

Numerical Approximations to Optimal Nonlinear Filters

Harold J.Kushner*
Brown University, Applied Mathematics

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Abstract

Two types of numerical algorithms for nonlinear filters are considered. The first is based on the Markov chain approximation method, a powerful approach to numerical problems in stochastic control. It yields an approximation to the weak-sense conditional density and converges in the weak sense as the approximation parameter goes to zero. Various forms are developed and convergence and robustness are shown. The second type of approximation is called the assumed density approach, where one supposes that the conditional density takes a given parametrized form, and the evolution equations for the parameters are developed. Most typically, this assumed density is Gaussian (more rarely, a Gaussian mixture) and the parameters are the conditional mean and covariance. The method is heuristic, but has been shown to give good results for many problems

1 Introduction

The usefulness of the theory of nonlinear filtering is limited by the availability of good practical approaches that well approximate the quantities of major interest, for example the conditional (weak-sense) density or the conditional mean and covariance. The mathematical theory is mainly concerned with diffusion-type models and white noise corrupted observations that are taken continuously in time. If the observations are taken in discrete time (as they tend to be in practical applications), then the theoretical issues are smaller, since one only needs to approximate the (weak-sense) solution

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to the Kolmogorov forward (the Fokker-Planck) equation between observations and then use Bayes' rule to incorporate the observations. In both cases, the computational issues and the development of the most effective approximation methods is still evolving. We will discuss two broad classes of approximations that have been successfully used on various classes of very nonlinear problems and hold considerable promise. The first approach is the so-called Markov chain approximation method [31]. It will be applied to both cases where the observations are taken continuously and in discrete time. At this time it seems to be the most appropriate approach to approximating the weak-sense conditional density, at least for low-dimensional problems. The basic idea is to use a filter for a Markov chain that approximates the diffusion, but with the actual physical observations. Convergence theorems can be proved as the approximation parameters go to zero.

The Markov chain approximation method is a widely used approach to the numerical solution of control and optimal control problems for general jump-diffusion models, with or without boundary reflection, and with all of the standard cost functions. It is the most general set of algorithms available at this time for the control problem, in terms of both construction of the actual numerical algorithm and the convergence proofs. The basic idea is the following. One constructs an approximating Markov chain on a finite state space, with approximation parameter h , and that is "locally consistent" with the diffusion or jump-diffusion. Then one solves for the cost or optimal cost function for the approximating chain, and finally proves that as $h \rightarrow 0$, these functions converge to those for the original model. The local consistency condition is not restrictive and should be satisfied by any approximating process. The approximating chains should be constructed so that the associated numerical problem is reasonable. Getting such constructions is straightforward, as can be seen in [31]. The same chains can be used for the filtering problem. Due to the growth of the size of the required state space as the dimension increases, at this time the method would be difficult to use if the dimension were greater than four.

Numerical methods for the approximate evaluation of the conditional distribution were given in [27] and related methods are in [9, 13, 15]. Robustness (locally Lipschitz continuity) of the numerical approximations with respect to the observation process was shown in [28] and this property is satisfied by the algorithm that is to be described. Extensions, including point process observations, are in [10, 14].

As noted, the Markov chain approximation method is used for the approximation of the conditional weak-sense density and can be "computationally intensive." Often one is interested in just the first few conditional

moments. Since, for nonlinear problems, a finite set of conditional moments will define the conditional distribution only under very restrictive conditions, one must resort to a heuristic procedure. The second method to be considered is the so-called “assumed form of the conditional density” approach, first proposed in [25] and developed and used in various ways since then [1, 8, 20, 38]. With this method, one assumes that the conditional density takes a particular parametrized form, and then approximates the evolution of the parameters, under this assumption. Most commonly, the assumed density is Gaussian (or a Gaussian mixture), where the parameters are the conditional mean and covariance. The numerical issues center about the approximation of integrals with respect to Gaussian kernels. With guidance from the literature on the numerical evaluation of integrals, there are many ways of doing this, and some methods of current interest will be discussed. Numerical data lend support to the value of the approach. The original observation that led to this method was the realization that the extended Kalman filter was equivalent to the assumption of a conditional “density” that could have negative values, and this was a major cause of its instability in truly nonlinear problems.

