1}||_{L^2(\Omega)}||(a_{\rho^m})_n||_{L^\infty(\Omega)}||\partial_s^\mu u_n^{m+1}||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^m}\mathcal{D}^{m+1}.\end{aligned} \tag{4.108}$$

Term $\int_{\mathbb{T}^{n-1}} S_{\mu,s}^m$: Note that $S_{\mu,s}^m$ is given by (3.70) where S is given by (2.41), $g_{\mu,s}^m$ by (3.64) and $h_{\mu,s}^m$ by (3.66). The first two terms on RHS of (2.41) are the cross-terms and in order to estimate them we shall exploit the part (c) of Lemma 3.3. It turns out that the constants on the right-hand side will depend on ϵ. Note that for any $\eta,\lambda > 0$

$$\begin{aligned}&\Big|\int_{\mathbb{T}^{n-1}} \partial_s^\mu\nabla\rho_t^{m+1}\Big(\partial_s^\mu\nabla\rho_t^m\langle\rho^m\rangle^{-1}-\partial_s^\mu\nabla\rho_t^{m+1}\langle\rho^m\rangle^{-1}\Big)\Big|\\ &\leq \frac{C}{\eta}||\partial_s^\mu\nabla\rho_t^{m+1}||_2^2+\eta||\partial_s^\mu\nabla\rho_t^m||_2^2\leq\frac{C}{\eta\lambda\epsilon^4}\mathcal{E}^{m+1}+\frac{C\lambda}{\eta}\mathcal{D}^{m+1}+\eta\mathcal{D}^m.\end{aligned} \tag{4.109}$$

In the last estimate we have used the estimate (3.82). Similarly,

$$\begin{aligned}&\Big|\epsilon\int_{\mathbb{T}^{n-1}} \partial_s^\mu\Delta\nabla\rho_t^{m+1}\Big(\partial_s^\mu\Delta\nabla\rho_t^m\langle\rho^m\rangle^{-1}-\partial_s^\mu\Delta\nabla\rho_t^{m+1}\langle\rho^m\rangle^{-1}\Big)\Big|\\ &\leq \frac{\epsilon C}{\eta}||\partial_s^\mu\Delta\nabla\rho_t^{m+1}||_2^2+\eta\epsilon||\partial_s^\mu\Delta\nabla\rho_t^m||_2^2\leq\frac{C}{\eta\lambda\epsilon^3}\mathcal{E}^{m+1}+\frac{C\epsilon\lambda}{\eta}\mathcal{D}^{m+1}+\eta\mathcal{D}^m.\end{aligned} \tag{4.110}$$

The third term of $\int_{\mathbb{T}^{n-1}} S_{\mu,s}^m$ is given by the third term on RHS of (2.41) and (3.70):

$$\begin{aligned}&\Big|\int_{\mathbb{T}^{n-1}} \partial_s^\mu\rho_t^{m+1}\partial_s^\mu\nabla\rho_t^m\cdot\nabla(\langle\rho^m\rangle^{-1})\Big|\leq\\ &||\partial_s^\mu\rho_t^{m+1}||_2||\partial_s^\mu\nabla\rho_t^m||_2||\nabla(\langle\rho^m\rangle^{-1})||_\infty\leq C\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{D}^m}\sqrt{\mathcal{E}^m}\\ &C\sqrt{\mathcal{E}^m}(\mathcal{D}^m+\mathcal{D}^{m+1}).\end{aligned} \tag{4.111}$$

Similarly, the fourth term in $\int_{\mathbb{T}^{n-1}} S^m_{\mu,s}$ (given by the fourth term on RHS of (2.41) and (3.70)) is bounded by:

$$\epsilon\Big|\int_{\mathbb{T}^{n-1}} \Delta\partial^\mu_s\rho^m_t \Delta\nabla\partial^\mu_s\rho^{m+1}_t\cdot\nabla(\langle\rho^m\rangle^{-1})\Big|\leq \\ ||\sqrt{\epsilon}\Delta\partial^\mu_s\rho^m_t||_2||\sqrt{\epsilon}\Delta\nabla\partial^\mu_s\rho^{m+1}_t||_2||\nabla(\langle\rho^m\rangle^{-1})||_\infty\leq C\sqrt{\mathcal{D}^m}\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^m} \\ \leq C\sqrt{\mathcal{E}^m}(\mathcal{D}^m+\mathcal{D}^{m+1}). \tag{4.112}$$

Note that the fifth term in $\int_{\mathbb{T}^{n-1}} S^m_{\mu,s}$, by (2.41) and (3.70)) involves the function $g^m_{\mu,s}$, where $g^m_{\mu,s}$ is given by (3.64). The crucial step in estimating this term in $\int_{\mathbb{T}^{n-1}} S^m$, is to observe that $||(g^m_{\mu,s})_t||_2\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}$. This is proved by first differentiating $g^m_{\mu,s}$ with respect to t, and then in each product estimating the terms with lower order space derivatives in L^∞-norm and the other one in L^2-norm. The same method applies to show $||\sqrt{\epsilon}\nabla(g^m_{\mu,s})_t||_2\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}$. Also observe that

$$||\Delta\partial^\mu_s\rho^m\langle\rho^m\rangle^{-1}_t||_2\leq||\Delta\partial^\mu_s\rho^m||_2||\langle\rho^m\rangle^{-1}_t||_\infty\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}.$$

Analogous proof shows that

$$||\sqrt{\epsilon}\nabla\big(\Delta\partial^\mu_s\rho^m\langle\rho^m\rangle^{-1}_t\big)||_2\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}.$$

Using the above inequalitites and the Cauchy-Schwarz inequality we establish

$$\Big|\int_{\mathbb{T}^{n-1}}\partial^\mu_s\rho^{m+1}_t\Big(\Delta\partial^\mu_s\rho^m\langle\rho^m\rangle^{-1}_t+(g^m_{\mu,s})_t\Big)\Big|\leq||\partial^\mu_s\rho^{m+1}_t||_2||\Delta\partial^\mu_s\rho^m\langle\rho^m\rangle^{-1}_t+(g^m_{\mu,s})_t||_2 \\ \leq C\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}\leq C\sqrt{\mathcal{E}^m}\big(\mathcal{D}^m+\mathcal{D}^{m+1}\big). \tag{4.113}$$

Integrating by parts and using the analogous argument as in (4.113) we get

$$\epsilon\Big|\int_{\mathbb{T}^{n-1}}\partial^\mu_s\Delta^2\rho^{m+1}_t\Big(\Delta\partial^\mu_s\rho^m\langle\rho^m\rangle^{-1}_t+(g^m_{\mu,s})_t\Big)\Big|=\epsilon\Big|\int_{\mathbb{T}^{n-1}}\partial^\mu_s\Delta\nabla\rho^{m+1}_t\cdot\nabla\Big(\Delta\partial^\mu_s\rho^m\langle\rho^m\rangle^{-1}_t+(g^m_{\mu,s})_t\Big)\Big| \\ \leq||\sqrt{\epsilon}\partial^\mu_s\Delta\nabla\rho^{m+1}_t||_2||\sqrt{\epsilon}\nabla\Big(\Delta\partial^\mu_s\rho^m\langle\rho^m\rangle^{-1}_t+(g^m_{\mu,s})_t\Big)||_2 \\ \leq C\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}\leq C\sqrt{\mathcal{E}^m}\big(\mathcal{D}^m+\mathcal{D}^{m+1}\big) \tag{4.114}$$

The sixth and the last term in $\int_{\mathbb{T}^{n-1}} S^m$ (given through the last term on RHS of (2.41) and (3.70)) involves the function $h^m_{\mu,s}$, where $h^m_{\mu,s}$ is given by (3.66). Since $\partial^\mu_s\kappa^m_t=\nabla\cdot\partial^\mu_s\big(\nabla\rho^m\langle\rho^m\rangle^{-1}_t\big)$, we can integrate by parts to obtain

$$\int_{\mathbb{T}^{n-1}}\langle\rho^m\rangle^2 h^m_{\mu,s}\partial^\mu_s\kappa^m_t=-\int_{\mathbb{T}^{n-1}}\nabla\big(\langle\rho^m\rangle^2 h^m_{\mu,s}\big)\cdot\big(\nabla\rho^m\langle\rho^m\rangle^{-1}\big)_t.$$

We split $h^m_{\mu,s} = h_1 + \epsilon h_2$ as in (4.101). For $|\mu'| + s' < |\mu| + s$

$$\begin{aligned} &||\nabla\big(\langle\rho^m\rangle^2 h_1\big)||_2 \leq ||\nabla(\langle\rho^m\rangle^2)||_\infty ||\partial^{\mu'}_{s'}\rho^{m+1}_t \partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-2})||_2 + \\ &+ ||\langle\rho^m\rangle^2||_\infty \Big(||\partial^{\mu'}_{s'}\nabla\rho^{m+1}_t \partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-2})||_2 + ||\partial^{\mu'}_{s'}\rho^{m+1}_t \partial^{\mu-\mu'}_{s-s'}\nabla(\langle\rho^m\rangle^{-2})||_2\Big) \\ &\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}} + C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}} \leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}. \end{aligned} \tag{4.115}$$

Here we have used $L^2 - L^\infty$ estimated by separating the cases $|\mu'| + 2s' \leq k$ and $|\mu'| + 2s' > k$. Since

$$||\sqrt{\epsilon}\nabla\big(\langle\rho^m\rangle^2 \partial^{\mu'}_{s'}\Delta^2\rho^{m+1}_t \partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-2})\big)||_2 \leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}},$$

we deduce

$$||\sqrt{\epsilon}\nabla(\langle\rho^m\rangle^2 h_2)||_2 \leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}, \tag{4.116}$$

where h_2 is given by (4.101). Similarly, $||\partial^\mu_{s+1}\big(\nabla\rho^m\langle\rho^m\rangle^{-1}\big)||_2 \leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}$ and also $||\sqrt{\epsilon}\partial^\mu_{s+1}\big(\nabla\rho^m\langle\rho^m\rangle^{-1}\big)||_2 \leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}$.

