

STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS DRIVEN BY PURELY SPATIAL NOISE

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ABSTRACT. We study bilinear stochastic parabolic and elliptic PDEs driven by purely spatial white noise. Even the simplest equations driven by this noise often do not have a square-integrable solution and must be solved in special weighted spaces. We demonstrate that the Cameron-Martin version of the Wiener chaos decomposition is an effective tool to study both stationary and evolution equations driven by space-only noise. The paper presents results about solvability of such equations in weighted Wiener chaos spaces and studies the long-time behavior of the solutions of evolution equations with space-only noise.

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1. INTRODUCTION

Stochastic PDEs of the form

$$\dot{u}(t, x) = \mathbf{A}u(t, x) + \mathbf{M}u(t, x) \cdot \dot{W}(t, x), \quad (1.1)$$

where \mathbf{A} and \mathbf{M} are linear partial differential operators and $\dot{W}(t, x)$ is space-time noise, are usually referred to as bilinear evolution SPDEs.¹ These equations are of interest in various applications: nonlinear filtering for hidden Markov models [30], propagation of magnetic field in random flow [2] stochastic transport [6, 7, 20]), porous media [3] and others. The theory and the applications of bilinear SPDEs have been actively investigated for a few decades now; see, for example, [4, 13, 23, 27, 28, 29, etc.].

In contrast, very little is known about bilinear parabolic and elliptic equations driven by purely spatial Gaussian white noise $\dot{W}(x)$. Important examples of these equations include:

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¹Bilinear SPDEs differ from linear by the term including multiplicative noise. Bilinear SPDEs are technically more difficult than linear. On the other hand, multiplicative models preserve many features of the unperturbed equation, such as positivity of the solution and conservation of mass, and are often more “physical”.

1. Heat equation with random potential modeled by spatial white noise:

$$\dot{u}(t, x) = \Delta u(t, x) + u(t, x) \diamond \dot{W}(x), \quad (1.2)$$

where \diamond denotes the Wick product, which, in this case, coincides with the Skorokhod integral in the sense of Malliavin calculus. A surprising discovery made in [9] was that the spatial regularity of the solution of equation (1.2) is better than in the case of similar equation driven by the space-time white noise.

2. Stochastic Poisson equations in random medium:

$$\nabla(A_\varepsilon(x) \diamond \nabla u(x)) = f(x), \quad (1.3)$$

where $A_\varepsilon(x) := (a(x) + \varepsilon \dot{W}(x))$, $a(x)$ is a deterministic positive-definite matrix, and ε is a positive number. (Note that $a(x) \diamond \nabla u(x) = a(x) \nabla u(x)$).

3. Heat equation in random medium:

$$\dot{v}(t, x) = \nabla(A_\varepsilon(x) \diamond \nabla v(t, x)) + g(t, x). \quad (1.4)$$

Note that the matrix A_ε in equations (1.3) and (1.4) is not necessarily positive definite; only its expectation $a(x)$ is.

Equations (1.3) and (1.4) are random perturbation of the deterministic Poisson and Heat equations. An important feature of these type of perturbation is that the resulting equations are *unbiased* in that they preserve the mean dynamics: the functions $u_0(x) := \mathbb{E}u(x)$ and $v_0 := \mathbb{E}v(t, x)$ solve the deterministic Poisson equation,

$$\nabla(a(x) \nabla u_0(t, x)) = \mathbb{E}f(x)$$

and the deterministic Heat equation

$$\dot{v}_0(t, x) = \nabla(a(x) v_0(t, x)) + \mathbb{E}g(t, x),$$

respectively.

The objective of this paper is to develop a systematic approach to bilinear SPDEs driven by purely spatial Gaussian noise. More specifically, we will investigate bilinear parabolic equations

$$\frac{\partial v(t, x)}{\partial t} = \mathbf{A}v(t, x) + \mathbf{M}v(t, x) \diamond \dot{W}(x) - f(x) \quad (1.5)$$

and elliptic equations

$$\mathbf{A}u(x) + \mathbf{M}u(x) \diamond \dot{W}(x) = f(x). \quad (1.6)$$

for a wide range of operators \mathbf{A} and \mathbf{M} .

Purely spatial white noise is an important type of stationary perturbations. However, except for elliptic equations with additive random forcing [5, 22, 26], SPDEs driven by spatial noise have not been investigated nearly as extensively as those driven by strictly temporal or space-time noise.

In the case of spatial white noise, there is no natural and convenient filtration, especially in the dimension $d > 2$. Therefore, it makes sense to consider anticipative solutions. This rules out Itô calculus and makes it necessary to rely on more nuanced techniques provided by Malliavin calculus.

In this paper, we deal with broad classes of operators \mathbf{A} and \mathbf{M} that were investigated previously for causal (nonanticipating) solutions of the equation (1.1) driven by space-time white noise.

The notion of ellipticity for SPDEs is more restrictive than in deterministic theory. Traditionally, nonanticipating solutions of equation (1.1) were studied under the following assumptions:

(i) *The operator $\mathbf{A} - \frac{1}{2}\mathbf{M}\mathbf{M}^*$ is “elliptic” (possibly degenerate coercive) operator.*

Of course, this assumption does not hold for equation (1.4) and other equations in which the operators \mathbf{A} and \mathbf{M} have the same order. Therefore, it is important to study equations (1.6) and (1.5) under weaker assumptions, for example

(ii) *The operator \mathbf{A} is coercive and $\text{ord}(\mathbf{M}) \leq \text{ord}(\mathbf{A})$.*

In 1981, it was shown by Krylov and Rozovskii [15] that, unless assumption (i) holds, equation (1.1) has no solutions in the space $L_2(\Omega; X)$ of square integrable (in probability) solutions in any reasonable functional space X . The same effect holds for bilinear SPDEs driven by space only white noise.

Numerous attempts to investigate solutions of stochastic PDEs violating the stochastic ellipticity conditions were made since then. In particular it was shown in [19, 21, 25] that if the operator \mathbf{A} is coercive (“elliptic”) and $\text{ord}(\mathbf{M})$ is *strictly* less than $\text{ord}(\mathbf{A})$ then there exists a unique generalized (Wiener Chaos) nonanticipative solution of equation (1.1). This generalized solution is a formal Wiener Chaos series $u = \sum_{|\alpha| < \infty} u_\alpha \xi_\alpha$, where $\{\xi_\alpha\}_{|\alpha| < \infty}$ is the Cameron-Martin orthonormal basis in the space $L_2(\Omega)$. Regularity of this solution is determined by system of positive weights $\{r_\alpha\}_{|\alpha| < \infty}$ and a function space X such that

$$\|u\|_{\mathcal{R}, X}^2 := \sum_{|\alpha| < \infty} r_\alpha^2 \|u_\alpha\|_{L_2((0, T); X)}^2 < \infty. \quad (1.7)$$

The stochastic Fourier coefficients u_α satisfy a lower-triangular system of deterministic PDEs. This system, called **propagator**, is uniquely determined by the underlying equation (1.1).

Stochastic spaces equipped with the norms similar to (1.7) have been known for quite some time; see, e.g. [11, 12, 25]. For historical remarks regarding other types of generalized solutions and applications to SPDEs see the review paper [18] and the references therein.

The Wiener Chaos is a bona fide generalization of the classical Itô solution: if exists, a non-antisipating square integrable Itô solution coincides with the Wiener Chaos solution.

In this paper, we establish existence and uniqueness of Wiener Chaos solutions for stationary (elliptic) equations of the type (1.6) and evolution (parabolic) equations of the type (1.5). These results are proved under assumption (ii) that allows us in particular to deal with equations like (1.3) and (1.4). In many cases we are able to find optimal or near-optimal systems of weights $\{r_\alpha\}_{|\alpha| < \infty}$ that guarantee (1.7)

Finally, we establish the convergence, as $t \rightarrow +\infty$, of the solution of the evolution equation to the solution of the related stationary equation.

The structure of the paper is as follows. Section 2 reviews the definition of the Skorokhod integral in the framework of the Malliavin calculus and shows how to compute the integral using Wiener chaos. Sections 3 and 4 deal with existence and uniqueness of solutions to abstract evolution and stationary equations, respectively, driven by a general (not necessarily white) spatial Gaussian noise; Section 4 describes also the limiting behavior of the solution of the evolution equation; Section 5 illustrates the general results for bilinear SPDEs driven by purely spatial white noise.

2. WEIGHTED WIENER CHAOS AND MALLIAVIN CALCULUS

Let $\mathbb{F} = (\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and \mathcal{U} , a real separable Hilbert space with inner product $(\cdot, \cdot)_{\mathcal{U}}$. On \mathbb{F} , consider a zero-mean Gaussian family

$$\dot{W} = \left\{ \dot{W}(h), h \in \mathcal{U} \right\}$$

so that

$$\mathbb{E} \left(\dot{W}(h_1) \dot{W}(h_2) \right) = (h_1, h_2)_{\mathcal{U}}.$$

It suffices, for our purposes, to assume that \mathcal{F} is the σ -algebra generated by \dot{W} . Given a real separable Hilbert space X , we denote by $L_2(\mathbb{F}; X)$ the Hilbert space of square-integrable \mathcal{F} -measurable X -valued random elements f . In particular,

$$(f, g)_{L_2(\mathbb{F}; X)}^2 := \mathbb{E}(f, g)_X^2.$$

When $X = \mathbb{R}$, we write $L_2(\mathbb{F})$ instead of $L_2(\mathbb{F}; \mathbb{R})$.

Definition 2.1. *A formal series*

$$\dot{W} = \sum_k \dot{W}(\mathbf{u}_k) \mathbf{u}_k, \tag{2.1}$$

where $\{\mathbf{u}_k, k \geq 1\}$ is a complete orthonormal basis in \mathcal{U} , is called (Gaussian) **white noise** on \mathcal{U} .

The white noise on $\mathcal{U} = L_2(G)$, where G is a domain in \mathbb{R}^d , is usually referred to as a **spatial or space white noise** (on $L_2(G)$). The space white noise is of central importance for this paper.

Below, we will introduce a class of spaces that are convenient for treating nonlinear functionals of white noise, in particular, solutions of SPDEs driven by white noise.

Given an orthonormal basis $\mathcal{U} = \{\mathbf{u}_k, k \geq 1\}$ in \mathcal{U} , define a collection $\{\xi_k, k \geq 1\}$ of independent standard Gaussian random variables so that $\xi_k = \dot{W}(\mathbf{u}_k)$. Denote by \mathcal{J} the collection of multi-indices α with $\alpha = (\alpha_1, \alpha_2, \dots)$ so that each α_k is a non-negative integer and $|\alpha| := \sum_{k \geq 1} \alpha_k < \infty$. For $\alpha, \beta \in \mathcal{J}$, we define

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots), \quad \alpha! = \prod_{k \geq 1} \alpha_k!$$

By (0) we denote the multi-index with all zeroes. By ε_i we denote the multi-index α with $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$. With this notation, $n\varepsilon_i$ is the multi-index α with $\alpha_i = n$ and $\alpha_j = 0$ for $j \neq i$.

Define the collection of random variables $\Xi = \{\xi_\alpha, \alpha \in \mathcal{J}\}$ as follows:

$$\xi_\alpha = \prod_k \left(\frac{H_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}} \right), \quad (2.2)$$

where

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \quad (2.3)$$

is Hermite polynomial of order n .

Theorem 2.2 (Cameron and Martin [1]). *The collection $\Xi = \{\xi_\alpha, \alpha \in \mathcal{J}\}$ is an orthonormal basis in $L_2(\mathbb{F})$: if $\eta \in L_2(\mathbb{F})$ and $\eta_\alpha = \mathbb{E}(\eta \xi_\alpha)$, then $\eta = \sum_{\alpha \in \mathcal{J}} \eta_\alpha \xi_\alpha$ and*

$$\mathbb{E}|\eta|^2 = \sum_{\alpha \in \mathcal{J}} \eta_\alpha^2.$$

Expansions with respect to the Cameron-Martin basis Ξ is usually referred to as Wiener Chaos. Next, we introduce a modification of the Wiener Chaos expansion which will be called *weighted Wiener Chaos*.