An alternative approach to filtering is that of batch processing, which we will not discuss since we are concerned with “dynamical systems” approximations. Loosely speaking, with observation process $y_n = g(x_n) +$ observation noise, at step n one constructs a penalty function of the form [2]

$$\sum_{i=1}^n |g(x_i) - y_i|^2 + \text{penalty on the driving and observation noises}$$

and seeks to minimize it with respect to the driving and observation noises. The approach can yield very good results, if the computation required for the minimization is not too onerous.

Section 2 describes the system model and the basic results from the theory of nonlinear filtering that will be needed. The proofs for the Markov approximation method depend on results from the theory of weak convergence, and a summary of the required results is given in Section 3. Section 4 gives a general result on the approximation of the conditional weak-sense density. The Markov chain approximation method is discussed in Section 5. It is applied to the filter approximation problem in Section 6, where the result of Section 4 will be crucial. The main issues in the construction concern the coordination of the time scale of the chain with that of the physical observation process, and there are several ways of doing this. The robustness

of the approximating filter (uniformly in the approximation parameter) is shown in Section 7. Sections 8 and 9 concern the assumed density approach and it is illustrated by a numerical example in Section 10.

2 The System Model and Filter Representations

The system. The system will evolve in continuous time. To keep the technicalities at a minimum, we will use the diffusion process defined by

$$x(t) = x(0) + \int_0^t b(x(s))ds + \int_0^t \sigma(x(s))dw(s), \quad (2.1)$$

where $x(t) \in \mathbb{R}^r$, Euclidean r -space, $w(\cdot)$ is a standard vector-valued Wiener process, and $b(\cdot)$ and $\sigma(\cdot)$ are bounded and continuous.¹ We suppose that (2.1) has a unique weak-sense solution for each initial condition. Define the matrix $a(x) = \{a_{i,j}(x); i, j\} = \sigma(x)\sigma'(x)$.

Notes on and extensions of the model. One can stop the process or reinject it when it exits some bounded set. Many physical problems have reflecting boundaries. Under the usual conditions on such boundaries and reflection directions (see, e.g., [30, 31]) all of the subsequent results will hold true, provided that the Markov chain approximation accounts for the boundary. We could also treat the jump-diffusion model where the term $\int_0^t \int q(x(s-), \gamma)N(ds d\gamma)$ is added to (2.1), where $N(\cdot)$ is a Poisson random measure with intensity measure $h(dt d\gamma) = \lambda dt \Pi(d\gamma)$, $\Pi(\cdot)$ is a bounded measure, and $q(\cdot)$ is bounded and continuous. The extra technicalities associated with such extensions shed little light on the problems of filter approximations, so we stay with (2.1). For expositional simplicity, we suppose that $b(\cdot), \sigma(\cdot), q(\cdot), g(\cdot)$ do not depend on time. The time-dependent case is a trivial modification.

The observations. The observations can be taken in continuous or in discrete time. For the first case, the data at t is $\mathcal{Y}_t = \{y(s), s \leq t\}$, where

$$dy(t) = g(x(t))dt + Vdw_0(t), \quad y(0) = 0, \quad y(t) \in \mathbb{R}^m, \quad (2.2)$$

where $w_0(\cdot)$ is a standard Wiener process that is independent of $x(\cdot)$, $g(\cdot)$ is bounded and continuous and the matrix VV' is positive-definite.

¹As a practical matter, all the functions can be taken to be bounded. Generally in applications, one tries to model the dynamics for state values in the range that is most common, and the models tend to become invalid as the absolute value of the state goes to infinity, where the model is often chosen for convenience in analysis.

The observations (2.2) are often approximated by supposing that for small $\delta > 0$ they are taken at times $n\delta, n = 1, 2, \dots$, and that at $n\delta$ is either

$$y_n^\delta = g(x(n\delta))\delta + V[w_0(n\delta) - w_0(n\delta - \delta)], \quad (2.3a)$$

or

$$y_n^\delta = \int_{n\delta-\delta}^{n\delta} g(x(s))ds + V[w_0(n\delta) - w_0(n\delta - \delta)] = y(n\delta) - y(n\delta - \delta). \quad (2.3b)$$

For (2.3a,b), let $\mathcal{Y}_{n\delta}^\delta$ denote the data available at time $n\delta$. When the observations are taken at arbitrary times $t_n, n = 1, 2, \dots$, let that at t_n be

$$y_n = g_i(x(t_n)) + V_n v_n, \quad (2.3c)$$

where $g_n(\cdot)$ is bounded and continuous, $V_n V_n'$ is positive-definite and $\{v_n\}$ are i.i.d. Gaussian vectors, independent of $x(\cdot)$, and with covariance matrix the identity. Then we define $\mathcal{Y}_n = \{y_{t_i}, i \leq n\}$. In all cases, since we lose nothing by working with the observation process $V^{-1}y(t)$, without loss of generality in what follows we will suppose that $V = V_i = \mathbb{I}$, the identity matrix.