In summary,

$$\begin{aligned} &\Big|\int_{\mathbb{T}^{n-1}} \langle\rho^m\rangle^2 h^m_{\mu,s}\partial^\mu_s\kappa^m_t\Big| \leq ||\nabla\big(\langle\rho^m\rangle^2 h_1\big)||_2 ||\partial^\mu_s\big(\nabla\rho^m\langle\rho^m\rangle^{-1}\big)||_2 + \\ &+ ||\sqrt{\epsilon}\big(\langle\rho^m\rangle^2 h_2\big)||_2 ||\sqrt{\epsilon}\partial^\mu_s\big(\nabla\rho^m\langle\rho^m\rangle^{-1}\big)||_2 \\ &\leq C\mathcal{E}^m\sqrt{\mathcal{D}^m}\sqrt{\mathcal{D}^{m+1}} \leq C\mathcal{E}^m\mathcal{D}^m + C\mathcal{E}^m\mathcal{D}^{m+1}. \end{aligned} \tag{4.117}$$

Term $\int_{\mathbb{T}^{n-1}} T^m_{\mu,s}$: Recall that $T^m_{\mu,s}$ is defined by (3.70) where T is given by (2.42), $G^m_{\mu,s}$ by (3.65) and $h^m_{\mu,s}$ by (3.66). In particular the term A- the first term on RHS of (2.42) is given by (2.43). The first two terms of the expression A are the cross-terms. Using integration by parts, we obtain

$$\begin{aligned} &\Big|\int_{\mathbb{T}^{n-1}} \partial^\mu_s\Delta\rho^{m+1}_t\big(\partial^\mu_s\Delta\rho^m - \partial^\mu_s\Delta\rho^{m+1}\big)\langle\rho^m\rangle^{-1}\Big| = \\ &\Big|\int_{\mathbb{T}^{n-1}} \partial^\mu_s\nabla\rho^{m+1}_t \cdot \nabla\Big(\big(\partial^\mu_s\Delta\rho^m - \partial^\mu_s\Delta\rho^{m+1}\big)\langle\rho^m\rangle^{-1}\Big)\Big| \\ &\leq \lambda||\partial^\mu_s\nabla\rho^{m+1}_t||_2^2 + \frac{C}{\lambda}||\nabla\big(\partial^\mu_s\Delta\rho^m\langle\rho^m\rangle^{-1}\big)||_2^2 + \\ &\frac{C}{\lambda}||\nabla\big(\partial^\mu_s\Delta\rho^{m+1}\langle\rho^m\rangle^{-1}\big)||_2^2 \leq \lambda\mathcal{D}^{m+1} + \frac{C}{\epsilon\lambda}(\mathcal{E}^m + \mathcal{E}^{m+1}), \end{aligned} \tag{4.118}$$

where we note that by the definition of $\mathcal{E}^m$, $||\partial^\mu_s(\nabla\rho^m\langle\rho^m\rangle^{-1})||_{H^3} \leq \frac{C}{\sqrt{\epsilon}}\sqrt{\mathcal{E}^m}$ for $|\mu| + 2s \leq 2k$. Similarly, for the second cross term:

$$\begin{aligned} &\Big|\int_{\mathbb{T}^{n-1}} \partial^\mu_s\Delta\rho^{m+1}_t\Big(\rho^m_i\rho^m_j\big(\partial^\mu_s\rho^m_{ij} - \partial^\mu_s\rho^{m+1}_{ij}\big)\Big)\langle\rho^m\rangle^{-3}\Big| \\ &\qquad \leq \lambda\mathcal{D}^{m+1} + \frac{C}{\epsilon\lambda}(\mathcal{E}^m + \mathcal{E}^{m+1}). \end{aligned} \tag{4.119}$$

The proof of (4.119) relies on the same idea as above; we first integrate by parts and then establish the estimate

$$||\nabla\Big(\rho_i^m\rho_j^m\big(\partial_s^\mu\rho_{ij}^m-\partial_s^\mu\rho_{ij}^{m+1}\big)\langle\rho^m\rangle^{-3}\Big)||_2^2\leq\frac{C}{\epsilon}(\mathcal{E}^m+\mathcal{E}^{m+1}).$$

By $A-$(crossterms) we denote the sum of all the remaining terms in the expression A (recall (2.43) and the fact that $\psi=\rho^m$ in our case). Terms of the form $\langle\rho^m\rangle_i^{-1}$, $\langle\rho^m\rangle_t^{-1}$, $\langle\rho^m\rangle_t^{-3}$, ρ_i^m and ρ_{ij}^m for $1\leq i,j\leq n-1$ are bounded in L^∞-norm by $C\sqrt{\mathcal{E}^m}$, by Lemma 3.3. Terms of the form $\langle\rho^m\rangle^{-3}$ are estimated by 1 in L^∞-norm. Note that in the last two terms in the expression A (2.43) the leading order derivatives cancel out after the the product rule has been applied within the parentheses. Using these observations and applying the Cauchy-Schwarz inequality, we conclude

$$|A-(\text{crossterms})|\leq C\sqrt{\mathcal{E}^m}(\mathcal{D}^m+\mathcal{D}^{m+1}). \tag{4.120}$$

Recall now that the term B (the second expression on RHS of (2.42)) is given by (2.44). The first two terms in the expression B are again the cross-terms. By part (c) of Lemma 3.3, we obtain

$$\begin{aligned}&\Big|\epsilon\int_{\mathbb{T}^{n-1}}\partial_s^\mu\Delta^2\rho_t^{m+1}\big(\partial_s^\mu\Delta^2\rho^m-\partial_s^\mu\Delta^2\rho^{m+1}\big)\langle\rho^m\rangle^{-1}\Big|\leq\\&\epsilon^2||\partial_s^\mu\Delta^2\rho_t^{m+1}||_2^2+C||\partial_s^\mu\Delta^2\rho^m||_2^2+C||\partial_s^\mu\Delta^2\rho^{m+1}||_2^2\leq\\&\leq\frac{C}{\lambda}\mathcal{E}^{m+1}+(\lambda+C\mathcal{E}^m)\mathcal{D}^{m+1}+\frac{C}{\epsilon}\big(\mathcal{E}^m+\mathcal{E}^{m+1}\big).\end{aligned} \tag{4.121}$$

Analogously, we establish

$$\begin{aligned}&\Big|\epsilon\int_{\mathbb{T}^{n-1}}\partial_s^\mu\Delta^2\rho_t^{m+1}\Big(\rho_i^m\rho_j^m\big(\partial_s^\mu\Delta\rho_{ij}^m-\partial_s^\mu\Delta\rho_{ij}^{m+1}\big)\Big)\langle\rho^m\rangle^{-3}\Big|\leq\\&\frac{C}{\lambda}\mathcal{E}^{m+1}+(\lambda+C\mathcal{E}^m)\mathcal{D}^{m+1}+\frac{C}{\epsilon}\big(\mathcal{E}^m+\mathcal{E}^{m+1}\big).\end{aligned} \tag{4.122}$$

We denote the sum of the remaining terms in the expression B by $B-$(crossterms). The same idea as in the estimates for $|A-(\text{crossterms})|$ works. It is important to note that we have canceling of the highest order derivatives within the parentheses in the last three expressions on RHS of (2.44). In addition to that, we factorize $\epsilon=\sqrt{\epsilon}\times\sqrt{\epsilon}$. By the Cauchy-Schwarz inequality,

$$|B-(\text{crossterms})|\leq C\sqrt{\mathcal{E}^m}(\mathcal{D}^m+\mathcal{D}^{m+1}). \tag{4.123}$$

The third term of $\int_{\mathbb{T}^{n-1}} T^m_{\mu,s}$ is given by the third term on RHS of (2.42) together with (3.70). Recall that $G^m_{\mu,s}$ is given by (3.65). In order to estimate it, we first integrate by parts.

$$\int_{\mathbb{T}^{n-1}} \Delta G^m_{\mu,s} \big(\partial^\mu_s \rho^{m+1}_t + \epsilon \partial^\mu_s \Delta^2 \rho^{m+1}_t\big) = \\ -\int_{\mathbb{T}^{n-1}} \nabla G^m_{\mu,s} \cdot \partial^\mu_s \nabla \rho^{m+1}_t - \epsilon \int_{\mathbb{T}^{n-1}} \Delta \nabla G^m_{\mu,s} \cdot \partial^\mu_s \Delta \nabla \rho^{m+1}_t.$$

The crucial observation is $||\nabla G^m_{\mu,s}||_2 \leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}$, $||\sqrt{\epsilon}\Delta\nabla G^m_{\mu,s}||_2 \leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}$. Both inequalities follow in the standard way, by using $L^\infty - L^2$ type estimates and (3.67). The third term of $\int_{\mathbb{T}^{n-1}} T^m_{\mu,s}$ is then bounded by:

$$\Big|\int_{\mathbb{T}^{n-1}} \Delta G^m_{\mu,s} \big(\partial^\mu_s \rho^{m+1}_t + \epsilon \partial^\mu_s \Delta^2 \rho^{m+1}_t\big)\Big| \leq ||\nabla G^m_{\mu,s}||_2 ||\partial^\mu_s \nabla \rho^{m+1}_t||_2 + \\ + ||\sqrt{\epsilon}\Delta\nabla G^m_{\mu,s}||_2 ||\sqrt{\epsilon}\partial^\mu_s \Delta \nabla \rho^{m+1}_t||_2 \leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}\sqrt{\mathcal{D}^{m+1}} \leq C\sqrt{\mathcal{E}^m}(\mathcal{D}^m + \mathcal{D}^{m+1}). \tag{4.124}$$

We integrate by parts in the fourth and the last term of $\int_{\mathbb{T}^{n-1}} T^m_{\mu,s}$ (given by the last term on RHS of (2.42) and (3.70)). Recall $\Delta \mathcal{U}|_{\mathbb{T}^{n-1}} = \Delta \partial^\mu_s u^{m+1}|_{\mathbb{T}^{n-1}} = \Delta \partial^\mu_s \kappa^m$.

$$\int_{\mathbb{T}^{n-1}} \langle \rho^m \rangle^2 h^m_{\mu,s} \Delta \partial^\mu_s \kappa^m = \\ -\int_{\mathbb{T}^{n-1}} \nabla(\langle \rho^m \rangle^2 h_1) \cdot \partial^\mu_s \nabla \kappa^m - \epsilon \int_{\mathbb{T}^{n-1}} \nabla(\langle \rho^m \rangle^2 h_2) \cdot \partial^\mu_s \nabla \kappa^m.$$

By (4.101), we set $h^m_{\mu,s} = h_1 + \epsilon h_2$. By the trace inequality,

$$||\nabla \partial^\mu_s \kappa^m||_2 = ||\partial^\mu_s \nabla_{x'} u^{m+1}||_{L^2(\mathbb{T}^{n-1})} \leq C||\partial^\mu_s \nabla u^{m+1}||_{H^1(\Omega)} \leq C\sqrt{\mathcal{D}^{m+1}}.$$

By (4.115), we obtain

$$\Big|\int_{\mathbb{T}^{n-1}} \nabla(\langle \rho^m \rangle^2 h_1) \cdot \partial^\mu_s \nabla \kappa^m\Big| \leq ||\nabla(\langle \rho^m \rangle^2 h_1)||_2 ||\partial^\mu_s \nabla \kappa^m||_2 \leq C\sqrt{\mathcal{E}^m}\mathcal{D}^{m+1}. \tag{4.125}$$