Let \mathcal{R} be a bounded linear operator on $L_2(\mathbb{F})$ defined by $\mathcal{R}\xi_\alpha = r_\alpha \xi_\alpha$ for every $\alpha \in \mathcal{J}$, where the *weights* $\{r_\alpha, \alpha \in \mathcal{J}\}$ are positive numbers. By Theorem 2.2, \mathcal{R} is bounded if and only if the weights r_α are uniformly bounded from above: $r_\alpha < C$ for all $\alpha \in \mathcal{J}$, with C independent of α . The inverse operator \mathcal{R}^{-1} is defined by $\mathcal{R}^{-1}\xi_\alpha = r_\alpha^{-1}\xi_\alpha$.

We extend \mathcal{R} to an operator on $L_2(\mathbb{F}; X)$ by defining $\mathcal{R}f$ as the unique element of $L_2(\mathbb{F}; X)$ so that, for all $g \in L_2(\mathbb{F}; X)$,

$$\mathbb{E}(\mathcal{R}f, g)_X = \sum_{\alpha \in \mathcal{J}} r_\alpha \mathbb{E}((f, g)_X \xi_\alpha).$$

Denote by $\mathcal{R}L_2(\mathbb{F}; X)$ the closure of $L_2(\mathbb{F}; X)$ with respect to the norm

$$\|f\|_{\mathcal{R}L_2(\mathbb{F}; X)}^2 := \|\mathcal{R}f\|_{L_2(\mathbb{F}; X)}^2.$$

Then the elements of $\mathcal{R}L_2(\mathbb{F}; X)$ can be identified with a formal series $\sum_{\alpha \in \mathcal{J}} f_\alpha \xi_\alpha$, where $f_\alpha \in X$ and $\sum_{\alpha \in \mathcal{J}} \|f_\alpha\|_X^2 r_\alpha^2 < \infty$.

We define the space $\mathcal{R}^{-1}L_2(\mathbb{F}; X)$ as the dual of $\mathcal{R}L_2(\mathbb{F}; X)$ relative to the inner product in the space $L_2(\mathbb{R}; X)$:

$$\mathcal{R}^{-1}L_2(\mathbb{F}; X) = \{g \in L_2(\mathbb{F}; X) : \mathcal{R}^{-1}g \in L_2(\mathbb{F}; X)\}.$$

For $f \in \mathcal{R}L_2(\mathbb{F}; X)$ and $g \in \mathcal{R}^{-1}L_2(\mathbb{F}; X)$ we define the scalar product

$$\langle\langle f, g \rangle\rangle := \mathbb{E}((\mathcal{R}f)(\mathcal{R}^{-1}g)) \in X. \quad (2.4)$$

In what follows, we will identify the operator \mathcal{R} with the corresponding collection $(r_\alpha, \alpha \in \mathcal{J})$. Note that if $u \in \mathcal{R}_1L_2(\mathbb{F}; X)$ and $v \in \mathcal{R}_2L_2(\mathbb{F}; X)$, then both u and v belong to $\mathcal{R}L_2(\mathbb{F}; X)$, where $r_\alpha = \min(r_{1,\alpha}, r_{2,\alpha})$. As usual, the argument X will be omitted if $X = \mathbb{R}$.

Important particular cases of the space $\mathcal{R}L_2(\mathbb{F}; X)$ correspond to the following weights:

(1)

$$r_\alpha^2 = \prod_{k=1}^{\infty} q_k^{\alpha_k},$$

where $\{q_k, k \geq 1\}$ is a non-increasing sequence of positive numbers with $q_1 \leq 1$ (see [19, 25]);

(2)

$$r_\alpha^2 = (\alpha!)^\rho (2\mathbb{N})^{\ell\alpha}, \quad \rho \leq 0, \quad \ell \leq 0, \quad \text{where } (2\mathbb{N})^{\ell\alpha} = \prod_{k \geq 1} (2k)^{\ell\alpha_k}. \quad (2.5)$$

This set of weights defines Kondratiev's spaces $(\mathcal{S})_{\rho, \ell}(X)$.

Now we will sketch the basics of Malliavin calculus on $\mathcal{R}L_2(\mathbb{F}; X)$.

Denote by \mathbf{D} the *Malliavin derivative* on $L_2(\mathbb{F})$ (see e.g. [24]). In particular, if $F : \mathbb{R}^N \rightarrow \mathbb{R}$ is a smooth function and $h_i \in \mathcal{U}$, $i = 1, \dots, N$, then

$$\mathbf{D}F(\dot{W}(h_1), \dots, \dot{W}(h_N)) = \sum_{i=1}^N \frac{\partial F}{\partial x_i}(\dot{W}(h_1), \dots, \dot{W}(h_N)) h_i \in L_2(\mathbb{F}; \mathcal{U}). \quad (2.6)$$

It is known [24] that the domain $\mathbb{D}^{1,2}(\mathbb{F})$ of the operator \mathbf{D} is a dense linear subspace of $L_2(\mathbb{F})$.

The adjoint of the Malliavin derivative on $L_2(\mathbb{F})$ is the Itô-Skorokhod integral and is traditionally denoted by δ [24]. We will keep this notation for the extension of this operator to $\mathcal{R}L_2(\mathbb{F}; X \otimes \mathcal{U})$.

For $f \in \mathcal{R}L_2(\mathbb{F}; X \otimes \mathcal{U})$, we define $\delta(f)$ as the unique element of $\mathcal{R}L_2(\mathbb{F}; X)$ with the property

$$\langle\langle \delta(f), \varphi \rangle\rangle = \mathbb{E}(\mathcal{R}f, \mathcal{R}^{-1}\mathbf{D}\varphi)_{\mathcal{U}} \quad (2.7)$$

for every φ satisfying $\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F})$ and $\mathbf{D}\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F}; \mathcal{U})$.

Next, we derive the expressions for the Malliavin derivative \mathbf{D} and its adjoint δ in the basis Ξ . To begin, we compute $\mathbf{D}(\xi_\alpha)$.

Proposition 2.3. *For each $\alpha \in \mathcal{J}$, we have*

$$\mathbf{D}(\xi_\alpha) = \sum_{k \geq 1} \sqrt{\alpha_k} \xi_{\alpha - \varepsilon_k} \mathbf{u}_k. \quad (2.8)$$

Proof. The result follows by direct computation using the property (2.6) of the Malliavin derivative and the relation $H'_n(x) = nH_{n-1}(x)$ for the Hermite polynomials (cf. [24]). \square

Obviously, the set \mathcal{J} is not invariant with respect to subtraction. In particular, the expression $\alpha - \varepsilon_k$ is undefined if $\alpha_k = 0$. In (2.8) and everywhere below in this paper where undefined expressions of this type appear, we use the following convention: if $\alpha_k = 0$, then $\sqrt{\alpha_k} \xi_{\alpha - \varepsilon_k} = 0$.

Proposition 2.4. For $\xi_\alpha \in \Xi$, $h \in X$, and $\mathbf{u}_k \in \mathfrak{U}$, we have

$$\delta(\xi_\alpha h \otimes \mathbf{u}_k) = h \sqrt{\alpha_k + 1} \xi_{\alpha + \varepsilon_k}. \quad (2.9)$$

Proof. It is enough to verify (2.7) with $f = h \otimes \mathbf{u}_k \xi_\alpha$ and $\varphi = \xi_\beta$, where $h \in X$. By (2.8),

$$\mathbb{E}(f, \mathbf{D}\varphi)_U = \sqrt{\beta_k} h \mathbb{E}(\xi_\alpha \xi_{\beta - \varepsilon_k}) = \begin{cases} \sqrt{\alpha_k + 1} h, & \text{if } \alpha = \beta - \varepsilon_k, \\ 0, & \text{if } \alpha \neq \beta - \varepsilon_k. \end{cases}$$

In other words,

$$\mathbb{E}(\xi_\alpha h \otimes \mathbf{u}_k, \mathbf{D}\xi_\beta)_U = h \mathbb{E}(\sqrt{\alpha_k + 1} \xi_{\alpha + \varepsilon_k} \xi_\beta)$$

for all $\beta \in \mathcal{J}$. □

Remark 2.5. The operator $\delta \mathbf{D}$ is linear and unbounded on $L_2(\mathbb{F})$; it follows from Propositions 2.3 and 2.4 that the random variables ξ_α are eigenfunctions of this operator:

$$\delta(\mathbf{D}(\xi_\alpha)) = |\alpha| \xi_\alpha. \quad (2.10)$$

To give an alternative characterization of the operator δ , we define a new operation on the elements of Ξ .

Definition 2.6. For ξ_α, ξ_β from Ξ , define the Wick product

$$\xi_\alpha \diamond \xi_\beta := \sqrt{\left(\frac{(\alpha + \beta)!}{\alpha! \beta!}\right)} \xi_{\alpha + \beta}. \quad (2.11)$$

In particular, taking in (2.6) $\alpha = k\varepsilon_i$ and $\beta = n\varepsilon_i$, and using (2.2), we get

$$H_k(\xi_i) \diamond H_n(\xi_i) = H_{k+n}(\xi_i). \quad (2.12)$$

By linearity, we define the Wick product $f \diamond \eta$ for $f \in \mathcal{RL}_2(\mathbb{F}; X)$ and $\eta \in \mathcal{RL}_2(\mathbb{F})$: if $f = \sum_{\alpha \in \mathcal{J}} f_\alpha \xi_\alpha$, $f_\alpha \in X$, and $\eta = \sum_{\alpha \in \mathcal{J}} \eta_\alpha \xi_\alpha$, $\eta_\alpha \in \mathbb{R}$, then

$$f \diamond \eta = \sum_{\alpha, \beta} f_\alpha \eta_\beta \xi_\alpha \diamond \xi_\beta.$$

Proposition 2.7. If $f \in \mathcal{RL}_2(\mathbb{F}; X)$ and $\eta \in \mathcal{RL}_2(\mathbb{F})$, then $f \diamond \eta$ is an element of $\mathcal{RL}_2(\mathbb{F}; X)$ for a suitable operator $\bar{\mathcal{R}}$.

Proof. It follows from (2.6) that $f \diamond \eta = \sum_{\alpha \in \mathcal{J}} F_\alpha \xi_\alpha$ and

$$F_\alpha = \sum_{\beta, \gamma \in \mathcal{J}: \beta + \gamma = \alpha} \sqrt{\left(\frac{(\beta + \gamma)!}{\beta! \gamma!}\right)} f_\beta \eta_\gamma.$$

Therefore, each $F_\alpha X$ is an element of X , because, for every $\alpha \in \mathcal{J}$, there are only finitely many multi-indices β, γ satisfying $\beta + \gamma = \alpha$. It is known [18, Proposition 7.1] that

$$\sum_{\alpha \in \mathcal{J}} (2\mathbb{N})^{q\alpha} < \infty \quad \text{if and only if } q < -1. \quad (2.13)$$

Therefore, $f \diamond \eta \in \bar{\mathcal{R}}L_2(\mathbb{F}; X)$, where the operator $\bar{\mathcal{R}}$ can be defined using the weights $\bar{r}_\alpha^2 = (2\mathbb{N})^{-2\alpha} / (1 + \|F_\alpha\|_X^2)$. \square

An immediate consequence of Proposition 2.4 and Definition 2.6 is the following identity:

$$\delta(\xi_\alpha h \otimes \mathbf{u}_k) = h \xi_\alpha \diamond \xi_k, \quad h \in X. \quad (2.14)$$

Below we summarize the properties of the operator δ .