The optimal nonlinear filter for (2.1), (2.2). For $\phi(\cdot)$ a bounded and measurable real-valued function, define the conditional expectation operator \mathbb{E}_t by $\mathbb{E}_t \phi(x(t)) = \mathbb{E}[\phi(x(t)) | \mathcal{Y}_t]$. Perhaps the key result in the theory of nonlinear filtering is the so-called representation theorem, which is a “limit” form of Bayes’ rule. We will use it in the form that was used in the original derivation of the optimal nonlinear filter [26], which is convenient for the types of approximation and weak convergence methods that will be used. That reference was the first to get the result for the diffusion case, and the development is similar for the reflected-jump-diffusion model. Subsequent developments via measure transformation and “martingale-innovations” techniques are in [18, 34], but the representations obtained by those techniques are exactly the same as were obtained in [26]. The case where $x(\cdot)$ is a continuous-time Markov chain on a finite state space case was first derived in [36, 39].

Let $\tilde{x}(\cdot)$ be a process with the same probability law as $x(\cdot)$, but that is independent of $(x(\cdot), y(\cdot))$. Define

$$R(t) = \exp \left[\int_0^t g(\tilde{x}(s))' dy(s) - \frac{1}{2} \int_0^t |g(\tilde{x}(s))|^2 ds \right]. \quad (2.4)$$

Then we have the representation [26] for the filter for model (2.1) and observation (2.2):

$$\mathbb{E}_t \phi(x(t)) = \frac{\mathbb{E}[R(t)\phi(\tilde{x}(t)) | \mathcal{Y}_t]}{\mathbb{E}[R(t) | \mathcal{Y}_t]}. \quad (2.5)$$

The form holds for the observations (2.3b), if t is restricted to $n\delta, n = 1, 2, \dots$. It also holds for (2.3a) if the sums $\sum_{i=1}^n g'(\tilde{x}(i\delta))[w_0(i\delta) - w_0(i\delta - i\delta)]$ and $(\delta/2) \sum_{i=1}^n |g(\tilde{x}(i\delta))|^2$ replace the integrals.

Dynamical equations for the conditional probability density. For the moment let us proceed purely formally. Define the differential operator \mathcal{L} of (2.1), acting on smooth real-valued functions $f(\cdot)$,

$$\mathcal{L}f(x) = f'_x(x)b(x) + \frac{1}{2} \sum_{i,j} a_{i,j}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j}.$$

The formal adjoint operator \mathcal{L}^* is

$$\mathcal{L}^*f(x) = - \sum_i \frac{\partial(b_i(x)f(x))}{\partial x_i} + \frac{1}{2} \sum_{i,j} \frac{\partial^2(a_{i,j}(x)f(x))}{\partial x_i \partial x_j}.$$

Suppose that there is a smooth density $p(\tilde{x}, t, x)$ of $x(t)$ given initial condition $x(0) = \tilde{x}$. This satisfies the Kolmogorov forward (or Fokker-Planck) equation (with \mathcal{L}^* acting on the functions of x) [17, Chapter 5]

$$\frac{\partial p(\tilde{x}, t, x)}{\partial t} = \mathcal{L}^*p(\tilde{x}, t, x). \quad (2.6)$$

If the density is not smooth, then use the weak-sense solution.