On the other hand, by Lemma 3.3 and $L^\infty - L^2$ type estimates, $||\sqrt{\epsilon}\partial^\mu_s \nabla \kappa^m||_2 \leq C\sqrt{\mathcal{D}^m}$. By (4.116),

$$\Big|\epsilon \int_{\mathbb{T}^{n-1}} \nabla(\langle \rho^m \rangle^2 h_2) \cdot \partial^\mu_s \nabla \kappa^m\Big| \leq ||\sqrt{\epsilon}\nabla(\langle \rho^m \rangle^2 h_2)||_2 ||\sqrt{\epsilon}\partial^\mu_s \nabla \kappa^m||_2 \\ \leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{D}^m} \leq C\sqrt{\mathcal{E}^m}(\mathcal{D}^m + \mathcal{D}^{m+1}), \tag{4.126}$$

where we recall (4.101) again. From the estimates (4.125) and (4.126), the last term in $\int_{\mathbb{T}^{n-1}} T^m_{\mu,s}$ is bounded by

$$\Big|\int_{\mathbb{T}^{n-1}} \langle \rho^m\rangle^2 h^m_{\mu,s}\Delta\partial^\mu_s \kappa^m\Big| \leq C\sqrt{\mathcal{E}^m}(\mathcal{D}^m+\mathcal{D}^{m+1}). \tag{4.127}$$

Using the identity (3.68) and summing the estimates (4.92) - (4.114) and (4.117) - (4.127) to get a bound on the right-hand side of the identity (3.68), we arrive at

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}^{m+1}(t)+\mathcal{D}^{m+1}(t)\leq C\sqrt{\mathcal{E}^m}(\mathcal{D}^m+\mathcal{D}^{m+1})+C\sqrt{\mathcal{D}^m}\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^{m+1}}\\&+C(\eta+\lambda)\mathcal{D}^{m+1}+\frac{C}{\eta}(\mathcal{E}^m+\mathcal{E}^{m+1})+\frac{C}{\lambda}\mathcal{E}^{m+1}+\eta\mathcal{D}^m\\&+\frac{C}{\eta\lambda\epsilon^4}\mathcal{E}^{m+1}+\frac{C\lambda\epsilon}{\eta}\mathcal{D}^{m+1}+\frac{C}{\lambda\epsilon}(\mathcal{E}^m+\mathcal{E}^{m+1})+C\mathcal{E}^{m+1}\mathcal{D}^m.\end{aligned} \tag{4.128}$$

Now integrate in time over the interval $[0,t]$ to get

$$\begin{aligned}&\mathcal{E}^{m+1}(t)+\int_0^t\mathcal{D}^{m+1}(\tau)\,d\tau\leq\mathcal{E}^{m+1}(0)+C\sup_{s\leq t}\sqrt{\mathcal{E}^m(s)}\Big(\int_0^t\mathcal{D}^m(\tau)\,d\tau+\int_0^t\mathcal{D}^{m+1}(\tau)\,d\tau\Big)+\\&+C\sup_{s\leq t}\sqrt{\mathcal{E}^{m+1}(s)}\int_0^t\sqrt{\mathcal{D}^m(s)}\sqrt{\mathcal{D}^{m+1}(s)}\,ds+C(\eta+\lambda)\int_0^t\mathcal{D}^{m+1}(\tau)\,d\tau\\&+Ct\big(\frac{1}{\eta}+\frac{1}{\lambda}\big)\sup_{s\leq t}\mathcal{E}^{m+1}(s)+\frac{Ct}{\eta}\sup_{s\leq t}\mathcal{E}^m(s)+\frac{Ct}{\eta\lambda\epsilon^4}\sup_{s\leq t}\mathcal{E}^{m+1}(s)+\\&+\frac{C\lambda\epsilon}{\eta}\int_0^t\mathcal{D}^{m+1}(\tau)\,d\tau+\eta\int_0^t\mathcal{D}^m(\tau)\,d\tau+\frac{Ct}{\lambda\epsilon}\Big(\sup_{s\leq t}\mathcal{E}^m(s)+\sup_{s\leq t}\mathcal{E}^{m+1}(s)\Big)+\\&+\sup_{s\leq t}\mathcal{E}^{m+1}(s)\int_0^t\mathcal{D}^m(\tau)\,d\tau.\end{aligned} \tag{4.129}$$

Note that by Cauchy-Schwarz inequality we have

$$\begin{aligned}&\sup_{s\leq t}\sqrt{\mathcal{E}^{m+1}(s)}\int_0^t\sqrt{\mathcal{D}^m(s)}\sqrt{\mathcal{D}^{m+1}(s)}\,ds\leq\\&\leq\sup_{s\leq t}\sqrt{\mathcal{E}^{m+1}(s)}\sqrt{\int_0^t\mathcal{D}^m(s)\,ds}\sqrt{\int_0^t\mathcal{D}^{m+1}(s)\,ds}\\&\leq\big(\int_0^t\mathcal{D}^m(\tau)\,d\tau\big)^{1/2}\big(\sup_{s\leq t}\mathcal{E}^{m+1}(s)+\int_0^t\mathcal{D}^{m+1}(\tau)\,d\tau\big).\end{aligned} \tag{4.130}$$

By assumption $\sup_{s\le t}\mathcal{E}^m(s)+\int_0^t\mathcal{D}^m(\tau)\,d\tau\le L$, and thus from (4.129) and (4.130), for any $t'\le t$:

$$\mathcal{E}^{m+1}(t')+\int_0^{t'}\mathcal{D}^{m+1}(\tau)\,d\tau\le\frac{L}{2}+L^{3/2}+\frac{Ct}{\eta}L+\eta L+\frac{CtL}{\lambda\epsilon}+$$
$$\big(\sup_{s\le t}\mathcal{E}^{m+1}(s)+\int_0^t\mathcal{D}^{m+1}(\tau)\,d\tau\big)\Big\{L^{1/2}+C(\eta+\lambda)+Ct(\frac{1}{\eta}+\frac{1}{\lambda})+\frac{Ct}{\eta\lambda\epsilon^4}+\frac{C\lambda\epsilon}{\eta}+\frac{Ct}{\lambda\epsilon}+L\Big\}.$$

Since the above inequality holds for any $t'\le t$, we obtain

$$\big(\sup_{s\le t}\mathcal{E}^{m+1}(s)+\int_0^t\mathcal{D}^{m+1}(\tau)\,d\tau\big)\Big\{1-\Big(L^{1/2}+C(\eta+\lambda)+Ct(\frac{1}{\eta}+\frac{1}{\lambda})+\frac{Ct}{\eta\lambda\epsilon^4}+\frac{C\lambda\epsilon}{\eta}+\frac{Ct}{\lambda\epsilon}+L\Big)\Big\}$$
$$\le\frac{L}{2}+CL^{3/2}+\frac{Ct}{\eta}L+\eta L+\frac{CtL}{\lambda\epsilon}.$$

We first choose η small and then λ small so that $C(\lambda+\eta)+\frac{C\lambda\epsilon}{\eta}$ is small. Further, we choose t (t depends on ϵ) and L small so that

$$L^{1/2}+C(\eta+\lambda)+Ct(\frac{1}{\eta}+\frac{1}{\lambda})+\frac{Ct}{\eta\lambda\epsilon^4}+\frac{C\lambda\epsilon}{\eta}+\frac{Ct}{\lambda\epsilon}+L<\frac{1}{3},$$

and

$$\frac{L}{2}+C(L)^{3/2}+\frac{Ct}{\eta}L+\eta L+\frac{CtL}{\lambda\epsilon}\le\frac{2}{3}L,\quad L\le\theta.$$

With such a choice of L and $t=:\mathcal{T}^\epsilon$, we obtain $\sup_{s\le\mathcal{T}^\epsilon}\mathcal{E}^{m+1}(s)+\int_0^{\mathcal{T}^\epsilon}\mathcal{D}^{m+1}(\tau)\,d\tau\le L$ and this finishes the proof of Lemma 4.1. □

5 Regularized Stefan problem.

The principal goal of this section is the following local existence theorem:

Theorem 5.1 *For any sufficiently small $L>0$ there exists $t^\epsilon>0$ depending on L and ϵ such that if for given initial data $(u_0^\epsilon,\rho_0^\epsilon)$*

$$\mathcal{E}_\epsilon(u_0^\epsilon,\rho_0^\epsilon;\rho_0^\epsilon)+\Big|\int_{\mathbb{T}^{n-1}}\rho_0^\epsilon-\int_\Omega u_0^\epsilon(1+\phi'\rho_0^\epsilon)\Big|\le\frac{L}{2}$$

then there exists a unique solution $(u^\epsilon,\rho^\epsilon)$ to the regularized Stefan problem (1.11)- (1.15) and (1.17) defined on the time interval $[0,t^\epsilon]$. Moreover,

$$\sup_{0\le t\le t^\epsilon}\mathcal{E}_\epsilon(u^\epsilon,\rho^\epsilon;\rho^\epsilon)(t)+\int_0^{t^\epsilon}\mathcal{D}_\epsilon(u^\epsilon,\rho^\epsilon;\rho^\epsilon)(\tau)\,d\tau\le L$$

and $\mathcal{E}_\epsilon(u^\epsilon,\rho^\epsilon)(\cdot)$ *is continuous on* $[0,t^\epsilon[$.

Remark. Note that the constant L is independent of ϵ.