Theorem 2.8. *If f is an element of $\mathcal{R}L_2(\mathbb{F}; X \otimes \mathcal{U})$ so that $f = \sum_{k \geq 1} f_k \otimes \mathbf{u}_k$, with $f_k = \sum_{\alpha \in \mathcal{J}} f_{k,\alpha} \xi_\alpha \in \mathcal{R}L_2(\mathbb{F}; X)$, then*

$$\delta(f) = \sum_{k \geq 1} f_k \diamond \xi_k, \quad (2.15)$$

and

$$(\delta(f))_\alpha = \sum_{k \geq 1} \sqrt{\alpha_k} f_{k,\alpha - \varepsilon_k}. \quad (2.16)$$

Proof. By linearity and (2.14),

$$\delta(f) = \sum_{k \geq 1} \sum_{\alpha \in \mathcal{J}} \delta(\xi_\alpha f_{k,\alpha} \otimes \mathbf{u}_k) = \sum_{k \geq 1} \sum_{\alpha \in \mathcal{J}} f_{k,\alpha} \xi_\alpha \diamond \xi_k = \sum_{k \geq 1} f_k \diamond \xi_k,$$

which is (2.15). On the other hand, by (2.9),

$$\delta(f) = \sum_{k \geq 1} \sum_{\alpha \in \mathcal{J}} f_{k,\alpha} \sqrt{\alpha_k + 1} \xi_{\alpha + \varepsilon_k} = \sum_{k \geq 1} \sum_{\alpha \in \mathcal{J}} f_{k,\alpha - \varepsilon_k} \sqrt{\alpha_k} \xi_\alpha,$$

and (2.16) follows. \square

Remark 2.9. *It is not difficult to show that the operator δ can be considered as an extension of the Skorokhod integral to the weighted spaces $\mathcal{R}L_2(\mathbb{F}; X \otimes \mathcal{U})$.*

One way to describe a multi-index α with $|\alpha| = n > 0$ is by its characteristic set K_α , that is, an ordered n -tuple $K_\alpha = \{k_1, \dots, k_n\}$, where $k_1 \leq k_2 \leq \dots \leq k_n$ characterize the locations and the values of the non-zero elements of α . More precisely, k_1 is the index of the first non-zero element of α , followed by $\max(0, \alpha_{k_1} - 1)$ of entries with the same value. The next entry after that is the index of the second non-zero element of α , followed by $\max(0, \alpha_{k_2} - 1)$ of entries with the same value, and so on. For example, if $n = 7$ and $\alpha = (1, 0, 2, 0, 0, 1, 0, 3, 0, \dots)$, then the non-zero elements of α are $\alpha_1 = 1$, $\alpha_3 = 2$, $\alpha_6 = 1$, $\alpha_8 = 3$. As a result, $K_\alpha = \{1, 3, 3, 6, 8, 8, 8\}$, that is, $k_1 = 1$, $k_2 = k_3 = 3$, $k_4 = 6$, $k_5 = k_6 = k_7 = 8$.

Using the notion of the characteristic set, we now state the following analog of the well-known result of Itô [10] connecting multiple Wiener integrals and Hermite polynomials.

Proposition 2.10. *Let $\alpha \in \mathcal{J}$ be a multi-index with $|\alpha| = n \geq 1$ and characteristic set $K_\alpha = \{k_1, \dots, k_n\}$. Then*

$$\xi_\alpha = \frac{\xi_{k_1} \diamond \xi_{k_2} \diamond \dots \diamond \xi_{k_n}}{\sqrt{\alpha!}}. \quad (2.17)$$

Proof. This follows from (2.2) and (2.12), because by (2.12), for every i and k ,

$$H_k(\xi_i) = \underbrace{\xi_i \diamond \cdots \diamond \xi_i}_{k \text{ times}}.$$

□

3. EVOLUTION EQUATIONS DRIVEN BY WHITE NOISE

3.1. The setting. In this section we study anticipating solutions of stochastic evolution equations driven by Gaussian white noise on a Hilbert space \mathcal{U} .

Definition 3.1. *The triple (V, H, V') of Hilbert spaces is called **normal** if and only if*

- (1) $V \hookrightarrow H \hookrightarrow V'$ and both embeddings $V \hookrightarrow H$ and $H \hookrightarrow V'$ are dense and continuous;
- (2) The space V' is the dual of V relative to the inner product in H ;
- (3) There exists a constant $C > 0$ so that $|(h, v)_H| \leq C \|v\|_V \|h\|_{V'}$ for all $v \in V$ and $h \in H$.

For example, the Sobolev spaces $(H_2^{\ell+\gamma}(\mathbb{R}^d), H_2^\ell(\mathbb{R}^d), H_2^{\ell-\gamma}(\mathbb{R}^d))$, $\gamma > 0$, $\ell \in \mathbb{R}$, form a normal triple.

Denote by $\langle v', v \rangle$, $v' \in V'$, $v \in V$, the duality between V and V' relative to the inner product in H . The properties of the normal triple imply that $|\langle v', v \rangle| \leq C \|v\|_V \|v'\|_{V'}$, and, if $v' \in H$ and $v \in V$, then $\langle v', v \rangle = (v', v)_H$.

We will also use the following notation:

$$\mathcal{V} = L_2((0, T); V), \quad \mathcal{H} = L_2((0, T); H), \quad \mathcal{V}' = L_2((0, T); V'). \quad (3.1)$$

Given a normal triple (V, H, V') , let $\mathbf{A} : V \rightarrow V'$ and $\mathbf{M} : V \rightarrow V' \otimes \mathcal{U}$ be bounded linear operators.

Definition 3.2. *The solution of the stochastic evolution equation*

$$\dot{u} = \mathbf{A}u + f + \delta(\mathbf{M}u), \quad 0 < t \leq T, \quad (3.2)$$

with $f \in \mathcal{R}L_2(\mathbb{F}; \mathcal{V}')$ and $u|_{t=0} = u_0 \in \mathcal{R}L_2(\mathbb{F}; H)$, is a process $u \in \mathcal{R}L_2(\mathbb{F}; \mathcal{V})$ so that, for every φ satisfying $\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F})$ and $\mathbf{D}\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F}; \mathcal{U})$, the equality

$$\langle\langle u(t), \varphi \rangle\rangle = \langle\langle u_0, \varphi \rangle\rangle + \int_0^t \langle\langle \mathbf{A}u(s) + f(s) + \delta(\mathbf{M}u)(s), \varphi \rangle\rangle ds \quad (3.3)$$

holds in \mathcal{V}' ; see (2.4) for the definition of $\langle\langle \cdot, \cdot \rangle\rangle$.

Remark 3.3. (a) *The solutions described by Definitions 3.2 and 4.1 belong to the class of “variational solutions”, which is quite typical for partial differential equations (see [14, 16, 17, 28], etc.)*

(b) *Since $\langle\langle u(t), \varphi \rangle\rangle \in \mathcal{V}$ and $\langle\langle u(t), \varphi \rangle\rangle_t \in \mathcal{V}'$, by the standard embedding theorem (see e.g. [17, Section 1.2.2]) there exists a version of $\langle\langle u(t), \varphi \rangle\rangle \in \mathbf{C}([0, T]; H)$. Clearly, one could also select a version of $u(t)$ such that $\langle\langle u(t), \varphi \rangle\rangle \in \mathbf{C}([0, T]; H)$. In the*

future, we will consider only this version of the solution. By doing this we ensure that formula (3.3) which is understood as an equality in \mathcal{V}' yields $u|_{t=0} = u_0 \in \mathcal{R}L_2(\mathbb{F}; H)$.

Remark 3.4. To simplify the notations and the overall presentation, we assume that \mathbf{A} and \mathbf{M} do not depend on time, even though many of the results in this paper can easily be extended to time-dependent operators.

Fix an orthonormal basis \mathfrak{U} in \mathcal{U} . Then, for every $v \in V$, there exists a collection $v_k \in V'$, $k \geq 1$, so that

$$\mathbf{M}v = \sum_{k \geq 1} v_k \otimes \mathbf{u}_k.$$

We therefore define the operators $\mathbf{M}_k : V \rightarrow V'$ by setting $\mathbf{M}_k v = v_k$ and write

$$\mathbf{M}v = \sum_{k \geq 1} (\mathbf{M}_k v) \otimes \mathbf{u}_k.$$

By (2.15), equation (3.2) becomes

$$\dot{u}(t) = \mathbf{A}u(t) + f(t) + \mathbf{M}u(t) \diamond \dot{W}, \quad (3.4)$$

where

$$\mathbf{M}v \diamond \dot{W} := \sum_{k \geq 1} (\mathbf{M}_k v) \diamond \xi_k. \quad (3.5)$$

3.2. Equivalence Theorem. In this section we investigate stochastic Fourier representation of equation (3.4).

Recall that every process $u = u(t)$ from $\mathcal{R}L_2(\mathbb{F}; \mathcal{V})$ is represented by a formal series $u(t) = \sum_{\alpha \in \mathcal{J}} u_\alpha(t) \xi_\alpha$, with $u_\alpha \in \mathcal{V}$ and

$$\sum_{\alpha} r_\alpha^2 \|u_\alpha\|_{\mathcal{V}}^2 < \infty. \quad (3.6)$$

Theorem 3.5. Let $u = \sum_{\alpha \in \mathcal{J}} u_\alpha \xi_\alpha$ be an element of $\mathcal{R}L_2(\mathbb{F}; \mathcal{V})$. The process u is a solution of equation (3.2) if and only if the functions u_α have the following properties:

- (1) every u_α is an element of $\mathbf{C}([0, T]; H)$
- (2) the system of equalities

$$u_\alpha(t) = u_{0,\alpha} + \int_0^t \left(\mathbf{A}u_\alpha(s) + f_\alpha(s) + \sum_{k \geq 1} \sqrt{\alpha_k} \mathbf{M}_k u_{\alpha - \varepsilon_k}(s) \right) ds \quad (3.7)$$

holds in V' for all $t \in [0, T]$ and $\alpha \in \mathcal{J}$.

Proof. Let u be a solution of (3.2) in $\mathcal{R}L_2(\mathbb{F}; \mathcal{V})$. Taking $\varphi = \xi_\alpha$ in (3.3) and using relation (2.16), we obtain equation (3.7). By Remark 3.3 $u_\alpha \in \mathcal{V} \cap \mathbf{C}([0, T]; H)$.

Conversely, let $\{u_\alpha, \alpha \in \mathcal{J}\}$ be a collection of functions from $\mathcal{V} \cap \mathbf{C}([0, T]; H)$ satisfying (3.6) and (3.7). Set $u(t) := \sum_{\alpha \in \mathcal{J}} u_\alpha(t) \xi_\alpha$. Then, by Theorem 2.8, equation (3.7) yields that, for every $\alpha \in \mathcal{J}$,

$$\langle\langle u(t), \xi_\alpha \rangle\rangle = \langle\langle u_0, \xi_\alpha \rangle\rangle + \int_0^t \langle\langle \mathbf{A}u(s) + f(s) + \delta(\mathbf{M}u)(s), \xi_\alpha \rangle\rangle ds.$$

By continuity, we conclude that for any $\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F})$ such that $\mathbf{D}\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F}; \mathcal{U})$, equality

$$\langle\langle u(t), \varphi \rangle\rangle = \langle\langle u_0, \varphi \rangle\rangle + \int_0^t \langle\langle \mathbf{A}u(s) + f(s) + \delta(\mathbf{M}u)(s), \varphi \rangle\rangle ds$$

holds in \mathcal{V}' . By Remark 3.3 $\langle\langle u(t), \varphi \rangle\rangle \in \mathbf{C}([0, T]; H)$. \square

This simple but very helpful result establishes the equivalence of the “physical” (3.4) and the (stochastic) Fourier (3.7) forms of equation (3.2). System of equations (3.7) is often referred in the literature as the *propagator* of equation (3.4). Note that the propagator is lower-triangular and can be solved by induction on $|\alpha|$.

3.3. Existence and uniqueness. Below, we will present several results on existence and uniqueness of evolution equations driven by Gaussian white noise.

Before proceeding with general existence-uniqueness problems, we will introduce two simple examples that indicate the limits of the “quality” of solutions of bi-linear SPDEs driven by general Gaussian white noise.