Consider the filtering problem with model (2.1), (2.2). Let $\phi(\cdot)$ be a bounded and continuous real-valued function whose partial derivatives up to second order are bounded and continuous. Then (recall that we are taking $V = \mathbb{I}$, the identity matrix) the conditional moments evolve as [26, 34]

$$\begin{aligned} d[\mathbb{E}_t \phi(x(t))] &= \mathbb{E}_t \mathcal{L} \phi(x(t)) dt \\ &+ (\mathbb{E}_t g(x(t)) \phi(x(t)) - \mathbb{E}_t g(x(t)) \mathbb{E}_t \phi(x(t)))' (dy(t) - \mathbb{E}_t g(x(t)) dt). \end{aligned} \quad (2.7)$$

Let π_t denote the (weak-sense) conditional density of $x(t)$ given \mathcal{Y}_t . Then, purely formally, the normalized density evolves as [18, 24, 34]

$$d\pi_t(x) = \mathcal{L}^* \pi_t(x) dt + \pi_t(x) (g(x) - \mathbb{E}_t g(x(t)))' (dy(t) - \mathbb{E}_t g(x(t)) dt). \quad (2.8)$$

Write the numerator of (2.5) as $\mathbb{E}_t^0 \phi(x(t))$, the unnormalized conditional expectation. Then

$$d[\mathbb{E}_t^0 \phi(x(t))] = \mathbb{E}_t^0 \mathcal{L} \phi(x(t)) dt + \mathbb{E}_t^0 g(x(t)) \phi(x(t)) dy(t).$$

This equation, known as the Zakai equation ([40], [18, Section 14], and [26, Equation 15]), is simpler than (2.7), but does not seem to provide simpler numerical algorithms. The Markov chain approximation method already starts by computing the unnormalized density, and the assumed density approach uses a normalized assumed density.

The optimal filter for a Markov chain signal process. Let $\{\xi_n, n < \infty\}$ be a finite-state Markov chain with one-step transition probabilities $p(x, y)$. Suppose that $\{v_n, n < \infty\}$ is a sequence of mutually independent normally distributed random variables with mean zero and covariance V_0 , and which is also independent of the $\{\xi_n, n < \infty\}$. Suppose that we observe the white noise corrupted data $y_n^0 = g_0(\xi_n) + v_n$ at time step n , for some bounded and measurable function $g_0(\cdot)$. Define $\mathcal{Y}_n^0 = \{y_i^0, i \leq n\}$ and the conditional distribution $Q_n^0(x) = P\{\xi_n = x | \mathcal{Y}_n^0\}$.

We now use Bayes' rule to define a recursive formula for $Q_n^0(\cdot)$. Let $P\{y_n^0 | \xi_n = x\}$ denote the conditional (normal, with given mean $g_0(x)$ and covariance V_0) density of the observation at the value y_n^0 . Note that

$$\begin{aligned} Q_n^0(x) &= \sum_z P\{\xi_n = x, \xi_{n-1} = z | y_n^0, \mathcal{Y}_{n-1}^0\} \\ &= \frac{\sum_z P\{y_n^0 | \xi_n = x, \xi_{n-1} = z\} P\{\xi_n = x | \xi_{n-1} = z\} Q_{n-1}^0(z)}{\text{normalization}}. \end{aligned} \tag{2.9}$$

Substituting in the conditional density function of y_n^0 , rewrite the last expression as

$$\frac{\sum_z \exp\left[g_0'(x)V_0^{-1}y_n^0 - \frac{1}{2}g_0'(x)V_0^{-1}g_0(x)\right] p(z, x) Q_{n-1}^0(z)}{\text{normalization}}. \tag{2.10}$$

The normalization is the numerator summed over x and might differ from expression to expression. Let $\phi(\cdot)$ be bounded, measurable and real-valued, and let $\{\tilde{\xi}_n\}$ have the same probability law as $\{\xi_n\}$, but independent of $\{\xi_n, y_n\}$. Then, iterating (2.10) yields the expression

$$\mathbb{E}_{\mathcal{Y}_n^0} \phi(\xi_n) = \frac{\mathbb{E}_{\mathcal{Y}_n^0} \phi(\tilde{\xi}_n) \exp\left\{\sum_{i=0}^n \left[g_0'(\tilde{\xi}_i)V_0^{-1}y_i^0 - \frac{1}{2}g_0'(\tilde{\xi}_i)V_0^{-1}g_0(\tilde{\xi}_i)\right]\right\}}{\text{normalization}}. \tag{2.11}$$

3 Weak Convergence

Expressions like (2.11) will be basis of the numerical approximation, but where the y_n^0 come from the original (2.2) or (2.3). The proofs of convergence

depend on the theory of weak convergence of probability measures [4, 19] and we will give a brief description of the essential facts. The probability measures of interest are those on the path space of $x(\cdot)$ or of the processes to be used to approximate it.