Proof. Convergence. Combining Lemmas 3.2 and 4.1, we obtain a uniform-in-m bound on the sequence $\big\{(u^m,\rho^m)\big\}_m$. Our goal is to show that $\big\{(u^m,\rho^m)\big\}_m$ is a Cauchy sequence in the energy space. For any $l\in\mathbb{N}$ let $v^{l+1}:=u^{l+1}-u^l$ and $\sigma^{l+1}=\rho^{l+1}-\rho^l$. By subtracting two consecutive equations in the iteration process, we obtain

$$v_t^{m+1}-\Delta_{x'}v^{m+1}-a_{\rho^m}v_{nn}^{m+1}=f_m^\circ, \tag{5.131}$$

$$\begin{aligned} v^{m+1}&=\Delta\sigma^m\langle\rho^m\rangle^{-1}+g_m^\circ\\ &=\Delta\sigma^m\langle\rho^m\rangle^{-1}-\rho_i^m\rho_j^m\sigma_{ij}^m\langle\rho^m\rangle^{-1}+G_m^\circ\quad\text{on}\quad\mathbb{T}^{n-1}\times\{x_n=0\}, \end{aligned} \tag{5.132}$$

$$\partial_n v^{m+1}=0\qquad\text{on}\quad\mathbb{T}^{n-1}\times\big\{x_n=\pm1\big\}, \tag{5.133}$$

$$[v_n^{m+1}]_+^-=\big(\sigma_t^{m+1}+\epsilon\Delta^2\sigma_t^{m+1}\big)\langle\rho^m\rangle^{-2}+h_m^\circ\quad\text{on}\quad\mathbb{T}^{n-1}\times\{x_n=0\}. \tag{5.134}$$

Here

$$\begin{aligned} f_m^\circ&=-B_{\rho^m}\cdot\nabla_{x'}v_n^{m+1}-c_{\rho^m}v_n^{m+1}+u_{nn}^m(a_{\rho^m}-a_{\rho^{m-1}})\\ &-\nabla_{x'}u_n^m\cdot\big(B_{\rho^m}-B_{\rho^{m-1}}\big)-u_n^m(c_{\rho^m}-c_{\rho^{m-1}}), \end{aligned} \tag{5.135}$$

$$\begin{aligned} G_m^\circ=\Delta\rho^{m-1}\big(\langle\rho^m\rangle^{-1}-\langle\rho^{m-1}\rangle\big)+\rho_{ij}^{m-1}\big(&\sigma_i^m\rho_j^m\langle\rho^m\rangle^{-1}+\rho_i^{m-1}\sigma_j^m\langle\rho^m\rangle^{-1}+\\ &\rho_i^{m-1}\rho_j^{m-1}(\langle\rho^m\rangle^{-1}-\langle\rho^{m-1}\rangle)\big), \end{aligned}$$

$$\begin{aligned} g_m^\circ&=-\rho_i^m\rho_j^m\sigma_{ij}^m\langle\rho^m\rangle^{-1}+G_m^\circ,\\ h_m^\circ&=-\big(|\nabla\rho^m|^2-|\nabla\rho^{m-1}|^2\big)[u_n^m]_+^-\langle\rho^m\rangle^{-2}. \end{aligned} \tag{5.136}$$

After applying the differential operator ∂_s^μ to the equations (5.131), (5.132) and (5.134) and singling out the leading-order terms we arrive at:

$$\partial_s^\mu v_t^{m+1}-\Delta_{x'}v^{m+1}-a_{\rho^m}v_{nn}^{m+1}=f_m', \tag{5.137}$$

$$\partial_s^\mu v^{m+1}=\partial_s^\mu\Delta\sigma^m\langle\rho^m\rangle^{-1}+g_m'=\partial_s^\mu\Delta\sigma^m\langle\rho^m\rangle^{-1}-\rho_i^m\rho_j^m\partial_s^\mu\sigma_{ij}^m\langle\rho^m\rangle^{-1}+G_m', \tag{5.138}$$

$$[\partial_s^\mu v_n^{m+1}]_+^-=\big(\partial_s^\mu\sigma_t^{m+1}+\epsilon\partial_s^\mu\Delta^2\sigma_t^{m+1}\big)\langle\rho^m\rangle^{-2}+h_m', \tag{5.139}$$

where

$$f'_m = \partial_s^\mu f_m^\circ + \big(\partial_s^\mu(a_{\rho^m}\partial_s^\mu v_{nn}^{m+1}) - a_{\rho^m}\partial_s^\mu v_{nn}^{m+1}\big), \tag{5.140}$$

$$g'_m = \partial_s^\mu g_m^\circ + \sum_{\substack{|\mu'|+s' \\ <|\mu|+s}} \partial_{s'}^{\mu'} \Delta\sigma^m \partial_{s-s'}^{\mu-\mu'}(\langle\rho^m\rangle^{-1}), \tag{5.141}$$

$$\begin{aligned} G'_m = \partial_s^\mu G_m^\circ + \sum_{\substack{|\mu'|+s' \\ <|\mu|+s}} \partial_{s'}^{\mu'} \Delta\sigma^m \partial_{s-s'}^{\mu-\mu'}(\langle\rho^m\rangle^{-1}) - \\ \Big(\partial_s^\mu\big(\rho_i^m\rho_j^m\sigma_{ij}^m\langle\rho^m\rangle^{-1}\big) - \rho_i^m\rho_j^m\partial_s^\mu\sigma_{ij}^m\langle\rho^m\rangle^{-1}\Big), \end{aligned} \tag{5.142}$$

$$h'_m = \partial_s^\mu h_m^\circ + \sum_{\substack{|\mu'|+s' \\ <|\mu|+s}} \partial_{s'}^{\mu'}\big(\sigma_t^{m+1} + \epsilon\Delta^2\sigma_t^{m+1}\big)\partial_{s-s'}^{\mu-\mu'}(\langle\rho^m\rangle^{-2}). \tag{5.143}$$

As a next step, we use the identities from Chapter 2 to obtain the energy identities for the problem (5.137) - (5.139). Respecting the notations of Chapter 2 we set for any $l \in \mathbb{N}$

$$f = f'_l,\ g = g'_l,\ G = G'_l,\ h = h'_l,\ \mathcal{U} = \partial_s^\mu v^{l+1},\ \omega = \partial_s^\mu \sigma^{l+1},\ \chi = \partial_s^\mu \sigma^l \text{ and } \psi = \rho^l. \tag{5.144}$$

Additionally, we introduce the notations

$$e^l := \mathcal{E}_\epsilon(v^l, \sigma^l; \rho^{l-1}), \quad d^l := \mathcal{D}_\epsilon(v^l, \sigma^l; \rho^{l-1}),$$

where $\mathcal{E}_\epsilon$ and $\mathcal{D}_\epsilon$ are defined by (1.26) and (1.27) respectively. Using (2.37) and (5.144), we arrive at

$$\frac{d}{dt}e^{m+1} + d^{m+1} = \int_\Omega \{p^m + r^m\} - \int_{\mathbb{T}^{n-1}} \{q^m + s^m + t^m\}, \tag{5.145}$$

where

$$\begin{aligned} p^m &:= \sum_{|\mu|+2s\le 2k} P(\partial_s^\mu v^{m+1}, \rho^m, f'_m), \quad r^m := \sum_{|\mu|+2s\le 2k} R(\partial_s^\mu v^{m+1}, \rho^m, f'_m) \\ q^m &:= \sum_{|\mu|+2s\le 2k} Q(\partial_s^\mu \sigma^m, \partial_s^\mu \sigma^{m+1}, \rho^m, g'_m, h'_m), \\ s^m &:= \sum_{|\mu|+2s\le 2k} S(\partial_s^\mu \sigma^m, \partial_s^\mu \sigma^{m+1}, \rho^m, g'_m, h'_m), \\ t^m &:= \sum_{|\mu|+2s\le 2k} T(\partial_s^\mu \sigma^m, \partial_s^\mu \sigma^{m+1}, \rho^m, g'_m, h'_m). \end{aligned} \tag{5.146}$$

Here P, Q, R, S and T are defined by (2.38), (2.40), (2.39), (2.41) and (2.42) respectively. Our aim is to prove that for suitably small $t\leq t(\epsilon)$ there exists a $\Lambda<1$ such that

$$e^{m+1}(t)+\int_0^t d^{m+1}(\tau)\,d\tau\leq\Lambda\big(e^m(t)+\int_0^t d^m(\tau)\,d\tau\big).$$

We shall accomplish this by estimating the terms p^m, r^m, q^m, s^m and t^m on RHS of the identity (5.145). These estimates will be largely analogous to the estimates from Chapter 4. However, due to the formally new terms $\partial_s^\mu f_m^\circ$, $\partial_s^\mu g_m^\circ$, $\partial_s^\mu G_m^\circ$, $\partial_s^\mu h_m^\circ$ appearing in the definitions (5.140), (5.141), (5.142) and (5.143) of f_m', g_m', G_m' and h_m' respectively, we need to make several preparatory steps. First, for any $l\in\mathbb{N}$

$$||a_{\rho^m}-a_{\rho^{m-1}}||_{H^l(\Omega)}+||B_{\rho^m}-B_{\rho^{m-1}}||_{H^l(\Omega)}\leq C||\sigma^m||_{H^{l+1}(\Omega)},\tag{5.147}$$

and

$$||c_{\rho^m}-c_{\rho^{m-1}}||_{H^l(\Omega)}\leq C||\sigma_t^m||_{H^l}+C||\sigma^m||_{H^{l+2}(\Omega)}.\tag{5.148}$$

Note that $\sup_{0\leq t\leq t^\epsilon}\mathcal{E}^m(t)+\int_0^{t^\epsilon}\mathcal{D}^m(\tau)\,d\tau\leq L$. In particular, for $|\mu|+2s\leq 2k$

$$||\partial_s^\mu u^m||_{H^1(\Omega)}\,,||\partial_s^\mu a_{\rho^m}||_{L^2(\Omega)}\,,||\partial_s^\mu B_{\rho^m}||_{L^2(\Omega)}\leq C\sqrt{L}.\tag{5.149}$$

Furthermore, $||\partial_s^\mu c_{\rho^m}||_{L^2(\Omega)}\leq C\sqrt{L}$ for $|\mu|+2s\leq 2k-1$. We can now use the Sobolev inequality to bound the lower order derivatives of u^m, a_{ρ^m}, B_{ρ^m} and c_{ρ^m} in L^∞-norm by $C\sqrt{L}$. The major step is to provide the analogoues of part (c) of Lemma 3.3 for the function σ^{m+1} instead of ρ^{m+1} and Lemma 3.5 for the function v^{m+1} instead of u^{m+1}. By the boundary condition (5.139) and the proof of Lemma 3.3, part (c), we deduce

$$||\partial_s^\mu\sigma_t^{m+1}||_2^2+2\epsilon||\partial_s^\mu\Delta\sigma_t^{m+1}||_2^2+\epsilon^2||\partial_s^\mu\Delta^2\sigma_t^{m+1}||_2^2\leq\lambda d^{m+1}+\frac{C}{\lambda}e^{m+1}+e^m(\mathcal{E}^m+\mathcal{D}^m)\tag{5.150}$$

and

$$||\partial_s^\mu\sigma_t^{m+1}||_2^2+2\epsilon||\partial_s^\mu\Delta\sigma_t^{m+1}||_2^2+\epsilon^2||\partial_s^\mu\Delta^2\sigma_t^{m+1}||_2^2\leq Cd^{m+1}+e^m\mathcal{D}^m.\tag{5.151}$$

As in Lemma 3.5,

$$||\partial_s^\mu\partial_{n^r}v^{m+1}||_{L^2(\Omega)}^2\leq Ce^{m+1}+Ce^m.\tag{5.152}$$