Example 3.6. Consider equation

$$u(t) = \phi + \int_0^t (bu(s) \diamond \xi - \lambda u(s)) ds, \quad (3.8)$$

where ϕ, λ are real numbers, b is a complex number, and ξ is a standard Gaussian random variable. In other words ξ is Gaussian white noise on $\mathcal{U} = \mathbb{R}$. With only one Gaussian random variable ξ , the set \mathcal{J} becomes $\{0, 1, 2, \dots\}$ so that $u(t) = \sum_{n \geq 0} u_{(n)}(t) H_n(\xi) / \sqrt{n!}$, where H_n is Hermite polynomial of order n (2.3). According to (3.7),

$$u_{(n)}(t) = \phi I_{(n=0)} - \int_0^t \lambda u_{(n)}(s) ds + \int_0^t b \sqrt{n} u_{(n-1)}(s) ds.$$

It follows that $u_{(0)}(t) = \phi e^{-\lambda t}$ and then, by induction, $u_{(n)}(t) = \phi \frac{(bt)^n}{\sqrt{n!}} e^{-\lambda t}$. As a result,

$$u(t) = e^{-\lambda t} \left(\phi + \sum_{n \geq 1} \frac{(bt)^n}{n!} H_n(\xi) \right) = \phi e^{-\lambda t + (bt\xi - |b|^2 t^2 / 2)}.$$

Obviously, the solution of the equation is square integrable on any fixed time interval. However, as the next example indicates, the solutions of SPDEs driven by stationary noise are much more intricate than the non-anticipating, or adapted, solutions of SPDEs driven by space-time white noise.

Example 3.7. With ξ as in the previous examples, consider a partial differential equation

$$u_t(t, x) = au_{xx}(t, x) + (\beta u(t, x) + \sigma u_x(t, x)) \diamond \xi, \quad t > 0, \quad x \in \mathbb{R}, \quad (3.9)$$

with some initial condition $u_0 \in L_2(\mathbb{R})$. By taking the Fourier transform and using the results of Example 3.6 with $\phi = \hat{u}_0(y)$, $\lambda = -ay^2$, $b = \beta + \sqrt{-1}y\sigma$, we find

$$\begin{aligned}\hat{u}_t(t, y) &= -y^2 a \hat{u}(t) + (\beta + \sqrt{-1}y\sigma) \hat{u}(t, y) \diamond \xi; \\ \hat{u}(t, y) &= \hat{u}_0(y) \exp(-tay^2 + (\sigma^2 y^2 - \beta^2) t^2/2 + \sqrt{-1}\beta\sigma y t^2 + (\sqrt{-1}\sigma y + \beta) t\xi).\end{aligned}$$

If $\sigma = 0$, i.e. the ‘‘diffusion’’ operator in equation (3.9) is of order zero, then the solution belongs to $L_2(\mathbb{F}; L_2(\mathbb{R}))$ for all t . However, if $\sigma > 0$, then the solution $u(t, \cdot)$ will, in general, belong to $L_2(\mathbb{F}; L_2(\mathbb{R}))$ only for $t \leq 2a/\sigma^2$. This blow-up in finite time is in sharp contrast with the solution of the equation

$$u_t = au_{xx} + \sigma u_x \diamond \dot{w}, \quad (3.10)$$

driven by the standard one-dimensional white noise $\dot{w}(t) = \partial_t W(t)$, where $W(t)$ is the one-dimensional Brownian motion; a more familiar way of writing (3.10) is in the Itô form

$$du = au_{xx}dt + \sigma u_x dW(t). \quad (3.11)$$

It is well known (see, for example, [28]) that the solution of (3.11) belongs to $L_2(\mathbb{F}; L_2(\mathbb{R}))$ for every $t > 0$ as long as $u_0 \in L_2(\mathbb{R})$ and

$$a - \sigma^2/2 \geq 0. \quad (3.12)$$

The existence of a square integrable (global) solution of an Itô’s SPDE with square integrable initial condition hinges on the parabolic condition which in the case of equation (3.10) is given by (3.12). Example 3.7 shows that this condition is not in any way sufficient for SPDEs involving a Skorokhod-type integral. The next theorem provides sufficient conditions for the existence and uniqueness of a solution to equation (3.4) in the space $\mathcal{R}L_2(\mathbb{F}; \mathcal{V})$, which appears to be a reasonable extension of the class of square integrable solutions.

Firstly, we introduce an additional assumption on the operator \mathbf{A} that will be used throughout this section:

(A): For every $U_0 \in H$ and $F \in \mathcal{V}' := L_2((0, T); V')$, there exists a function $U \in \mathcal{V}$ that solve the deterministic equation

$$\partial_t U(t) = \mathbf{A}U(t) + F(t), \quad U(0) = U_0, \quad (3.13)$$

and there exists a constant $C = C(\mathbf{A}, T)$ so that

$$\|U\|_{\mathcal{V}} \leq C(\mathbf{A}, T)(\|U_0\|_H + \|F\|_{\mathcal{V}'}). \quad (3.14)$$

Remark 3.8. Assumption (A) implies that a solution of equation (3.13) is unique and belongs to $\mathbf{C}((0, T); H)$ (cf. Remark 3.3). The assumption also implies that the operator \mathbf{A} generates a semi-group $\Phi = \Phi_t$, $t \geq 0$, and, for every $v \in \mathcal{V}$,

$$\int_0^T \left\| \int_0^t \Phi_{t-s} \mathbf{M}_k v(s) ds \right\|_{\mathcal{V}}^2 dt \leq C_k^2 \|v\|_{\mathcal{V}}^2, \quad (3.15)$$

with numbers C_k independent of v .

Remark 3.9. *There are various types of assumptions on the operator \mathbf{A} that yield the statement of the assumption (A). In particular, (A) holds if the operator \mathbf{A} is coercive in (V, H, V') :*

$$\langle \mathbf{A}v, v \rangle + \gamma \|v\|_V^2 \leq C \|v\|_H^2$$

for every $v \in V$, where $\gamma > 0$ and $C \in \mathbb{R}$ are both independent of v .

Theorem 3.10. *Assume (A). Consider equation (3.4) in which $u_0 \in \bar{\mathcal{R}}L_2(\mathbb{F}; H)$, $f \in \bar{\mathcal{R}}L_2(\mathbb{F}; \mathcal{V}')$ for some operator \mathcal{R} , and each \mathbf{M}_k is a bounded linear operator from V to V' .*

Then there exist an operator \mathcal{R} and a unique solution $u \in \mathcal{R}L_2(\mathbb{F}; \mathcal{V})$ of (3.4).

Proof. By Theorem 3.5, it suffices to prove that the propagator (3.7) has a unique solution $(u_\alpha(t))_{\alpha \in \mathcal{J}}$ such that for each α , $u_\alpha \in \mathcal{V} \cap \mathbf{C}([0, T]; H)$ and $u := \sum_{\alpha \in \mathcal{J}} u_\alpha \xi_\alpha \in \mathcal{R}L_2(\mathbb{F}; \mathcal{V})$.

For $\alpha = (0)$, that is, when $|\alpha| = 0$, equation (3.7) reduces to

$$u_{(0)}(t) = u_{0,(0)} + \int_0^t (\mathbf{A}u_{(0)} + f_{(0)})(s) ds.$$

By (A), this equation has a unique solution and

$$\|u_{(0)}\|_{\mathcal{V}} \leq C(\mathbf{A}, T) (\|u_{0,(0)}\|_H + \|f_{(0)}\|_{\mathcal{V}'}).$$

Using assumption (A), it follows by induction on $|\alpha|$ that, for every $\alpha \in \mathcal{J}$, equation

$$\partial_t u_\alpha(t) = \mathbf{A}u_\alpha(t) + f_\alpha(t) + \sum_{k \geq 1} \sqrt{\alpha_k} \mathbf{M}_k u_{\alpha - \varepsilon_k}(t), \quad u_\alpha(0) = u_{0,\alpha} \quad (3.16)$$

has a unique solution in $\mathcal{V} \cap \mathbf{C}([0, T]; H)$. Moreover, by (3.14),

$$\|u_\alpha\|_{\mathcal{V}} \leq \bar{C}(\mathbf{A}, \mathbf{M}, T) \left(\|u_{0,\alpha}\|_H + \|f_\alpha\|_{\mathcal{V}'} + \sum_{k \geq 1} \sqrt{\alpha_k} \|u_{\alpha - \varepsilon_k}\|_{\mathcal{V}} \right).$$

Since only finitely many of α_k are different from 0, we conclude that $\|u_\alpha\|_{\mathcal{V}} < \infty$ for all $\alpha \in \mathcal{J}$.

Define the operator \mathcal{R} on $L_2(\mathbb{F})$ using the weights

$$r_\alpha = \min \left(\bar{r}_\alpha, \frac{(2\mathbb{N})^{-\kappa\alpha}}{1 + \|u_\alpha\|_{\mathcal{V}}} \right),$$

where $\kappa > 1/2$ (cf. (2.5)). Then $u(t) := \sum_{\alpha \in \mathcal{J}} u_\alpha(t) \xi_\alpha$ is a solution of (3.4) and, by (2.13), belongs to $\mathcal{R}L_2(\mathbb{F}; \mathcal{V})$. \square

While Theorem 3.10 establishes that under very broad assumptions one can find an operator \mathcal{R} such that equation (3.4) has a unique solution in $\mathcal{R}L_2(\mathbb{F}; \mathcal{V})$, the choice of the operator \mathcal{R} is not sufficiently explicit (because of the presence of $\|u_\alpha\|_{\mathcal{V}}$) and is not necessarily optimal.

Consider equation (3.4) with non-random f and u_0 . In this situation, it is possible to find more constructive expression for r_α and to derive explicit formulas, both for $\mathcal{R}u$ and for each individual u_α .

Theorem 3.11. *If u_0 and f are non-random, then the following holds:*

- (1) *the coefficient u_α , corresponding to the multi-index α with $|\alpha| = n \geq 1$ and characteristic set $K_\alpha = \{k_1, \dots, k_n\}$, is given by*

$$u_\alpha(t) = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_0^t \int_0^{s_n} \dots \int_0^{s_2} \Phi_{t-s_n} \mathbf{M}_{k_{\sigma(n)}} \dots \Phi_{s_2-s_1} \mathbf{M}_{k_{\sigma(1)}} u_{(0)}(s_1) ds_1 \dots ds_n, \quad (3.17)$$

where

- \mathcal{P}_n is the permutation group of the set $(1, \dots, n)$;
 - Φ_t is the semigroup generated by \mathbf{A} ;
 - $u_{(0)}(t) = \Phi_t u_0 + \int_0^t \Phi_{t-s} f(s) ds$.
- (2) *the weights r_α can be taken in the form*

$$r_\alpha = \frac{q^\alpha}{2^{|\alpha|} \sqrt{|\alpha|!}}, \quad \text{where } q^\alpha = \prod_{k=1}^{\infty} q_k^{\alpha_k}, \quad (3.18)$$

and the numbers q_k , $k \geq 1$, are chosen so that $\sum_{k \geq 1} q_k^2 k^2 C_k^2 < 1$, with C_k from (3.15).