Let S denote a metric space with metric d , with $C_b(S)$ the bounded continuous real-valued functions on S . Suppose that $X_n, n < \infty$, and X are S -valued random variables, with measures $P_n, n < \infty$, and P , resp. If $Ef(X_n) \rightarrow Ef(X)$ for all $f \in C_b(S)$, then we say that P_n converges weakly to P , and this defines the weak topology on $\mathcal{P}(S)$, the set of measures on the Borel sets of S . We also say that X_n converges weakly to X . If X_n converges weakly to X , then $Ef(X_n) \rightarrow Ef(X)$ for bounded measurable $f(\cdot)$ which are continuous only almost everywhere with respect to P [4, Theorem 5.1].

A set $\{P_\alpha\} \subset \mathcal{P}(S)$ is called *tight* if for each $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset S$ such that $\inf_\alpha P_\alpha(K_\varepsilon) \geq 1 - \varepsilon$. If the P_α are the measures which induce random variables X_α , then we will also refer to the collection $\{X_\alpha\}$ as *tight*. The following theorem is fundamental.

Theorem 3.1 [Prokhorov's Theorem] [4, 19]. *If S is a complete and separable metric space, then a set $\{P_\alpha\} \subset \mathcal{P}(S)$ has compact closure in the metric of weak convergence if and only if $\{P_\alpha\}$ is tight.*

Criterion for tightness. Let $D^r[0, \infty)$ denote the space of \mathbb{R}^r -valued functions which are right-continuous and have left-hand limits, with the Skorokhod topology. (See [4, 19] for the precise definition.) Then it is a complete and separable metric space. The topology is weaker than that induced by the metric of uniform convergence on bounded intervals since it allows for discontinuities to be "shifted slightly" when measuring the distance between functions. Also if $f_n(\cdot)$ converges to a continuous $f(\cdot)$ in the Skorokhod topology, then the convergence is uniform on each bounded time interval. The following criteria for tightness will be used.

Theorem 3.2. [23, Theorem 2.7b][19]. *Let $\{x^n(\cdot)\}$, defined on (Ω, \mathcal{F}, P) , take values in $D^r[0, \infty)$. Assume that for each t in a dense set in $[0, \infty)$ and each $\delta > 0$ there exist compact $K_{t,\delta} \subset \mathbb{R}^r$ such that $\sup_n P\{x^n(t) \notin K_{t,\delta}\} \leq \delta$. Define \mathcal{F}_t^n to be the σ -algebra generated by $\{x^n(s), s \leq t\}$. Let \mathcal{T}_T^n be the set of $\{\mathcal{F}_t^n\}$ -stopping times which are less than or equal to T w.p.1. Then $\{x^n(\cdot)\}$ is tight if for each $T \in [0, \infty)$,*

$$\lim_{\delta \rightarrow 0} \limsup_n \sup_{\tau \in \mathcal{T}_T^n} \mathbb{E}(1 \wedge |x^n(\tau + \delta) - x^n(\tau)|) = 0. \quad (3.1)$$

Since we will have need for weak convergence of functions of the approximating process, with the observation taking the role of a “parameter,” the following extension of weak convergence will be useful

Theorem 3.3. *Let X_n converge weakly to X . Suppose (w.l.o.g.) that $\{X_n, n < \infty\}$ and X are defined on the same probability space, and that Y (with measure P_Y) is independent of (X_n, X) for each n . Let $f(\cdot, \cdot)$ be a real-valued measurable function, such that for P_Y -almost all y , $f(\cdot, y)$ is continuous with probability one with respect to the measure P_X of X . Let P_n denote the measure of X_n . Suppose that for each y in a set of P_Y -probability one $\{f(X_n, y)\}$ is uniformly integrable. Then*

$$\int f(x, Y)P_n(dx) \rightarrow \int f(x, Y)P_X(dx), \quad \text{w.p.1 } (P_Y). \quad (3.2)$$

4 Approximations to the Optimal Filter

The expression (2.5) is difficult to evaluate owing to the expectation over $\tilde{x}(\cdot)$. Recall that $\tilde{x}(\cdot)$ is a copy of $x(\cdot)$, and is independent of $y(\cdot)$. For computational purposes, one replaces $\tilde{x}(\cdot)$ by an approximation $x^h(\cdot)$ which is selected so that the evaluation can be carried out with a reasonable amount of work, and then one proves that the result converges to that defined by (2.5) as the approximation parameter $h \rightarrow 0$. We will suppose that the path is confined to a compact set, either by its own dynamics, a reflecting boundary, or by being stopped or reinjected on leaving a given compact set. There is considerable flexibility in the choice of the approximating process, as seen by the next theorem.