The proof of (5.152) is completely analogous to the proof of Lemma 3.5 whereby, due to the addition of the formally new term $\partial_s^\mu f_m^\circ$ in the definition of f_m' we need to exploit the relations (5.147) and (5.148), which are responsible for the occurence of the term Ce^m on RHS of (5.152). We proceed fully analogously to the energy estimates in Chapter 4 to estimate the right-hand side of the energy identity (5.145). The terms involving $\partial_s^\mu f_m^\circ$ and $\partial_s^\mu h_m^\circ$ require an additional care. In the estimate for the term $\int_\Omega p^m = \int_\Omega f_m' \partial_s^\mu v^{m+1}$, where f_m' is given by (5.140), we single out the term $\int_\Omega \partial_s^\mu f_m^\circ \partial_s^\mu v^{m+1}$. Here f_m° is given by (5.135). Writing

$$\partial_s^\mu(c_{\rho^m} v_n^{m+1}) = \sum_{\mu',s'} C_{s'}^{\mu'} \partial_{s'}^{\mu'} c_{\rho^m} \partial_{s-s'}^{\mu-\mu'} v_n^{m+1} = \sum_{\mu'+2s'\leq k} + \sum_{\mu'+2s'>k} \tag{5.153}$$

we estimate the lower-order terms in both sums in L^∞-norm and the higher-order terms in L^2-norm. By the Cauchy-Schwarz inequality and (5.152),

$$\Big|\int_\Omega \partial_s^\mu(c_{\rho^m} v_n^{m+1}) \partial_s^\mu v^{m+1}\Big| \leq C\sqrt{\mathcal{D}^m}\sqrt{e^{m+1}}(\sqrt{e^m}+\sqrt{e^{m+1}}). \tag{5.154}$$

Similarly, $|\int_\Omega \partial_s^\mu(B_{\rho^m}\cdot\nabla_{x'} v_n^{m+1})\partial_s^\mu v^{m+1}| \leq C\sqrt{\mathcal{E}^m}(e^{m+1}+d^{m+1})$. In analogous fashion, we find

$$\begin{aligned}&\Big|\int_\Omega \partial_s^\mu\big(u_{nn}^m(a_{\rho^m}-a_{\rho^{m-1}}) - \nabla_{x'}u_n^m\cdot\big(B_{\rho^m}-B_{\rho^{m-1}}\big) - u_n^m(c_{\rho^m}-c_{\rho^{m-1}})\big)\partial_s^\mu v^{m+1}\Big| \\ &\leq C\sqrt{\mathcal{D}^m}\sqrt{e^m}\sqrt{e^{m+1}} + C\sqrt{\mathcal{E}^m}\sqrt{d^m}\sqrt{e^{m+1}},\end{aligned} \tag{5.155}$$

where we rely on (5.147) and (5.148). The term r^m is of the form $r^m = (f_m')^2 + (\text{rest})$, ($r^m$ is defined in (5.146)). The formally new term to estimate has the form $\int_\Omega(\partial_s^\mu f_m^\circ)^2$, where f_m° is given by (5.135). By the same splitting idea as in (5.153) and the estimate (5.152),

$$\Big|\int_\Omega (\partial_s^\mu f_m^\circ)^2\Big| \leq C\mathcal{E}^m(e^m+e^{m+1}+d^{m+1}) + C\mathcal{D}^m e^{m+1} + C\mathcal{D}^m e^m + C\mathcal{E}^m d^m.$$

Recall now that s^m and t^m are defined by (5.146). The last term in each of the expressions (2.41) and (2.42) has the form $\langle\rho^m\rangle^2 h_m' v_t^{m+1}|_{\mathbb{T}^{n-1}}$ and $\langle\rho^m\rangle^2 h_m' \Delta_{x'} v^{m+1}|_{\mathbb{T}^{n-1}}$, respectively. By (5.143), the formally new term (with respect to $h_{\mu,s}^m$ defined by (3.66)) is $\partial_s^\mu h_m^\circ$. By (5.136), we conclude

$$\partial_s^\mu h_m^\circ = -\sum_{\mu',s'} C_{s'}^{\mu'} \partial_{s'}^{\mu'}([u_n^m]_+^-) \partial_{s-s'}^{\mu-\mu'}\big((|\nabla\rho^m|^2 - |\nabla\rho^{m-1}|^2)\langle\rho^m\rangle^{-2}\big) = \sum_{\mu'+2s'\leq k} + \sum_{\mu'+2s'>k}$$

Hitting the lower order terms with L^∞-norm and the higher order terms with L^2-norm and using the trace inequality to estimate $||\partial_{s'}^{\mu'}[u_n^m]_+^-||_2$, we obtain

$$||\partial_s^\mu h_m^\circ||_2 \le (\lambda\sqrt{\mathcal{D}^m} + \frac{C}{\lambda}\sqrt{\mathcal{E}^m})\sqrt{e^m}, \qquad ||\partial_s^\mu h_m^\circ||_2 \le C\sqrt{\mathcal{D}^m}\sqrt{e^m}. \quad (5.156)$$

Note that $v_t^{m+1}|_{\mathbb{T}^{n-1}} = (\kappa_{\rho^m} - \kappa_{\rho^{m-1}})_t$. It is easy to check that

$$||\partial_s^\mu v_t^{m+1}|_{\mathbb{T}^{n-1}}||_2 \le C(||\nabla^2\sigma_t^m||_2 + ||\nabla\sigma_t^m||_2 + ||\nabla^2\sigma^m||_2 + ||\nabla\sigma^m||_2).$$

By the definitions of d^{m+1} and e^{m+1}, $||\partial_s^\mu v_t^{m+1}|_{\mathbb{T}^{n-1}}||_2 \le \frac{C}{\sqrt{\epsilon}}\sqrt{d^m} + C\sqrt{e^m}$. Combining this with (5.156) and the Cauchy-Schwarz inequality, we obtain

$$\Big|\int_{\mathbb{T}^{n-1}} \langle\rho^m\rangle^2 \partial_s^\mu h_m^\circ v_t^{m+1}|_{\mathbb{T}^{n-1}}\Big| \le (\lambda\sqrt{\mathcal{D}^m} + \frac{C}{\lambda}\sqrt{\mathcal{E}^m})\sqrt{e^m}(\frac{C}{\sqrt{\epsilon}}\sqrt{d^m} + C\sqrt{e^m}).$$

Choosing $\lambda = \sqrt{\epsilon}$ we arrive at

$$\begin{aligned}\Big|\int_{\mathbb{T}^{n-1}} \langle\rho^m\rangle^2 \partial_s^\mu h_m^\circ v_t^{m+1}|_{\mathbb{T}^{n-1}}\Big| &\le C\sqrt{\mathcal{D}^m}\sqrt{e^m}\sqrt{d^m} + C\sqrt{\epsilon}\sqrt{\mathcal{D}^m}e^m + \\ &\frac{C}{\epsilon}\sqrt{\mathcal{E}^m}\sqrt{e^m}\sqrt{d^m} + \frac{C}{\sqrt{\epsilon}}\sqrt{\mathcal{E}^m}e^m.\end{aligned} \quad (5.157)$$

We want to estimate the time integral of the right-hand side of the inequality (5.157). For any $t \le t^\epsilon, \lambda > 0$, we have

$$\int_0^t \sqrt{\mathcal{D}^m}\sqrt{e^m}\sqrt{d^m}\, d\tau \le \lambda\int_0^t d^m(\tau)\, d\tau + \frac{C}{\lambda}\sup_{0\le s\le t} e^m(s)\int_0^t \mathcal{D}^m(\tau)\, d\tau. \quad (5.158)$$

Similarly,

$$\int_0^t \sqrt{\mathcal{D}^m}e^m\, d\tau \le \int_0^t e^m + \sup_{0\le s\le t} e^m(s)\int_0^t \mathcal{D}^m(\tau)\, d\tau \le t\sup_{0\le s\le t} e^m(s) + L\sup_{0\le s\le t} e^m(s). \quad (5.159)$$

Since $\mathcal{E}^m \le L$, for any $\lambda > 0$, we obtain

$$\int_0^t \frac{C}{\epsilon}\sqrt{\mathcal{E}^m}\sqrt{e^m}\sqrt{d^m}\, d\tau \le \lambda\int_0^t d^m(\tau)\, d\tau + \frac{C}{\lambda\epsilon^2}Lt\sup_{0\le s\le t} e^m(s), \quad (5.160)$$

and similarly,

$$\int_0^t \frac{C}{\sqrt{\epsilon}}\sqrt{\mathcal{E}^m}e^m\, d\tau \le \frac{C}{\sqrt{\epsilon}}t\sqrt{L}\sup_{0\le s\le t} e^m(s). \quad (5.161)$$

Noting that $||\Delta_{x'} v^{m+1}|_{\mathbb{T}^{n-1}}||_2 \le \frac{C}{\sqrt{\epsilon}}\sqrt{e^m}$, we can use the Cauchy-Schwarz inequality and the estimate (5.156) to get

$$\Big| \int_{\mathbb{T}^{n-1}} \langle \rho^m \rangle^2 \partial_s^\mu h_m^\circ \Delta_{x'} v^{m+1}|_{\mathbb{T}^{n-1}} \Big| \le \frac{C}{\sqrt{\epsilon}} \sqrt{\mathcal{D}^m} e^m. \tag{5.162}$$

We get

$$\int_0^t \frac{C}{\sqrt{\epsilon}} \sqrt{\mathcal{D}^m} e^m \, d\tau \le \frac{C}{\epsilon} t \sup_{0\le s\le t} e^m(s) + L \sup_{0\le s\le t} e^m(s). \tag{5.163}$$

In analogy to the estimate (4.128) together with (5.154), (5.155), (5.157) and (5.162), we obtain

$$\begin{aligned}
&\frac{d}{dt} e^{m+1} + d^{m+1} \le C e^m \Big(\frac{1}{\eta} + \frac{C}{\lambda\epsilon}\Big) + \eta d^m + C e^{m+1}\Big(\frac{1}{\eta} + \frac{1}{\lambda} + \frac{C}{\eta\lambda\epsilon^4} + \frac{C}{\lambda\epsilon}\Big) + \\
&C d^{m+1}\Big(\sqrt{L} + \eta + \lambda + \frac{C\lambda\epsilon}{\eta}\Big) + C\sqrt{\mathcal{D}^m}\sqrt{d^{m+1}}\sqrt{e^{m+1}} + C\mathcal{D}^m e^{m+1} + \\
&C\sqrt{\mathcal{D}^{m+1}}\sqrt{e^{m+1}}\big(\sqrt{e^m} + \sqrt{e^{m+1}}\big) + C\sqrt{\mathcal{D}^m}\sqrt{e^m}\sqrt{e^{m+1}} + C\sqrt{\mathcal{E}^m}\sqrt{d^m}\sqrt{e^{m+1}} + \\
&C\sqrt{\mathcal{D}^m}\sqrt{e^m}\sqrt{d^m} + C\sqrt{\epsilon}\sqrt{\mathcal{D}^m} e^m + \frac{C}{\epsilon}\sqrt{\mathcal{E}^m}\sqrt{e^m}\sqrt{d^m} + \frac{C}{\sqrt{\epsilon}}\sqrt{\mathcal{E}^m} e^m + \frac{C}{\sqrt{\epsilon}}\sqrt{\mathcal{D}^m} e^m.
\end{aligned}$$