- (3) *With q_k and r_α from (3.18), we have*

$$\sum_{|\alpha|=n} q^\alpha u_\alpha(t) \xi_\alpha = \int_0^t \int_0^{s_n} \dots \int_0^{s_2} \Phi_{t-s_n} \delta(\overline{\mathbf{M}} \Phi_{s_n-s_{n-1}} \delta(\dots \delta(\overline{\mathbf{M}} u_{(0)})) \dots) ds_1 \dots ds_{n-1} ds_n, \quad (3.19)$$

where $\overline{\mathbf{M}} = (q_1 \mathbf{M}_1, q_2 \mathbf{M}_2, \dots)$, and

$$\begin{aligned} \mathcal{R}u(t) &= u_{(0)}(t) \\ &+ \sum_{n=1}^{\infty} \frac{1}{2^n \sqrt{n!}} \int_0^t \int_0^{s_n} \dots \int_0^{s_2} \Phi_{t-s_n} \delta(\overline{\mathbf{M}} \Phi_{s_n-s_{n-1}} \delta(\dots \delta(\overline{\mathbf{M}} u_0(s_1))) \dots) ds_1 \dots ds_{n-1} ds_n. \end{aligned} \quad (3.20)$$

Proof. If u_0 and f are deterministic, then equation (3.7) becomes

$$u_{(0)}(t) = u_0 + \int_0^t \mathbf{A} u_{(0)}(s) ds + \int_0^t f(s) ds, \quad |\alpha| = 0; \quad (3.21)$$

$$u_\alpha(t) = \int_0^t \mathbf{A} u_\alpha(s) ds + \sum_{k \geq 1} \sqrt{\alpha_k} \int_0^t \mathbf{M}_k u_{\alpha-\varepsilon_k}(s) ds, \quad |\alpha| > 0. \quad (3.22)$$

Define $\tilde{u}_\alpha = \sqrt{\alpha!} u_\alpha$. Then $\tilde{u}_{(0)} = u_{(0)}$ and, for $|\alpha| > 0$, (3.22) implies

$$\tilde{u}_\alpha(t) = \int_0^t \mathbf{A} \tilde{u}_\alpha(s) ds + \sum_{k \geq 1} \int_0^t \alpha_k \mathbf{M}_k \tilde{u}_{\alpha-\varepsilon_k}(s) ds$$

or

$$\tilde{u}_\alpha(t) = \sum_{k \geq 1} \alpha_k \int_0^t \Phi_{t-s} \mathbf{M}_k \tilde{u}_{\alpha-\varepsilon_k}(s) ds = \sum_{k \in K_\alpha} \int_0^t \Phi_{t-s} \mathbf{M}_k \tilde{u}_{\alpha-\varepsilon_k}(s) ds.$$

By induction on n ,

$$\tilde{u}_\alpha(t) = \sum_{\sigma \in \mathcal{P}_n} \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} \Phi_{t-s_n} \mathbf{M}_{k_{\sigma(n)}} \cdots \Phi_{s_2-s_1} \mathbf{M}_{k_{\sigma(1)}} u_{(0)} ds_1 \cdots ds_n,$$

and (3.17) follows.

Since (3.20) follows directly from (3.19), it remains to establish (3.19). To this end, define

$$U_n(t) = \sum_{|\alpha|=n} q^\alpha u_\alpha(t) \xi_\alpha, \quad n \geq 0.$$

Let us first show that, for each $n \geq 1$, $U_n \in L_2(\mathbb{F}; \mathcal{V})$. Indeed, for $\alpha = (0)$, $u_\alpha(0) = u_0$, $f_\alpha = f$ and

$$u_{(0)}(t) = \Phi_t u_0 + \int_0^t \Phi_{t-s} f(s) ds.$$

By (3.14), we have

$$\|u_{(0)}\|_{\mathcal{V}} \leq C(\mathbf{A}, T) (\|u_0\|_H + \|f\|_{\mathcal{V}}). \quad (3.23)$$

When $|\alpha| \geq 1$, $f_\alpha = 0$ and the solution of (3.22) is given by

$$u_\alpha(t) = \sum_{k \geq 1} \sqrt{\alpha_k} \int_0^t \Phi_{t-s} \mathbf{M}_k u_{\alpha - \varepsilon_k}(s) ds. \quad (3.24)$$

By (3.17), together with (3.14), (3.23), and (3.15), we have

$$\|u_\alpha\|_{\mathcal{V}}^2 \leq C^2(\mathbf{A}, T) \frac{(|\alpha|!)^2}{\alpha!} (\|u_0\|_H^2 + \|f\|_{\mathcal{V}}^2) \prod_{k \geq 1} C_k^{2\alpha_k}. \quad (3.25)$$

By Lemma A.1 in Appendix,

$$|\alpha|! \leq \alpha! (2\mathbb{N})^{2\alpha}; \quad (3.26)$$

recall that

$$(2\mathbb{N})^{2\alpha} = 2^{2|\alpha|} \prod_{k \geq 1} k^{2\alpha_k}.$$

Then

$$\begin{aligned} \sum_{|\alpha|=n} q^{2\alpha} \|u_\alpha\|_{\mathcal{V}}^2 &\leq C^2(\mathbf{A}, T) (\|u_0\|_H^2 + \|f\|_{\mathcal{V}}^2) 2^{2n} n! \sum_{|\alpha|=n} \prod_{k \geq 1} (k C_k q_k)^{2\alpha_k} \\ &= C_A^2 2^{2n} n! \left(\sum_{k \geq 1} k^2 C_k^2 q_k^2 \right)^n < \infty, \end{aligned}$$

because of the selection of q_k , and so $U_n \in L_2(\mathbb{F}; \mathcal{V})$. Moreover, if the weights r_α are defined by (3.18), then

$$\sum_{\alpha \in \mathcal{J}} r_\alpha^2 \|u_\alpha\|_{\mathcal{V}}^2 = \sum_{n \geq 0} \sum_{|\alpha|=n} r_\alpha^2 \|u_\alpha\|_{\mathcal{V}}^2 \leq C_A^2 \sum_{n \geq 1} \left(\sum_{k \geq 1} k^2 C_k^2 q_k^2 \right)^n < \infty$$

because of the assumption $\sum_{k \geq 1} k^2 C_k^2 q_k^2 < 1$.

Next, the definition of $U_n(t)$ and (3.24) imply that (3.19) is equivalent to

$$U_n(t) = \int_0^t \Phi_{t-s} \delta(\overline{\mathbf{M}}U_{n-1}(s)) ds, \quad n \geq 1. \quad (3.27)$$

Accordingly, we will prove (3.27). For $n = 1$, we have

$$U_1(s) = \sum_{k \geq 1} q_k u_{\varepsilon_k}(t) \xi_k = \sum_{k \geq 1} \int_0^t q_k \Phi_{t-s} \mathbf{M}_k u_{(0)} \xi_k dt = \int_0^t \Phi_{t-s} \delta(\overline{\mathbf{M}}U_0(s)) ds,$$

where the last equality follows from (2.15). More generally, for $n > 1$ we have by definition of U_n that

$$(U_n)_\alpha(t) = \begin{cases} q^\alpha u_\alpha(t), & \text{if } |\alpha| = n, \\ 0, & \text{otherwise.} \end{cases}$$

From the equation

$$q^\alpha u_\alpha(t) = \int_0^t \mathbf{A} q^\alpha u_\alpha(s) ds + \sum_{k \geq 1} \int_0^t q_k \sqrt{\alpha_k} \mathbf{M}_k q^{\alpha - \varepsilon_k} u_{\alpha - \varepsilon_k}(s) ds$$

we find

$$\begin{aligned} (U_n(t))_\alpha &= \begin{cases} \sum_{k \geq 1} \sqrt{\alpha_k} q_k \int_0^t \Phi_{t-s} \mathbf{M}_k q^{\alpha - \varepsilon_k} u_{\alpha - \varepsilon_k}(s) ds, & \text{if } |\alpha| = n, \\ 0, & \text{otherwise.} \end{cases} \\ &= \sum_{k \geq 1} \sqrt{\alpha_k} \int_0^t \Phi_{t-s} \overline{\mathbf{M}}_k (U_{n-1}(s))_{\alpha - \varepsilon_k} ds, \end{aligned}$$

and then (3.27) follows from (2.16). Theorem 4.5 is proved. \square

Formula (3.19) is similar to the multiple Wiener integral representation of the solution of a stochastic parabolic equation driven by the Wiener process; see [19, Theorem 3.8].

Example 3.12. Consider the equation

$$u(t, x) = u_0(x) + \int_0^t u_{xx}(s, x) ds + \sum_{k \geq 1} \int_0^t \sigma_k u_{xx}(s, x) \diamond \xi_k ds. \quad (3.28)$$

With no loss of generality assume that $\sigma_k \neq 0$ for all k . Standard properties of the heat kernel imply assumption (A) and inequality (3.15) with $C_k = \sigma_k^2$. Then the conclusions of Theorem 3.11 hold, and we can take $q_k^2 = k^{-2} 4^{-k} (1 + \sigma_k^2)^{-k}$. Note that Theorem 3.11 covers equation (3.28) with no restrictions on the numbers σ_k .

In the existing literature on the subject, equations of the type (3.4) are considered only under the assumption

(H): each \mathbf{M}_k is a bounded linear operators from V to H .

Obviously this assumption rules out equation (3.28) but still covers equation (3.9).

Of course, Theorem 3.11 does not rule out a possibility of a better-behaving solution under additional assumptions on the operators \mathbf{M}_k . Indeed, it was shown in [18] that

if (H) is assumed and the space-only Gaussian noise in equation (3.4) is replaced by the space-time white noise, then a more delicate analysis of equation (3.4) is possible. In particular, the solution can belong to a much smaller Wiener chaos space even if u_0 and f are not deterministic.

If the operators \mathbf{M}_k are bounded in H (see e.g. equation (3.9) with $\sigma = 0$), then, as the following theorem shows, the solutions can be square integrable (cf. [9]).

Theorem 3.13. *Assume that the operator \mathbf{A} satisfies*

$$\langle \mathbf{A}v, v \rangle + \kappa \|v\|_V^2 \leq C_A \|v\|_H^2 \quad (3.29)$$

for every $v \in V$, with $\kappa > 0$, $C_A \in \mathbb{R}$ independent of v , and assume that each \mathbf{M}_k is a bounded operator on H so that $\|\mathbf{M}_k\|_{H \rightarrow H} \leq c_k$ and

$$C_M := \sum_{k \geq 1} c_k^2 < \infty. \quad (3.30)$$

If $f \in \mathcal{V}'$ and $u_0 \in H$ are non-random, then there exists a unique solution u of (3.4) so that $u(t) \in L_2(\mathbb{F}; H)$ for every t and

$$\mathbb{E} \|u(t)\|_H^2 \leq C(C_A, C_M, \kappa, t) \left(\int_0^t \|f(s)\|_V^2 ds + \|u_0\|_H^2 \right). \quad (3.31)$$

Proof. Existence and uniqueness of the solution follow from Theorem 3.10 and Remark 3.9, and it remains to establish (3.31).

It follows from (3.7) that

$$u_\alpha = \frac{1}{\sqrt{\alpha!}} \sum_{k \in K_\alpha} \int_0^t \Phi_{t-s} \mathbf{M}_k u_{\alpha - \varepsilon_k}(s) ds, \quad (3.32)$$

where Φ is the semi-group generated by \mathbf{A} and K_α is the characteristic set of α . Assumption (3.29) implies that $\|\Phi_t\|_{H \rightarrow H} \leq e^{pt}$ for some $p \in \mathbb{R}$. A straightforward calculation using relation (3.32) and induction on $|\alpha|$ shows that

$$\|u_\alpha(t)\|_H \leq e^{pt} \frac{t^{|\alpha|} c^\alpha}{\sqrt{\alpha!}} \|u_{(0)}\|_H, \quad (3.33)$$

where $c^\alpha = \prod_k c_k^{\alpha_k}$ and $u_{(0)}(t) = \Phi_t u_0 + \int_0^t \Phi_{t-s} f(s) ds$. Assumption (3.29) implies that $\|u_{(0)}\|_H^2 \leq C(C_A, \kappa, t) \left(\int_0^t \|f(s)\|_V^2 ds + \|u_0\|_H^2 \right)$. To establish (3.31), it remains to observe that

$$\sum_{\alpha \in \mathcal{J}} \frac{c^{2\alpha} t^{2|\alpha|}}{\alpha!} = e^{C_M t^2}.$$

Theorem 3.13 is proved. \square

Remark 3.14. Taking $\mathbf{M}_k u = c_k u$ shows that, in general, bound (3.33) cannot be improved. When condition (3.30) does not hold, a bound similar to (3.31) can be established in a weighted space $\mathcal{R}L_2(\mathbb{F}; H)$, for example with $r_\alpha = q^\alpha$, where $q_k = 1/(2^k(1 + c_k))$. For special operators \mathbf{M}_k , a more delicate analysis might be possible; see, for example, [9].