Let $\{x^h(\cdot)\}$ be a sequence that is independent of $y(\cdot)$, and that converges weakly to $x(\cdot)$ as $h \rightarrow 0$. Define

$$R^h(t) = \exp \left[\int_0^t g'(x^h(s))dy(s) - \frac{1}{2} \int_0^t |g(x^h(s))|^2 ds \right]. \quad (4.1)$$

Define the operator E_t^h by

$$E_t^h \phi(x^h(t)) = \frac{\mathbb{E}_t \phi(x^h(t)) R^h(t)}{\mathbb{E}_t R^h(t)}. \quad (4.2)$$

Comment. E_t^h is not a conditional expectation of $\phi(x^h(t))$, since $x^h(\cdot)$ does not have the probability law of the true signal process. But, by Theorem 4.1, (4.2) converges to (2.5).

Theorem 4.1. *Let $\phi(\cdot)$ be a continuous real-valued function. Assume the above conditions on $x^h(\cdot), x(\cdot), y(\cdot)$, with model (2.2) for the observations. Then, for any $T < \infty$,*

$$\limsup_{h \rightarrow 0} \sup_{t \leq T} \left| E_t^h \phi(x^h(t)) - \mathbb{E}_t \phi(x(t)) \right| = 0 \quad (4.3)$$

in the senses of probability and mean. The result holds for the observation models (2.3a) and (2.3b) at times $n\delta, n = 1, 2, \dots$

Proof. We will neglect the $|g|^2$ terms in (4.1), since they are easy to deal with. First, some preliminary computations. Let $\zeta_i(\cdot), i = 1, 2$, and $\zeta(\cdot)$ be bounded processes that are independent of the standard Wiener process $w_1(\cdot)$ of the same dimension. The analysis will make use of an estimate of

$$W = \mathbb{E} \sup_{t \leq T} \left| \exp \left[\int_0^t \zeta_1'(s) dw_1(s) \right] - \exp \left[\int_0^t \zeta_2'(s) dw_1(s) \right] \right|. \quad (4.4)$$

We will use the inequality, for real numbers A and B ,

$$|e^A - e^B| \leq |A - B| (e^A + e^B). \quad (4.5)$$

For any real-valued submartingale $N(\cdot)$ [16, Chapter VII, Thm. 3.4],

$$\mathbb{E} \sup_{t \leq T} N^2(t) \leq 4\mathbb{E}N^2(T). \quad (4.6)$$

The inequality (4.5) and Schwarz's inequality applied to (4.4) yield

$$\begin{aligned} W^2 &\leq \mathbb{E} \sup_{t \leq T} \left| \int_0^t (\zeta_1'(s) - \zeta_2'(s)) dw_1(s) \right|^2 \\ &\quad \times \mathbb{E} \sup_{t \leq T} \left| \exp \left[\int_0^t \zeta_1'(s) dw_1(s) \right] + \exp \left[\int_0^t \zeta_2'(s) dw_1(s) \right] \right|^2. \end{aligned} \quad (4.7)$$

By (4.6) the first factor in (4.7) is bounded by $4\mathbb{E} \int_0^T |\zeta_1(s) - \zeta_2(s)|^2 ds$. To bound the second factor, use the fact that

$$\mathbb{E} \exp \left[\int_0^T \zeta'(s) dw_1(s) \right] \leq \mathbb{E} \exp \left[\int_0^T |\zeta(s)|^2 ds / 2 \right]$$

for any bounded process $\zeta(\cdot)$ that is independent of $w(\cdot)$, together with (4.6) and the fact that the $\exp[\int_0^t \zeta_i'(s) dw_1(s)]$ are submartingales. Finally, for a constant C_1 that depends on T and on the bounds on the $\zeta_i(\cdot)$, we have

$$W^2 \leq C_1 \mathbb{E} \int_0^T |\zeta_1(s) - \zeta_2(s)|^2 ds. \quad (4.8)$$

It suffices to show that $\sup_{t \leq T} |\mathbb{E}_t \phi(x^h(t))R^h(t) - \mathbb{E}_t \phi(\tilde{x}(t))R(t)| \rightarrow 0$ in probability for each bounded and continuous $\phi(\cdot)$. The above computations imply that, for each $T < \infty$,

$$\mathbb{E} \sup_{s \leq T} [R^h(t)]^2 + \mathbb{E} \sup_{s \leq T} [R(t)]^2 < \infty. \quad (4.9)$$