Integrating the above inequality over $[0,t]$ and proceeding as in (4.129), using (5.158), (5.159), (5.160), (5.161), (5.163), we conclude

$$\begin{aligned}
&e^{m+1}(t) + \int_0^t d^{m+1}(\tau)\, d\tau \le Ct\Big(\frac{1}{\eta} + \frac{C}{\lambda\epsilon}\Big) \sup_{0\le s\le t} e^m(s) + \eta \int_0^t d^m(\tau)\, d\tau + \\
&Ct\Big(\frac{1}{\eta} + \frac{1}{\lambda} + \frac{C}{\eta\lambda\epsilon^4} + \frac{C}{\lambda\epsilon}\Big) \sup_{0\le s\le t} e^{m+1}(s) + C\Big(\sqrt{L} + \eta + \lambda + \frac{C\lambda\epsilon}{\eta}\Big) \int_0^t d^{m+1}(\tau)\, d\tau \\
&+ C\lambda\Big(\int_0^t d^m(\tau)\, d\tau + \int_0^t d^{m+1}(\tau)\, d\tau\Big) + \\
&\Big(\frac{CL}{\lambda} + L + t + \frac{CL}{\lambda\epsilon^2} t + \frac{C\sqrt{L}}{\sqrt{\epsilon}} t + \frac{C}{\epsilon} t\Big) \sup_{0\le s\le t} e^m(s) + C(t + L + \sqrt{L}) \sup_{0\le s\le t} e^{m+1}(s)
\end{aligned} \tag{5.164}$$

We choose η, λ, L and $t =: t^\epsilon \le \mathcal{T}^\epsilon$ small, so that $\eta + C\lambda < \chi < \frac{1}{5}$

$$Ct\Big(\frac{1}{\eta} + \frac{C}{\lambda\epsilon}\Big) + \frac{CL}{\lambda} + L + t^\epsilon + \frac{CL}{\lambda\epsilon^2} t^\epsilon + \frac{C\sqrt{L}}{\sqrt{\epsilon}} t^\epsilon + \frac{C}{\epsilon} t^\epsilon < \chi < \frac{1}{5},$$

$$Ct^\epsilon\Big(\frac{1}{\eta} + \frac{1}{\lambda} + \frac{C}{\eta\lambda\epsilon^4} + \frac{C}{\lambda\epsilon}\Big) + C(t^\epsilon + L + \sqrt{L}) < \chi < \frac{1}{5}$$

and $C(\sqrt{L}+\eta+\lambda+\frac{C\lambda\epsilon}{\eta})<\chi<\frac{1}{5}$. Taking supremum over $[0,t^\epsilon]$, we arrive at

$$\sup_{0\le\tau\le t^\epsilon} e^{m+1}(\tau)+\int_0^{t^\epsilon} d^{m+1}(s)\,ds \le \chi\Big\{\sup_{0\le\tau\le t^\epsilon} e^{m}(\tau)+\int_0^{t^\epsilon} d^{m}(s)\,ds\Big\}+$$
$$+\chi\Big\{\sup_{0\le\tau\le t^\epsilon} e^{m+1}(\tau)+\int_0^{t^\epsilon} d^{m+1}(s)\Big\}\,ds.$$

Therefore,

$$\sup_{0\le\tau\le t^\epsilon} e^{m+1}(\tau)+\int_0^{t^\epsilon} d^{m+1}(s)\,ds \le \frac{\chi}{1-\chi}\Big\{\sup_{0\le\tau\le t^\epsilon} e^{m}(\tau)+\int_0^{t^\epsilon} d^{m}(s)\,ds\Big\}. \tag{5.165}$$

We observe now that the conservation law (3.71) and the fact that $\sigma^{m+1}(0)=v^{m+1}(0)=0$ imply

$$\int_{\mathbb{T}^{n-1}}\sigma^{m+1}=\int_\Omega v^{m+1}(1+\phi'\rho^m)+\int_\Omega \phi' u^m\sigma^m.$$

By the Poincaré inequality, previous identity and the uniform bounds on ρ^m and u^m we get

$$\begin{aligned} &||\sigma^{m+1}||_2^2\le C||\nabla\sigma^{m+1}||_2^2+C||\int_{\mathbb{T}^{n-1}}\sigma^{m+1}||_2^2\\ &\le C||\nabla\sigma^{m+1}||_2^2+C||v^{m+1}||_2^2+C||u^m||_{L^2(\Omega)}^2||\sigma^m||_2^2\\ &\le Ce^{m+1}+C\mathcal{E}^m||\sigma^m||_2^2\le Ce^{m+1}+CL||\sigma^m||_2^2.\end{aligned} \tag{5.166}$$

With L and χ so small that $CL+2C\frac{\chi}{1-\chi}=:\Lambda<1$, we obtain by (5.165),

$$\sup_{0\le\tau\le t^\epsilon}||\sigma^{m+1}(\tau)||_2^2\le C\frac{\chi}{1-\chi}\Big\{\sup_{0\le\tau\le t^\epsilon} e^{m}(\tau)+\int_0^{t^\epsilon} d^{m}(s)\,ds\Big\}+$$
$$+CL\sup_{0\le\tau\le t^\epsilon}||\sigma^m(\tau)||_2^2\le\frac{\Lambda}{2}\Big\{\sup_{0\le\tau\le t^\epsilon} e^{m}(\tau)+\int_0^{t^\epsilon} d^{m}(s)\,ds\Big\}+\Lambda\sup_{0\le\tau\le t^\epsilon}||\sigma^m(\tau)||_2^2.$$

Adding $\big\{\sup_{0\le\tau\le t^\epsilon} e^{m+1}(\tau)+\int_0^{t^\epsilon} d^{m+1}(s)\,ds\big\}$ to the both sides of the above inequality and using (5.165) again we get

$$\sup_{0\le\tau\le t^\epsilon}\Big\{e^{m+1}(\tau)+\int_0^{t^\epsilon} d^{m+1}(s)\,ds+||\sigma^{m+1}(\tau)||_2^2\Big\}\le$$
$$C\frac{\chi}{1-\chi}\Big\{\sup_{0\le\tau\le t^\epsilon} e^{m}(\tau)+\int_0^{t^\epsilon} d^{m}(s)\,ds\Big\}+\frac{\Lambda}{2}\Big\{\sup_{0\le\tau\le t^\epsilon} e^{m}(\tau)+\int_0^{t^\epsilon} d^{m}(s)\,ds\Big\}+$$
$$\sup_{0\le\tau\le t^\epsilon}\Lambda||\sigma^m(\tau)||_2^2\le\Lambda\sup_{0\le\tau\le t^\epsilon}\Big\{e^{m}(\tau)+\int_0^{t^\epsilon} d^{m}(s)\,ds+||\sigma^m(\tau)||_2^2\Big\}.$$

Define the Banach spaces

$$X := \Big\{(v,\sigma)\Big|\ ||(v,\sigma)||_{\mathcal{E}_\epsilon} + ||\sigma||_2^2 < \infty\Big\}, \quad Y := \Big\{(v,\sigma)\Big|\ ||(v,\sigma)||_{\mathcal{D}_\epsilon} < \infty\Big\},$$

where $||(\cdot,\cdot)||_{\mathcal{E}_\epsilon}$ and $||(\cdot,\cdot)||_{\mathcal{D}_\epsilon}$ are defined by (1.28) and (1.29) respectively. Let $Z := L^\infty\big(X;[0,t^\epsilon]\big) \oplus L^2\big(Y;[0,t^\epsilon]\big)$. Since Λ can be chosen arbitrarily small, we have proven that the sequence $\big((v^l,\sigma^l)\big)_{l\in\mathbb{N}}$ satisfies $||(v^{m+1},\sigma^{m+1})||_Z \leq \Lambda'||(v^m,\sigma^m)||_Z$, for some $\Lambda' < 1$. This implies that $\big((u^l,\rho^l)\big)_{l\in\mathbb{N}}$ is a Cauchy sequence and converges strongly in Z. Thus the *whole* sequence $\big((u^l,\rho^l)\big)_{l\in\mathbb{N}}$ converges to a solution $(u^\epsilon,\rho^\epsilon)$ of the regularized Stefan problem in the original energy space. In addition to this, passing to a limit in (3.71), we obtain the conservation law

$$\partial_t\Big\{\int_\Omega u(1+\phi'\rho)\Big\} = \partial_t\Big\{\int_{\mathbb{T}^{n-1}} \rho\Big\}. \tag{5.167}$$

Uniqueness. We want to prove uniqueness in the class of functions (u,ρ) satisfying $\sup_{0\leq t<t^\epsilon} \mathcal{E}_\epsilon(u,\rho)(t) + \int_0^{t^\epsilon} \mathcal{D}_\epsilon(\tau)\,d\tau \leq L$, where L may be chosen smaller if necessary. Let us assume that there exists another solution (v,σ) satisfying the same initial conditions $(v(x,0),\sigma(x',0)) = (u_0(x),\rho_0(x'))$ and the bound $\sup_{0\leq t<t^\epsilon} \mathcal{E}_\epsilon(v,\sigma)(t) + \int_0^{t^\epsilon} \mathcal{D}_\epsilon(v,\sigma)(\tau)\,d\tau \leq L$. After subtracting them and setting $w := u - v$, $\tau := \rho - \sigma$, we obtain

$$w_t - \Delta_{x'} w - a_\rho w_{nn} = f^*, \tag{5.168}$$

$$w = \Delta\tau\,\langle\rho\rangle^{-1} + g^* = \Delta\tau\,\langle\rho\rangle^{-1} - \rho_i\rho_j\tau_{ij}\,\langle\rho\rangle^{-1} + G^* \quad \text{on} \quad \mathbb{T}^{n-1}\times\{x_n = 0\}, \tag{5.169}$$

$$[w_n]_+^- = \big(\tau_t + \epsilon\Delta^2\tau_t\big)\,\langle\rho\rangle^{-2} + h^* \quad \text{on} \quad \mathbb{T}^{n-1}\times\{x_n = 0\}, \tag{5.170}$$

where

$$\begin{aligned}
&f^* = -B_\rho\cdot\nabla_{x'} w_n - c_\rho w_n + (a_\rho - a_\sigma)v_{nn} + (B_\rho - B_\sigma)\nabla_{x'} v_n + (c_\sigma - c_\rho)v_n,\\
&G^* = \Delta\sigma\big(\langle\rho\rangle^{-1} - \langle\sigma\rangle^{-1}\big) + \sigma_{ij}\big(\tau_i\rho_j\,\langle\rho\rangle^{-1} + \sigma_i\tau_j\,\langle\rho\rangle^{-1} + \sigma_i\sigma_j\big(\langle\rho\rangle^{-1} - \langle\sigma\rangle^{-1}\big)\big),\\
&g^* = -\rho_i\rho_j\tau_{ij}\,\langle\rho\rangle^{-1} + G^*\\
&h^* = -(|\nabla\rho|^2 - |\nabla\sigma|^2)[v_n]_+^-\,\langle\rho\rangle^{-2}.
\end{aligned}$$