If f and u_0 are not deterministic, then the solution of (3.4) might not satisfy

$$\mathbb{E}\|u(t)\|_H^2 \leq C(C_A, C_M, \kappa, t) \left(\int_0^t \mathbb{E}\|f(s)\|_{V'}^2 ds + \mathbb{E}\|u_0\|_H^2 \right)$$

even if all other conditions of Theorem 3.13 are fulfilled. An example can be constructed similar to Example 9.7 in [18]: an interested reader can verify that the solution of the equation $u(t) = u_0 + \int_0^t u(s) \diamond \xi ds$, where ξ is a standard Gaussian random variable and $u_0 = \sum_{n \geq 0} a_n \frac{H_n(\xi)}{\sqrt{n!}}$, satisfies $\mathbb{E}u^2(1) \geq \frac{1}{10} \sum_{n \geq 1} a_n^2 e^{\sqrt{n}}$. For equations with random input, one possibility is to use the spaces $(\mathcal{S})_{-1,q}$; see (2.5). Examples of the corresponding results are Theorems 4.6 and 5.1 below and Theorem 9.8 in [18].

4. STATIONARY EQUATIONS

4.1. Definitions and Analysis. The objective of this section is to study stationary stochastic equation

$$\mathbf{A}u + \delta(\mathbf{M}u) = f. \quad (4.1)$$

Definition 4.1. *The solution of equation (4.1) with $f \in \mathcal{R}L_2(\mathbb{F}; V')$, is a random element $u \in \mathcal{R}L_2(\mathbb{F}; V)$ so that, for every φ satisfying $\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F})$ and $\mathbf{D}\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F}; \mathcal{U})$, the equality*

$$\langle \mathbf{A}u, \varphi \rangle + \langle \delta(\mathbf{M}u), \varphi \rangle = \langle f, \varphi \rangle \quad (4.2)$$

holds in V' .

As with evolution equations, we fix an orthonormal basis \mathcal{U} in \mathcal{U} and use (2.15) to rewrite (4.1) as

$$\mathbf{A}u + (\mathbf{M}u) \diamond \dot{W} = f, \quad (4.3)$$

where

$$\mathbf{M}u \diamond \dot{W} := \sum_{k \geq 1} \mathbf{M}_k u \diamond \xi_k. \quad (4.4)$$

Taking $\varphi = \xi_\alpha$ in (4.2) and using relation (2.16) we conclude, as in Theorem 3.5, that $u = \sum_{\alpha \in \mathcal{J}} u_\alpha \xi_\alpha$ is a solution of equation (4.1) if and only if u_α satisfies

$$\mathbf{A}u_\alpha + \sum_{k \geq 1} \sqrt{\alpha_k} \mathbf{M}_k u_{\alpha - \varepsilon_k} = f_\alpha \quad (4.5)$$

in the normal triple (V, H, V') . This system of equation is lower-triangular and can be solved by induction on $|\alpha|$.

The following example illucidates the limitations on the “quality” of the solution of equation (4.1).

Example 4.2. *Consider equation*

$$u = 1 + u \diamond \xi. \quad (4.6)$$

Similar to Example 3.6, we write $u = \sum_{n \geq 0} u_{(n)} H_n(\xi) / \sqrt{n!}$, where H_n is Hermite polynomial of order n (2.3). Then (4.5) implies $u_{(n)} = I_{(n=0)} + \sqrt{n} u_{(n-1)}$ or $u_{(0)} = 1$,

$u_{(n)} = \sqrt{n!}$, $n \geq 1$, or $u = 1 + \sum_{n \geq 1} H_n(\xi)$. Clearly, the series does not converge in $L_2(\mathbb{F})$, but does converge in $(\mathcal{S})_{-1,q}$ for every $q < 0$ (see (2.5)). As a result, even a simple stationary equation (4.6) can be solved only in weighted spaces.

Theorem 4.3. Consider equation (4.3) in which $f \in \bar{\mathcal{R}}L_2(\mathbb{F}; V')$ for some $\bar{\mathcal{R}}$.

Assume that the deterministic equation $\mathbf{A}U = F$ is uniquely solvable in the normal triple (V, H, V') , that is, for every $F \in V'$, there exists a unique solution $U = \mathbf{A}^{-1}F \in V$ so that $\|U\|_V \leq C_A \|F\|_{V'}$. Assume also that each \mathbf{M}_k is a bounded linear operator from V to V' so that, for all $v \in V$

$$\|\mathbf{A}^{-1}\mathbf{M}_k v\|_V \leq C_k \|v\|_V, \quad (4.7)$$

with C_k independent of v .

Then there exists an operator \mathcal{R} and a unique solution $u \in \mathcal{R}L_2(\mathbb{F}; V)$ of (3.4).

Proof. The argument is identical to the proof of Theorem 3.10. \square

Remark 4.4. The assumption of the theorem about solvability of the deterministic equation holds if the operator \mathbf{A} satisfies $\langle \mathbf{A}v, v \rangle \geq \kappa \|v\|_V^2$ for every $v \in V$, with $\kappa > 0$ independent of v .

An analog of Theorem 3.11 exists if f is non-random. With no time variable, we introduce the following notation to write multiple integrals in the time-independent setting:

$$\delta_{\mathbf{B}}^{(0)}(\eta) = \eta, \quad \delta_{\mathbf{B}}^{(n)}(\eta) = \delta(\mathbf{B}\delta_{\mathbf{B}}^{(n-1)}(\eta)), \quad \eta \in \mathcal{R}L_2(\mathbb{F}; V),$$

where \mathbf{B} is a bounded linear operator from V to $V \otimes \mathcal{U}$.

Theorem 4.5. Under the assumptions of Theorem 4.3, if f is non-random, then the following holds:

- (1) the coefficient u_α , corresponding to the multi-index α with $|\alpha| = n \geq 1$ and the characteristic set $K_\alpha = \{k_1, \dots, k_n\}$, is given by

$$u_\alpha = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \mathbf{B}_{k_{\sigma(n)}} \cdots \mathbf{B}_{k_{\sigma(1)}} u_{(0)}, \quad (4.8)$$

where

- \mathcal{P}_n is the permutation group of the set $(1, \dots, n)$;
- $\mathbf{B}_k = -\mathbf{A}^{-1}\mathbf{M}_k$;
- $u_{(0)} = \mathbf{A}^{-1}f$.

- (2) the operator \mathcal{R} can be defined by the weights r_α in the form

$$r_\alpha = \frac{q^\alpha}{2^{|\alpha|} \sqrt{|\alpha|!}}, \quad \text{where } q^\alpha = \prod_{k=1}^{\infty} q_k^{\alpha_k}, \quad (4.9)$$

where the numbers q_k , $k \geq 1$ are chosen so that $\sum_{k \geq 1} q_k^2 k^2 C_k^2 < 1$, and C_k are defined in (4.7).

(3) With r_α and q_k defined by (4.9),

$$\sum_{|\alpha|=n} q^\alpha u_\alpha \xi_\alpha = \delta_{\bar{\mathbf{B}}}^{(n)}(\mathbf{A}^{-1}f), \quad (4.10)$$

where $\bar{\mathbf{B}} = -(q_1 \mathbf{A}^{-1} \mathbf{M}_1, q_2 \mathbf{A}^{-1} \mathbf{M}_2, \dots)$, and

$$\mathcal{R}u = \mathbf{A}^{-1}f + \sum_{n \geq 1} \frac{1}{2^n \sqrt{n!}} \delta_{\bar{\mathbf{B}}}^{(n)}(\mathbf{A}^{-1}f), \quad (4.11)$$

Proof. While the proofs of Theorems 3.11 and 4.5 are similar, the complete absence of time makes equation (4.3) different from either (3.4) or anything considered in [19]. Accordingly, we present a complete proof.

Define $\tilde{u}_\alpha = \sqrt{\alpha!} u_\alpha$. If f is deterministic, then $\tilde{u}_{(0)} = \mathbf{A}^{-1}f$ and, for $|\alpha| \geq 1$,

$$\mathbf{A} \tilde{u}_\alpha + \sum_{k \geq 1} \alpha_k \mathbf{M}_k \tilde{u}_{\alpha - \varepsilon_k} = 0,$$

or

$$\tilde{u}_\alpha = \sum_{k \geq 1} \alpha_k \mathbf{B}_k \tilde{u}_{\alpha - \varepsilon_k} = \sum_{k \in K_\alpha} \mathbf{B}_k \tilde{u}_{\alpha - \varepsilon_k},$$

where $K_\alpha = \{k_1, \dots, k_n\}$ is the characteristic set of α and $n = |\alpha|$. By induction on n ,

$$\tilde{u}_\alpha = \sum_{\sigma \in \mathcal{P}_n} \mathbf{B}_{k_{\sigma(n)}} \cdots \mathbf{B}_{k_{\sigma(1)}} u_{(0)},$$

and (4.8) follows.

Next, define

$$U_n = \sum_{|\alpha|=n} q^\alpha u_\alpha \xi_\alpha, \quad n \geq 0.$$

Let us first show that, for each $n \geq 1$, $U_n \in L_2(\mathbb{F}; V)$. By (4.8) we have

$$\|u_\alpha\|_V^2 \leq C_A^2 \frac{(|\alpha|!)^2}{\alpha!} \|f\|_V^2, \prod_{k \geq 1} C_k^{\alpha_k}. \quad (4.12)$$

By (3.26),

$$\sum_{|\alpha|=n} q^{2\alpha} \|u_\alpha\|_V^2 \leq C_A^2 2^{2n} n! \sum_{|\alpha|=n} \prod_{k \geq 1} (k C_k q_k)^{2\alpha_k} = C_A^2 2^{2n} n! \left(\sum_{k \geq 1} k^2 C_k^2 q_k^2 \right)^n < \infty,$$

because of the selection of q_k , and so $U_n \in L_2(\mathbb{F}; V)$. If the weights r_α are defined by (4.9), then

$$\sum_{\alpha \in \mathcal{J}} r_\alpha^2 \|u\|_V^2 = \sum_{n \geq 0} \sum_{|\alpha|=n} r_\alpha^2 \|u\|_V^2 \leq C_A^2 \sum_{n \geq 0} \left(\sum_{k \geq 1} k^2 C_k^2 q_k^2 \right)^n < \infty,$$

because of the assumption $\sum_{k \geq 1} k^2 C_k^2 q_k^2 < 1$.

Since (4.11) follows directly from (4.10), it remains to establish (4.10), that is,

$$U_n = \delta_{\bar{\mathbf{B}}}(U_{n-1}), \quad n \geq 1. \quad (4.13)$$

For $n = 1$ we have

$$U_1 = \sum_{k \geq 1} q_k u_{\varepsilon_k} \xi_k = \sum_{k \geq 1} \bar{\mathbf{B}}_k u_{(0)} \xi_k = \delta_{\bar{\mathbf{B}}}(U_0),$$

where the last equality follows from (2.15). More generally, for $n > 1$ we have by definition of U_n that

$$(U_n)_\alpha = \begin{cases} q^\alpha u_\alpha, & \text{if } |\alpha| = n, \\ 0, & \text{otherwise.} \end{cases}$$

From the equation

$$q^\alpha \mathbf{A} u_\alpha + \sum_{k \geq 1} q_k \sqrt{\alpha_k} \mathbf{M}_k q^{\alpha - \varepsilon_k} u_{\alpha - \varepsilon_k} = 0$$

we find

$$\begin{aligned} (U_n)_\alpha &= \begin{cases} \sum_{k \geq 1} \sqrt{\alpha_k} q_k \mathbf{B}_k q^{\alpha - \varepsilon_k} u_{\alpha - \varepsilon_k}, & \text{if } |\alpha| = n, \\ 0, & \text{otherwise.} \end{cases} \\ &= \sum_{k \geq 1} \sqrt{\alpha_k} \bar{\mathbf{B}}_k (U_{n-1})_{\alpha - \varepsilon_k}, \end{aligned}$$

and then (4.13) follows from (2.16). Theorem 4.5 is proved. \square

Here is another result about solvability of (4.3), this time with random f . We use the space $(\mathcal{S})_{\rho, q}$, defined by the weights (2.5).

Theorem 4.6. *In addition to the assumptions of Theorem 4.3, let $C_A \leq 1$ and $C_k \leq 1$ for all k . If $f \in (\mathcal{S})_{-1, -\ell}(V')$ for some $\ell > 1$, then there exists a unique solution $u \in (\mathcal{S})_{-1, -\ell-4}(V)$ of (4.3) and*

$$\|u\|_{(\mathcal{S})_{-1, -\ell-4}(V)} \leq C(\ell) \|f\|_{(\mathcal{S})_{-1, -\ell}(V')}. \quad (4.14)$$

Proof. Denote by $u(g; \gamma)$, $\gamma \in \mathcal{J}$, $g \in V'$, the solution of (4.3) with $f_\alpha = g I_{(\alpha=\gamma)}$, and define $\bar{u}_\alpha = (\alpha!)^{-1/2} u_\alpha$. Clearly, $u_\alpha(g; \gamma) = 0$ if $|\alpha| < |\gamma|$ and so

$$\sum_{\alpha \in \mathcal{J}} \|u_\alpha(f_\gamma; \gamma)\|_V^2 r_\alpha^2 = \sum_{\alpha \in \mathcal{J}} \|u_{\alpha+\gamma}(f_\gamma; \gamma)\|_V^2 r_{\alpha+\gamma}^2. \quad (4.15)$$

It follows from (4.5) that

$$\bar{u}_{\alpha+\gamma}(f_\gamma; \gamma) = \bar{u}_\alpha(f_\gamma(\gamma!)^{-1/2}; (0)). \quad (4.16)$$

Now we use (4.12) to conclude that

$$\|\bar{u}_{\alpha+\gamma}(f_\gamma; \gamma)\|_V \leq \frac{|\alpha|!}{\sqrt{\alpha! \gamma!}} \|f\|_{V'}. \quad (4.17)$$

Coming back to (4.15) with $r_\alpha^2 = (\alpha!)^{-1} (2\mathbb{N})^{(-\ell-4)\alpha}$ and using inequality (3.26) we find:

$$\|u(f_\gamma; \gamma)\|_{(\mathcal{S})_{-1, -\ell-4}(V)} \leq C(\ell) (2\mathbb{N})^{-2\gamma} \frac{\|f_\gamma\|_{V'}}{(2\mathbb{N})^{(\ell/2)\gamma} \sqrt{\gamma!}},$$

where

$$C(\ell) = \left(\sum_{\alpha \in \mathcal{J}} \left(\frac{|\alpha|!}{\alpha!} \right)^2 (2\mathbb{N})^{(-\ell-4)\alpha} \right)^{1/2};$$

(2.13) and (3.26) imply $C(\ell) < \infty$. Then (4.14) follows by the triangle inequality after summing over all γ and using the Cauchy-Schwartz inequality. \square

Remark 4.7. *Example 4.2, in which $f \in (\mathcal{S})_{0,0}$ and $u \in (\mathcal{S})_{-1,q}$, $q < 0$, shows that, while the results of Theorem 4.6 are not sharp, a bound of the type $\|u\|_{(\mathcal{S})_{\rho,q}(V)} \leq C\|f\|_{(\mathcal{S})_{\rho,\ell}(V')}$ is, in general, impossible if $\rho > -1$ or $q \geq \ell$.*

4.2. Convergence to Stationary Solution. Let (V, H, V') be a normal triple of Hilbert spaces. Consider equation

$$\dot{u}(t) = (\mathbf{A}u(t) + f(t)) + \mathbf{M}_k u(t) \diamond \xi_k, \quad (4.18)$$

where the operators \mathbf{A} and \mathbf{M}_k do not depend on time, and assume that there exists an $f^* \in \mathcal{RL}_2(\mathbb{F}; H)$ such that $\lim_{t \rightarrow \infty} \|f(t) - f^*\|_{\mathcal{RL}_2(\mathbb{F}; H)} = 0$. The objective of this section is to study convergence, as $t \rightarrow +\infty$, of the solution of (4.18) to the solution u^* of the stationary equation

$$-\mathbf{A}u^* = f^* + \mathbf{M}_k u^* \diamond \xi_k. \quad (4.19)$$

Theorem 4.8. *Assume that*

(C1) *Each \mathbf{M}_k is a bounded linear operator from H to H , and \mathbf{A} is a bounded linear operator from V to V' with the property*

$$\langle \mathbf{A}v, v \rangle + \kappa \|v\|_V^2 \leq -c \|v\|_H^2 \quad (4.20)$$

for every $v \in V$, with $\kappa > 0$ and $c > 0$ both independent of v .

(C2) *$f \in \bar{\mathcal{R}}L_2(\mathbb{F}; \mathcal{H})$ and there exists an $f^* \in \bar{\mathcal{R}}L_2(\mathbb{F}; H)$ such that $\lim_{t \rightarrow +\infty} \|f(t) - f^*\|_{\bar{\mathcal{R}}L_2(\mathbb{F}; H)} = 0$.*

Then, for every $u_0 \in \bar{\mathcal{R}}L_2(\mathbb{F}; H)$, there exists an operator \mathcal{R} so that

- (1) *There exists a unique solution $u \in \mathcal{RL}_2(\mathbb{F}; \mathcal{V})$ of (4.18),*
- (2) *There exists a unique solution $u^* \in \mathcal{RL}_2(\mathbb{F}; V)$ of (4.19), and*
- (3) *The following convergence holds:*

$$\lim_{t \rightarrow +\infty} \|u(t) - u^*\|_{\mathcal{RL}_2(\mathbb{F}; H)} = 0. \quad (4.21)$$

Proof (1) Existence and uniqueness of the solution of (4.18) follow from Theorem 3.10 and Remark 3.9.

(2) Existence and uniqueness of the solution of (4.19) follow from Theorem 4.3 and Remark 4.4.

(3) The proof of (4.21) is based on the following result.

Lemma 4.9. *Assume that the operator \mathbf{A} satisfies (4.20) and $F = F(t)$ is a deterministic function such that $\lim_{t \rightarrow +\infty} \|F(t)\|_H = 0$. Then, for every $U_0 \in H$, the solution $U = U(t)$ of the equation $U(t) = U_0 + \int_0^t \mathbf{A}U(s)ds + \int_0^t F(s)ds$ satisfies $\lim_{t \rightarrow +\infty} \|U(t)\|_H = 0$.*

Proof. If $\Phi = \Phi_t$ is the semi-group generated by the operator \mathbf{A} (which exists because of (4.20)), then

$$U(t) = \Phi_t U_0 + \int_0^t \Phi_{t-s} F(s) ds.$$

Condition (4.20) implies $\|\Phi_t U_0\|_H \leq e^{-ct} \|U_0\|_H$, and then

$$\|U(t)\|_H \leq e^{-ct} \|U_0\|_H + \int_0^t e^{-c(t-s)} \|F(s)\|_H ds.$$

The convergence of $\|U(t)\|_H$ to zero now follows from the Toeplitz lemma (see Lemma A.2 in Appendix). Lemma 4.9 is proved. \square

To complete the proof of Theorem 4.8, we define $v_\alpha(t) = u_\alpha(t) - u_\alpha^*$ and note that

$$\dot{v}_\alpha(t) = \mathbf{A}v_\alpha(t) + (f_\alpha(t) - f_\alpha^*) + \sum_k \sqrt{\alpha_k} \mathbf{M}_k v_{\alpha-\varepsilon_k}.$$

By Theorem 4.3, $u_\alpha^* \in V$ and so $v_\alpha(0) \in H$ for every $\alpha \in \mathcal{J}$. By Lemma 4.9, $\lim_{t \rightarrow +\infty} \|v_{(0)}(t)\|_H = 0$. Using induction on $|\alpha|$ and the inequality $\|\mathbf{M}_k v_{\alpha-\varepsilon_k}(t)\|_H \leq c_k \|v_{\alpha-\varepsilon_k}(t)\|_H$, we conclude that $\lim_{t \rightarrow +\infty} \|v_\alpha(t)\|_H = 0$ for every $\alpha \in \mathcal{J}$. Since $v_\alpha \in \mathbf{C}((0, T); H)$ for every T , it follows that $\sup_{t \geq 0} \|v_\alpha(t)\|_H < \infty$. Define the operator \mathcal{R} on $L_2(\mathbb{F})$ so that $\mathcal{R}\xi_\alpha = r_\alpha \xi_\alpha$, where

$$r_\alpha = \frac{(2\mathbb{N})^{-\alpha}}{1 + \sup_{t \geq 0} \|v_\alpha(t)\|_H}.$$

Then (4.21) follows by the dominated convergence theorem.

Theorem 4.8 is proved. \square

5. BILINEAR PARABOLIC AND ELLIPTIC SPDES

Let G be a smooth bounded domain in \mathbb{R}^d and $\{h_k, k \geq 1\}$, an orthonormal basis in $L_2(G)$. We assume that

$$\sup_{x \in G} |h_k(x)| \leq c_k, \quad k \geq 1. \quad (5.1)$$

A space white noise on $L_2(G)$ is a formal series

$$\dot{W}(x) = \sum_{k \geq 1} h_k(x) \xi_k, \quad (5.2)$$

where $\xi_k, k \geq 1$, are independent standard Gaussian random variables.

5.1. Dirichlet Problem for parabolic SPDE of the Second Order. Consider the following equation:

$$\begin{aligned} u_t(t, x) &= a_{ij}(x) D_i D_j u(t, x) + b_i(x) D_i u(t, x) + c(x) u(t, x) + f(t, x) \\ &+ (\sigma_i(x) D_i u(t, x) + \nu(x) u(t, x) + g(t, x)) \diamond \dot{W}(x), \quad 0 < t \leq T, \quad x \in G, \end{aligned} \quad (5.3)$$

with zero boundary conditions and some initial condition $u(0, x) = u_0(x)$; the functions $a_{ij}, b_i, c, f, \sigma_i, \nu, g$, and u_0 are non-random. In (5.3) and in similar expressions below we assume summation over the repeated indices. Let (V, H, V') be the normal

triple with $V = \overset{\circ}{H}_2^1(G)$, $H = L_2(G)$, $V' = H_2^{-1}(G)$. In view of (5.2), equation (5.3) is a particular case of equation (3.4) so that

$$\mathbf{A}u = a_{ij}(x)D_iD_ju + b_i(x)D_iu + c(x)u, \quad \mathbf{M}_k u = (\sigma_i(x)D_iu + \nu(x)u)h_k(x), \quad (5.4)$$

and $f(t, x) + g(t, x) \diamond \dot{W}(x)$ is the free term.

We make the following assumptions about the coefficients:

D1 The functions a_{ij} are Lipschitz continuous in the closure \bar{G} of G , and the functions b_i , c , σ_i , ν are bounded and measurable in \bar{G} .

D2 There exist positive numbers A_1, A_2 so that $A_1|y|^2 \leq a_{ij}(x)y_iy_j \leq A_2|y|^2$ for all $x \in \bar{G}$ and $y \in \mathbb{R}^d$.

Given a $T > 0$, recall the notation $\mathcal{V} = L_2((0, T); V)$ and similarly for \mathcal{H} and \mathcal{V}' (see (3.1)).

Theorem 5.1. *Under the assumptions **D1** and **D2**, if $f \in \mathcal{V}'$, $g \in \mathcal{H}$, $u_0 \in H$, then there exists an $\ell > 1$ and a number $C > 0$, both independent of u_0, f, g , so that $u \in \mathcal{RL}_2(\mathbb{F}; \mathcal{V})$ and*

$$\|u\|_{\mathcal{RL}_2(\mathbb{F}; \mathcal{V})} \leq C \cdot (\|u_0\|_H + \|f\|_{\mathcal{V}'} + \|g\|_{\mathcal{H}}), \quad (5.5)$$

where the operator \mathcal{R} is defined by the weights

$$r_\alpha^2 = c^{-2\alpha}(|\alpha|!)^{-1}(2N)^{-2\ell\alpha} \quad (5.6)$$

and $c^\alpha = \prod_k c_k^{\alpha_k}$, with c_k from (5.1); the number ℓ in general depends on T .