We use Chapter 2 to derive the accompanying energy identities. To this end we set $\mathcal{U} = w$, $\psi = \rho$, $\omega = \chi = \tau$, $f = f^*$, $g = g^*$, $G = G^*$ and $h = h^*$. Here ρ takes the role of ρ^{m+1} and σ the role of ρ^m and additionally, the cross-terms vanish since $\omega=\chi=\tau$. With $k\geq n$ sufficiently large, the regularity assumptions of Lemma 2.1 are fulfilled. We are thus naturally led to the following energy quantities:

$$\mathcal{E}^*:=\mathcal{E}_\epsilon(w,\tau;\rho),\quad \mathcal{D}^*:=\mathcal{D}(w,\tau;\rho).$$

In addition to this we define $P^*=P(w,\rho,f^*)$ and analogously Q^*, R^*, S^* and T^*. Using the identity (2.37), we obtain

$$\frac{d}{dt}\mathcal{E}^*+\mathcal{D}^*=\int_\Omega\Big\{P^*+R^*\Big\}-\int_{\mathbb{T}^{n-1}}\Big\{Q^*+S^*+T^*\Big\}. \tag{5.171}$$

Our goal at this stage is to prove the inequality of the form

$$\frac{d}{dt}\mathcal{E}^*(t)+\mathcal{D}^*(t)\leq C\mathcal{E}^*(t)+C\sqrt{L}\mathcal{D}^*(t), \tag{5.172}$$

which would enable us to absorb the multiple of $\mathcal{D}^*$ on the right-hand side into the left-hand side and then use the Gronwall's inequality to conclude that $\mathcal{E}^*(t)=0$ for any $t\geq0$. It is essential that the constant C in the above estimate does not depend on ϵ so that the smallness bound on L remains independent of ϵ. That the identity (5.172) indeed holds, follows analogously to the energy estimates from the Chapter 4 applied to the right-hand side of (5.171). Here we strongly exploit the unfirom bounds on $\mathcal{E}_\epsilon(u,\rho)$ and $\mathcal{E}_\epsilon(v,\sigma)$. In particular we know that

$$||(a_\rho)_t||_\infty,\,||\nabla_{x'}(a_\rho)||_\infty,\,||(a_\rho)_n||_\infty,\,||B_\rho||_\infty,\,||c_\rho||_\infty,\,||B_\sigma||_\infty,\,||c_\sigma||_\infty\leq C\sqrt{L},$$

$$||v_n||_{L^\infty(\Omega)},\,||\nabla_{x'}v_n||_{L^\infty(\Omega)},\,||v_{nn}||_{L^\infty(\Omega)}\leq C\sqrt{L}$$

A major difference from the existence part of the proof is the absence of cross-terms in the energy identities (since $\omega=\chi$ in the notation of Chapter 2). In addition to that, we work in a lower order energy space and we can thus use the above uniform estimates to bound the term $[v_n]_+^-$ by $C\sqrt{L}$ in L^∞-norm. This observation is crucial when estimating h^*. Knowing that the ϵ-dependence comes only from the estimates of the cross-terms (cf. (4.109), (4.110), (4.118), (4.119), (4.121) and (4.122)), we conclude that the constants on the right-hand side of (5.172) *do not* depend on ϵ. Choosing L suitably small (5.172) implies $\frac{d}{dt}\mathcal{E}^*\leq C\mathcal{E}^*$ implying $\mathcal{E}^*(t)\leq C\int_0^t\mathcal{E}^*(s)\,ds$, since $\mathcal{E}^*(0)=0$. By Gronwall's inequality, we conclude $\mathcal{E}^*(t)=0$. In addition to this the conservation law (5.167) gives $||\tau||_2^2\leq C\mathcal{E}^*$. This estimate follows in the same way as (5.166). Thus $(u,\rho)=(v,\sigma)$. This finishes the proof of the uniqueness claim.

Continuity. Integrating the identity (3.68) over the time interval $[s,t]$, we obtain

$$\mathcal{E}^{m+1}(t)-\mathcal{E}^{m+1}(s)+\int_s^t \mathcal{D}^{m+1}(s)=\int_s^t\int_\Omega P^m+R^m-\int_s^t\int_{\mathbb{T}^{n-1}} Q^m+S^m+T^m. \tag{5.173}$$

However, since $(u^l,\rho^l)\to(u,\rho)$ strongly in the energy space, we may pass to the limit in (5.173) to conclude

$$\mathcal{E}_\epsilon(t)-\mathcal{E}_\epsilon(s)+\int_s^t \mathcal{D}_\epsilon(\tau)\,d\tau=\int_s^t\int_\Omega \bar{P}+\bar{R}-\int_s^t\int_{\mathbb{T}^{n-1}} \bar{Q}+\bar{S}+\bar{T}. \tag{5.174}$$

Here $\bar{P}=\sum_{|\mu|+2s\le 2k}P(\partial_s^\mu u,\rho,f_{\mu,s})$, where $f_{\mu,s}$ is defined by dropping the index m in the definition (3.63) of $f^m_{\mu,s}$. The terms $\bar{Q}$, $\bar{R}$, $\bar{S}$ and $\bar{T}$ are defined analogously. We claim that

$$\Big|\int_s^t\int_\Omega \bar{P}+\bar{R}-\int_s^t\int_{\mathbb{T}^{n-1}} \bar{Q}+\bar{S}+\bar{T}\Big|\le C\int_s^t \sqrt{\mathcal{E}_\epsilon(\tau)}\mathcal{D}_\epsilon(\tau)\,d\tau. \tag{5.175}$$

The inequality follows easily from the energy estimates in Chapter 4. We observe that the estimates involving ϵ on the right-hand side, are used only when estimating the cross-terms (cf. (4.109), (4.110), (4.118), (4.119), (4.121) and (4.122)). However, the cross-terms vanish as m goes to ∞ (since $\chi=\omega=\partial_s^\mu\rho$). As a result, we obtain the estimate (5.175) with the constant C on the right-hand side which does not depend on ϵ. Using (5.174) and (5.175), we obtain

$$\Big|\mathcal{E}_\epsilon(t)-\mathcal{E}_\epsilon(s)+\int_s^t \mathcal{D}_\epsilon(\tau)\,d\tau\Big|\le C\int_s^t \sqrt{\mathcal{E}_\epsilon(\tau)}\mathcal{D}_\epsilon(\tau)\,d\tau. \tag{5.176}$$

In addition to that, for any $0\le s<t\le t^\epsilon$ we have

$$\big|\mathcal{E}_\epsilon(t)-\mathcal{E}_\epsilon(s)\big|\le C\Big|\int_s^t \mathcal{D}_\epsilon(\tau)\,d\tau\Big|\Big(1+\sup_{0\le s\le t^\epsilon}\sqrt{\mathcal{E}_\epsilon(s)}\Big)\longrightarrow 0 \quad \text{as} \quad s\to t,$$

since $\sup_{0\le s\le t^\epsilon}\sqrt{\mathcal{E}_\epsilon(s)}\le\sqrt{L}$. This finishes the proof of Theorem 5.1. □

6 Global stability

Proof of Theorem 1.2: We exploit the estimate (5.176) to prove the theorem. We shall abbreviate $(\mathcal{E},\mathcal{D})(u,\rho;\rho)(t) =: (\mathcal{E},\mathcal{D})(t)$ and $(\mathcal{E}_\epsilon,\mathcal{D}_\epsilon)(u^\epsilon,\rho^\epsilon;\rho^\epsilon)(t) =: (\mathcal{E}_\epsilon,\mathcal{D}_\epsilon)(t)$.

Existence. Let $M \le L/2$ where L is given in Lemma 4.1. Let $(u^\epsilon,\rho^\epsilon)$ be the associated solution to the regularized Stefan problem on the time interval $[0,t^\epsilon]$ given by Theorem 5.1. Set

$$\mathcal{T} := \sup_t \Big\{ t : \sup_{0\le s\le t} \mathcal{E}_\epsilon(u^\epsilon,\rho^\epsilon)(s) + \int_0^t \mathcal{D}_\epsilon(u^\epsilon,\rho^\epsilon)(\tau)\,d\tau \le 2M \Big\}.$$

Theorem 5.1 guarantees $\mathcal{T} \ge t^\epsilon > 0$. For any $t < \mathcal{T}$, the estimate (5.176) with $s = 0$ implies

$$\mathcal{E}_\epsilon(t) + \int_0^t \mathcal{D}_\epsilon(\tau)\,d\tau \le \mathcal{E}_\epsilon(u_0^\epsilon,\rho_0^\epsilon) + C \sup_{0\le s\le \mathcal{T}} \mathcal{E}_\epsilon^{\frac{1}{2}}(s) \int_0^t \mathcal{D}_\epsilon(\tau)\,d\tau, \tag{6.177}$$

and thus

$$\sup_{0\le t\le \mathcal{T}} \mathcal{E}_\epsilon(t) + \int_0^{\mathcal{T}} \mathcal{D}_\epsilon(\tau)\,d\tau \le M + C\sqrt{2M} \int_0^{\mathcal{T}} \mathcal{D}_\epsilon(\tau)\,d\tau. \tag{6.178}$$

Choose $M < \min\{\frac{1}{32C^2}, L/2\} =: M_1$. Inequality (6.178) implies

$$\sup_{0\le t\le \mathcal{T}} \mathcal{E}_\epsilon(t) + \int_0^{\mathcal{T}} \mathcal{D}_\epsilon(\tau)\,d\tau \le \frac{4}{3} M < 2M,$$

which would contradict the choice of $\mathcal{T}$ in case $\mathcal{T}$ were finite. Thus $\mathcal{T} = \infty$ and the estimate (1.30) follows easily from (6.177) and the above choice of M. This proves the theorem.