Proof. We derive the result from Theorem 3.11. Consider the deterministic equation $\dot{U}(t) = \mathbf{A}U(t) + F$. Assumptions **D1** and **D2** imply that there exists a unique solution of this equation in the normal triple (V, H, V') , and the solution satisfies

$$\sup_{0 < t < T} \|U(t)\|_H + \|U\|_{\mathcal{V}} \leq C \cdot (\|U(0)\|_H + \|F\|_{\mathcal{V}'}), \quad (5.7)$$

where the number C depends on T and the operator \mathbf{A} . Moreover, (5.1) implies that (3.15) holds with $C_k = C_0 c_k$ for some positive number C_0 independent of k , but possibly depending on T .

To proceed, let us assume first that $g = 0$. Then the statement of the theorem follows directly from Theorem 3.11 if we take in (3.18) $q_k = c_k^{-1}(2k)^{-\ell}$ with sufficiently large ℓ .

It now remains to consider the case $g \neq 0$ and $f = u_0 = 0$. Even though g is non-random, $g\xi_k$ is, and therefore a direct application of Theorem 3.11 is not possible. Instead, let us look more closely at the corresponding equations for u_α . For $\alpha = (0)$,

$$u_{(0)}(t) = \int_0^t \mathbf{A}u_{(0)}(s)ds,$$

which implies $u_{(0)}(t) = 0$ for all t . For $\alpha = \varepsilon_k$,

$$u_{\varepsilon_k}(t) = \int_0^t \mathbf{A}u_{\varepsilon_k}(s)ds + h_k \int_0^t g(s)ds,$$

or

$$u_{\varepsilon_k}(t) = \int_0^t \Phi_{t-s} h_k g(s) ds,$$

so that

$$\|u_{\varepsilon_k}\|_{\mathcal{V}} \leq C_0 c_k \|g\|_{\mathcal{H}}. \quad (5.8)$$

If $|\alpha| > 1$, then

$$u_\alpha(t) = \int_0^t \mathbf{A} u_\alpha(s) ds + \sum_{k \geq 1} \sqrt{\alpha_k} \mathbf{M}_k u_{\alpha - \varepsilon_k},$$

which is the same as (3.22). In particular, if $|\alpha| = 2$ and $\{i, j\}$ is the characteristic set of α , then

$$u_\alpha(t) = \frac{1}{\sqrt{\alpha!}} \int_0^t \Phi_{t-s} (\mathbf{M}_i u_{\varepsilon_j}(s) + \mathbf{M}_j u_{\varepsilon_i}(s)) ds.$$

More generally, by analogy with (3.25), if $|\alpha| = n > 2$ and $\{k_1, \dots, k_n\}$ is the characteristic set of α , then

$$u_\alpha(t) = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_0^t \int_0^{s_n} \dots \int_0^{s_3} \Phi_{t-s_n} \mathbf{M}_{k_{\sigma(n)}} \dots \Phi_{s_3-s_2} \mathbf{M}_{k_{\sigma(2)}} u_{\varepsilon_{\sigma(1)}}(s_2) ds_2 \dots ds_n.$$

By the triangle inequality and (5.8),

$$\|u_\alpha\|_{\mathcal{V}} \leq \frac{|\alpha|! C_0^{|\alpha|} c^\alpha}{\sqrt{\alpha!}} \|g\|_{\mathcal{H}},$$

and then (5.5) follows from (3.26) if ℓ is sufficiently large.

This completes the proof of Theorem 5.1. \square

Theorem 5.2. *In addition to **D1** and **D2**, assume that*

- (1) $\sigma_i = 0$ for all i ;
- (2) the operator \mathbf{A} in G with zero boundary conditions satisfies (4.20).

If there exist functions f^* and g^* from H so that

$$\lim_{t \rightarrow +\infty} (\|f(t) - f^*\|_H + \|g(t) - g^*\|_H) = 0, \quad (5.9)$$

then the solution u of equation (5.3) satisfies

$$\lim_{t \rightarrow +\infty} \|u(t) - u^*\|_{\mathcal{R}L_2(\mathbb{F}; H)} = 0, \quad (5.10)$$

where the operator \mathcal{R} is defined by the weights (5.6) and u^* is the solution of the stationary equation

$$\begin{aligned} a_{ij}(x) D_i D_j u^*(x) + b_i(x) D_i u^*(x) + c(x) u^*(x) + f^*(x) \\ + (\nu(x) u^*(x) + g^*(x)) \diamond \dot{W}(x) = 0, \quad x \in G; \quad u|_{\partial G} = 0. \end{aligned} \quad (5.11)$$

Proof. This follows from Theorem 4.8. \square

Remark 5.3. *The operator \mathbf{A} satisfies (4.20) if, for example, each a_{ij} is twice continuously differentiable in \bar{G} , each b_i continuously differentiable in \bar{G} , and*

$$\inf_{x \in \bar{G}} c(x) - \sup_{x \in \bar{G}} (|D_i D_j a_{ij}(x)| + |D_i b_i(x)|) \geq \varepsilon > 0; \quad (5.12)$$

this is verified directly using integration by parts.

5.2. Elliptic SPDEs of the full second order. Consider the following Dirichlet problem:

$$\begin{aligned} & -D_i \left(a_{ij}(x) D_j u(x) \right) + \\ & D_i \left(\sigma_{ij}(x) D_j (u(x)) \right) \diamond \dot{W}(x) = f(x), \quad x \in G, \\ & u|_{\partial G} = 0, \end{aligned} \quad (5.13)$$

where \dot{W} is the space white noise (5.2). Assume that the functions a_{ij} , σ_{ij} , f , and g are non-random. Recall that according to our summation convention, in (5.13) and in similar expressions below we assume summation over the repeated indices.

We make the following assumptions:

- E1:** The functions $a_{ij} = a_{ij}(x)$ and $\sigma_{ij} = \sigma_{ij}(x)$ are measurable and bounded in the closure \bar{G} of G .
- E2:** There exist positive numbers A_1, A_2 so that $A_1|y|^2 \leq a_{ij}(x)y_i y_j \leq A_2|y|^2$ for all $x \in \bar{G}$ and $y \in \mathbb{R}^d$.
- E3:** The functions h_k in (5.2) are bounded and Lipschitz continuous.

Clearly, equation (5.13) is a particular case of equation (4.3) with

$$\mathbf{A}u(x) := -D_i \left(a_{ij}(x) D_j u(x) \right) \quad (5.14)$$

and

$$\mathbf{M}_k u(x) := h_k(x) D_i \left(\sigma_{ij}(x) D_j u(x) \right). \quad (5.15)$$

Assumptions **E1** and **E3** imply that each \mathbf{M}_k is a bounded linear operator from $\mathring{H}_2^1(G)$ to $H_2^{-1}(G)$. Moreover, it is a standard fact that under the assumptions **E1** and **E2** the operator \mathbf{A} is an isomorphism from V onto V' (see e.g. [17]). Therefore, for every k there exists a positive number C_k such that

$$\|\mathbf{A}^{-1} \mathbf{M}_k v\|_V \leq C_k \|v\|_{V'}, \quad v \in V. \quad (5.16)$$

Theorem 5.4. *Under the assumptions **E1** and **E2**, if $f \in H_2^{-1}(G)$, then there exists a unique solution of the Dirichlet problem (5.13) $u \in \mathcal{RL}_2(\mathbb{F}; \mathring{H}_2^1(G))$ such that*

$$\|u\|_{\mathcal{RL}_2(\mathbb{F}; \mathring{H}_2^1(G))} \leq C \cdot \|f\|_{H_2^{-1}(G)}. \quad (5.17)$$

The weights r_α can be taken in the form

$$r_\alpha = \frac{q^\alpha}{2^{|\alpha|} \sqrt{|\alpha|!}}, \quad \text{where } q^\alpha = \prod_{k=1}^{\infty} q_k^{\alpha_k}, \quad (5.18)$$

and the numbers q_k , $k \geq 1$ are chosen so that $\sum_{k \geq 1} C_k^2 q_k^2 k^2 < 1$, with C_k from (5.16).

Proof. This follows from Theorem 4.5. \square

Remark 5.5. *With an appropriate change of the boundary conditions, and with extra regularity of the basis functions h_k , the results of Theorem 5.4 can be extended to stochastic elliptic equations of order $2m$. The corresponding operators are*

$$\mathbf{A}u = (-1)^m D_{i_1} \cdots D_{i_m} \left(a_{i_1 \dots i_m j_1 \dots j_m}(x) D_{j_1} \cdots D_{j_m} u(x) \right) \quad (5.19)$$

and

$$\mathbf{M}_k u = h_k(x) D_{i_1} \cdots D_{i_m} \left(\sigma_{i_1 \dots i_m j_1 \dots j_m}(x) D_{j_1} \cdots D_{j_m} u(x) \right). \quad (5.20)$$

Since G is a smooth bounded domain, regularity of h_k is not a problem: we can take h_k as the eigenfunctions of the Dirichlet Laplacian in G .

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Appendix.

A FACTORIAL INEQUALITY.

Lemma A.1. *For every multi-index $\alpha \in \mathcal{J}$,*

$$|\alpha|! \leq \alpha!(2\mathbb{N})^{2\alpha}. \quad (\text{A1})$$

Proof. Recall that, for $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathcal{J}$,

$$|\alpha| = \sum_{\ell=1}^k \alpha_\ell, \quad \alpha! = \prod_{\ell=1}^k \alpha_\ell!, \quad \mathbb{N}^\alpha = \prod_{\ell=1}^k \ell^{\alpha_\ell}.$$

It is therefore clear that, if $|\alpha| = n$, then it is enough to establish (A1) for α with $\alpha_k = 0$ for $k \geq n + 1$, because a shift of a multi-index entry to the right increases the right-hand side of (A1) but does not change the left-hand side. For example, if $\alpha = (1, 3, 2, 0, \dots)$ and $\beta = (1, 3, 0, 2, 0, \dots)$, then $|\alpha| = |\beta|$, $\alpha! = \beta!$, but $\mathbb{N}^\alpha < \mathbb{N}^{\beta}$. Then

$$4^n \geq \left(1 + \frac{1}{2^2} + \dots + \frac{1}{n^2}\right)^n = \sum_{\alpha_1 + \dots + \alpha_n = n} \frac{|\alpha|!}{\alpha!} \frac{1}{\mathbb{N}^{2\alpha}},$$

where the equality follows by the multinomial formula. Since all the term in the sum are non-negative, we get (A2). □

The proof shows that inequality (A2) can be improved by observing that $\sum_{k \geq 1} k^{-2} = \pi^2/6 < 2$. One can also consider $\sum_{k \geq 1} k^{-q}$ for some $1 < q < 2$.

A VERSION OF THE TOEPLITZ LEMMA.

Lemma A.2. *Assume that $f = f(t)$ is an integrable function and $\lim_{t \rightarrow +\infty} |f(t)| = 0$. Then, for every $c > 0$, $\lim_{t \rightarrow +\infty} \int_0^t e^{-c(t-s)} f(s) ds = 0$.*

Proof. Given $\varepsilon > 0$, choose T so that $|f(t)| < \varepsilon$ for all $t > T$. Then $|\int_0^t e^{-c(t-s)} f(s) ds| \leq e^{-ct} \int_0^T e^{cs} |f(s)| ds + \varepsilon \int_T^t e^{-c(t-s)} ds$. Passing to the limit as $t \rightarrow +\infty$, we find $\lim_{t \rightarrow +\infty} |\int_0^t e^{-c(t-s)} f(s) ds| \leq \varepsilon/c$, which completes the proof. □

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