Uniqueness. We want to prove uniqueness in the class of functions (u,ρ) satisfying $\sup_{0\le t<\infty} \mathcal{E}_\epsilon(u,\rho)(t) + \int_0^t \mathcal{D}_\epsilon(u,\rho)(\tau)\,d\tau \le 2M$, where M may be chosen smaller if necessary. It is done in exactly the same way as the uniqueness proof in Theorem 5.1. □

Proof of Theorem 1.1: *Claim 1*: Let $K \le M$ be any positive number, where M is given by Theorem 1.2. If the initial data (u_0,ρ_0) satisfy

$$\mathcal{E}(u_0,\rho_0) + \Big| \int_{\mathbb{T}^{n-1}} \rho_0 - \int_\Omega u_0(1+\phi'\rho_0) \Big| \le \frac{K}{3},$$

then there exists a unique global solution to the Stefan problem (1.11) - (1.16). Moreover, we obtain the global bound $\sup_{0\le t<\infty}\mathcal{E}(t)+\int_0^t\mathcal{D}(\tau)\,d\tau\le K$.
Proof of Claim 1. Let $\{(u^\epsilon,\rho^\epsilon)\}_\epsilon$ be a family of solutions of the regularized Stefan problem satisfying the given initial condition $(u^\epsilon(x,0),\rho^\epsilon(x',0))=(u_0^\epsilon,\rho_0^\epsilon)$, where we choose $(u_0^\epsilon,\rho_0^\epsilon)$ so that

$$(u_0^\epsilon,\rho_0^\epsilon)\to(u_0,\rho_0),\quad \mathcal{E}(u_0^\epsilon,\rho_0^\epsilon;\rho_0^\epsilon)\to\mathcal{E}(u_0,\rho_0;\rho_0)\quad\text{as}\quad\epsilon\to0$$

and

$$\sum_{|\mu|+2s\le2k}\epsilon\int_{\mathbb{T}^{n-1}}|\nabla^2\partial_s^\mu\Delta\rho_0^\epsilon|\le\sqrt{\epsilon}.$$

Thus for ϵ small, we have $\mathcal{E}_\epsilon(u_0^\epsilon,\rho_0^\epsilon)+\left|\int_{\mathbb{T}^{n-1}}\rho_0^\epsilon-\int_\Omega u_0^\epsilon(1+\phi'\rho_0^\epsilon)\right|\le\frac{K}{2}$. Theorem 1.2 guarantees global existence of the solution $(u^\epsilon,\rho^\epsilon)$ and also gives the estimate $\sup_{0\le t<\infty}\mathcal{E}_\epsilon(t)+\int_0^\infty\mathcal{D}_\epsilon(\tau)\,d\tau\le K$. Since $\mathcal{E}\le\mathcal{E}_\epsilon$ and $\mathcal{D}\le\mathcal{D}_\epsilon$, we obtain $\sup_{0\le t<\infty}\mathcal{E}(u^\epsilon,\rho^\epsilon)(t)+\int_0^\infty\mathcal{D}(u^\epsilon,\rho^\epsilon)(\tau)\,d\tau\le K$. Passing to the limit as $\epsilon\to0$, we obtain the solution (u,ρ) to the original Stefan problem (1.11) - (1.16). The uniqueness claim follows by setting $\epsilon=0$ in the proof of the uniqueness statement of Theorem 1.2. This finishes the proof of *Claim 1.* In the same way as we derived the inequality (6.177), we deduce for any $T>0$:

$$\sup_{0\le t\le T}\mathcal{E}_\epsilon(t)+\int_0^T\mathcal{D}_\epsilon(\tau)\,d\tau\le\mathcal{E}_\epsilon(u_0^\epsilon,\rho_0^\epsilon)+C\sqrt{K}\int_0^T\mathcal{D}_\epsilon(\tau)\,d\tau. \tag{6.179}$$

If we choose $K<\min\{\frac{1}{(2C)^2},M\}=:M_1$, absorb the right-most term into the left-hand side and drop the supremum sign, we obtain for any $t>0$

$$\mathcal{E}_\epsilon(t)+\frac{1}{2}\int_0^t\mathcal{D}_\epsilon(s)\,ds\le\mathcal{E}_\epsilon(u_0^\epsilon,\rho_0^\epsilon).$$

We let $\epsilon\to0$ and by lower semicontinuity and the assumptions on initial data, we obtain

$$\mathcal{E}(t)+\frac{1}{2}\int_0^t\mathcal{D}(s)\,ds\le\mathcal{E}(u_0,\rho_0). \tag{6.180}$$

In addition to this, we obtain the conservation law

$$\partial_t\Big\{\int_{\mathbb{T}^{n-1}}\rho\Big\}=\partial_t\Big\{\int_\Omega u(1+\phi'\rho)\Big\}. \tag{6.181}$$

Let us set $M^*:=\frac{M_1}{12}$, where M_1 is defined in the line after (6.179), and assume $\mathcal{E}(u_0,\rho_0)+\left|\int_{\mathbb{T}^{n-1}}\rho_0-\int_\Omega u_0(1+\phi'\rho_0)\right|\le M^*$. *Claim 1* guarantees the global existence of the solution (u,ρ) and also gives the global bound $\sup_{0\le t<\infty}\mathcal{E}(t)+\int_0^\infty \mathcal{D}(\tau)\,d\tau \le \frac{M_1}{4}$. In order to prove (1.25), we first fix any $s>0$. The idea is to solve the Stefan problem with the new initial data $(u^1(x,0),\rho^1(x',0))=(u(x,s),\rho(x',s))$. The problem allows for unique solutions by *Claim 1*, since

$$\begin{aligned}&\mathcal{E}(u_0^1,\rho_0^1)+\Big|\int_{\mathbb{T}^{n-1}}\rho_0^1-\int_\Omega u_0^1(1+\phi'\rho_0^1)\Big|\\&=\mathcal{E}(u,\rho)(s)+\Big|\int_{\mathbb{T}^{n-1}}\rho_0-\int_\Omega u_0(1+\phi'\rho_0)\Big|\le\frac{M_1}{4}+\frac{M_1}{12}=\frac{M_1}{3}.\end{aligned}$$

In addition to this we have the global bound $\sup_{0\le t<\infty}\mathcal{E}(u^1,\rho^1)(t)+\int_0^\infty\mathcal{D}(u^1,\rho^1)(\tau)\,d\tau\le M_1$ (again by *Claim 1*). We are thus in the uniqueness regime and we conclude $(u^1,\rho^1)(t)=(u,\rho)(t+s)$ for any $t\ge 0$. We may now use the estimate (6.180) to obtain (1.25).

The second main ingredient in proving the decay is to control the instant energy in terms of the dissipation, i.e. to prove that there exists a constant $C>0$ such that $\mathcal{E}(t)\le C\mathcal{D}(t)$. We know that for $|\mu|+2s\le 2k$

$$||\partial_s^\mu\nabla\rho||_{H^1}\le C\sqrt{\mathcal{D}}. \tag{6.182}$$

Thus, the only non-trivial term left to estimate is $||u||_{L^2(\Omega)}$.

Claim 2: There exists a constant $C>0$ such that $||u||_{L^2(\Omega)}\le C\sqrt{\mathcal{D}}$.

Proof of Claim 2. Let $x\in\Omega$ and $x'\in\mathbb{T}^{n-1}$ be arbitrarily chosen. By the mean value theorem

$$u(x)=u(x')+\int_0^1\nabla u(tx+(1-t)x')\cdot(x-x')\,dt.$$

Note that $\int_{\mathbb{T}^{n-1}}u(x')\,dx'=0$ because $u=\nabla\cdot\left(\frac{\nabla\rho}{\langle\rho\rangle}\right)$ on $\mathbb{T}^{n-1}$. We thus integrate with respect to x' over $\mathbb{T}^{n-1}$ and then with respect to x over Ω to obtain

$$|\mathbb{T}^{n-1}|\int_\Omega u(x)\,dx=\int_\Omega\int_{\mathbb{T}^{n-1}}\int_0^1\nabla u(tx+(1-t)x')(x-x')\,dtdx'dx.$$

Therefore

$$\begin{aligned}\Big|\int_\Omega u(x)\,dx\Big|&\le\frac{1}{|\mathbb{T}^{n-1}|}\max_{\substack{x\in\Omega\\x'\in\mathbb{T}^{n-1}}}|x-x'||\Omega||\mathbb{T}^{n-1}|||\nabla u||_{L^\infty(\Omega)}\\&\le C||\nabla u||_{L^\infty(\Omega)}\le C||\nabla u||_{H^{\frac{n}{2}+1}(\Omega)}\le C\sqrt{\mathcal{D}}.\end{aligned}$$

By the Poincaré inequality,

$$||u||_{L^2(\Omega)} \le C|| \int_\Omega u||_{L^2(\Omega)} + C||\nabla u||_{L^2(\Omega)} \le C\sqrt{\mathcal{D}}.$$

This finishes the proof of *Claim 2*. As explained above, *Claim 2* and the estimate (6.182) together, imply that there exists $C>0$ such that

$$\mathcal{E} \le C\mathcal{D}. \tag{6.183}$$

Plugging (6.183) into (1.25) yields for any $s>0$ and some constant $\alpha>0$:

$$\mathcal{E}(t) + \alpha \int_s^t \mathcal{E}(\tau)\, d\tau \le \mathcal{E}(s). \tag{6.184}$$

As in [15], p. 135, define a function $V(s) := \int_s^\infty \mathcal{E}(\tau)\, d\tau$. From (6.184), $\alpha V(s) \le \mathcal{E}(s)$,

$$V'(s) = -\mathcal{E}(s) \le -\alpha V(s)$$

and thus $V(s) \le V(0)e^{-t\alpha}$. We integrate (6.184) with respect to s over the time interval $[t/2, t]$ to get

$$\mathcal{E}(t)\frac{t}{2} \le V(\frac{t}{2}).$$

Thus $\mathcal{E}(t) \le \frac{2C}{t} e^{-\frac{\alpha}{2}t}$. There exist $k_1, K_2 > 0$ such that for any $t>0$

$$\mathcal{E}(t) \le k_1 e^{-K_2 t}. \tag{6.185}$$

Integrating the conservation law (6.181) implies $\int_{\mathbb{T}^{n-1}} (\rho(x',t) - \bar{\rho})\, dx' = \int_\Omega u(1+\phi'\rho)$. By an argument analogous to (5.166), $||\rho(t) - \bar{\rho}||_2^2 \le C\mathcal{E}(u,\rho)(t)$. Combining this inequality with (6.185), we conclude

$$\mathcal{E}(u,\rho)(t) + ||\rho(t) - \bar{\rho}||_2^2 \le K_1 e^{-K_2 t}$$

for some new constant $K_1 > 0$, K_2 as in (6.185) and for all $t \ge 0$. This finishes the proof of the theorem. □

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