By the Skorokhod representation [19, Theorem 3.1.8.], w.l.o.g, we can suppose that all of the processes $x^h(\cdot), \tilde{x}(\cdot), x(\cdot), y(\cdot)$ are defined on the same probability space and that the convergence $x^h(\cdot) \rightarrow \tilde{x}(\cdot)$ is w.p.1 in the topologies of the paths. We have, where C_2 is a bound on $\phi(\cdot)$,

$$\begin{aligned} & \sup_{t \leq T} \mathbb{E}_t \left| \phi(x^h(t))R^h(t) - \phi(\tilde{x}(t))R(t) \right| \\ & \leq \sup_{t \leq T} \mathbb{E}_t \left| \phi(x^h(t)) - \phi(\tilde{x}(t)) \right| R^h(t) + C_2 \sup_{t \leq T} \mathbb{E}_t \left| R^h(t) - R(t) \right|. \end{aligned} \quad (4.10)$$

The first term on the right goes to zero as $h \rightarrow 0$ by the weak convergence of $x^h(\cdot)$ to $\tilde{x}(\cdot)$, the boundedness of these processes and (4.9). The convergence of the second term of (4.10) follows from the weak convergence and the inequality (4.8). ■

Comments on numerical approximations. It is important to keep in mind that all approximations to the nonlinear filtering problem are actually approximations to some representation of Bayes' rule, and (2.5) is the fundamental Bayes' rule formula. Except for a few special cases such as the classical Kalman-Bucy form or where $x(\cdot)$ is a finite-state Markov chain, the evaluation of (2.5) is not a finite calculation.

Theorem 4.1 provides an approach to the computation of the conditional density for nonlinear problems. The key to computational effectiveness is the choice of the approximating process $x^h(\cdot)$. There are many possibilities, but it will always be a Markov chain. One could consider a discrete-time approximation of (2.1) such as (for small $\delta > 0$)

$$X(n\delta + \delta) = X(n\delta) + \delta b(X(n\delta)) + \sigma(X(n\delta))[w(n\delta + \delta) - w(n\delta)]. \quad (4.11)$$

While the form is simple, the state space would have to be discretized and the transition functions (as a function of the initial and final values) computed. Then it will turn out to be a version of the Markov chain approximation method. Further comments are in Section 6.2.

If the observations are of the discrete-time form (2.3c), then the filtering problem is simple to describe. Suppose that $\pi_{t_n^-}(\cdot)$, the conditional weak-sense density just before the n th observation is taken, is available at time

t_n . Then use Bayes' rule to compute the conditional distribution $\pi_{t_n}(\cdot)$, taking the new observation into account. Then continue, starting with the density $\pi_{t_n}(\cdot)$, compute the weak-sense density $\pi_{t_{n+1}^-}(\cdot)$ at time t_{n+1} , just before the next observation is taken, etc. While the procedure is simple in principle, the actual computations can be difficult. Getting $\pi_{t_{n+1}^-}(\cdot)$ from $\pi_{t_n}(\cdot)$ involves approximating the solution of the (weak-sense) form of the Kolmogorov forward equation and the integrals involved in incorporating the observations will need to be suitably approximated.

5 The Markov Chain Approximation

From the perspective of the types of computation that is involved in computing the approximation to the conditional density, the simplest form of the Markov chain approximation method is analogous to methods for solving parabolic PDE's by finite differences or finite elements. If the computations are to be finite then the path must be confined to a bounded set, which might depend on time. Boundedness is often a consequence of the dynamics of (2.1). If $x(\cdot)$ is not bounded on the time interval of concern, then some type of truncation is needed. This can be done either by introducing a reflecting or stopping boundary. However, for expositional simplicity, we will generally ignore the boundedness issue since it will not affect the main ideas. The actual numerical approximating filter will be that for a Markov chain that approximates the diffusion, but with the actual physical observations (2.2) or (2.3) being used. The approximations to the conditional densities will converge to the weak-sense conditional density as the approximation parameters go to zero. The overall development is based on the methods in [27, 31].

The basis of the approximation is a discrete-time finite-state Markov chain whose "local properties" are "consistent" with those of (2.1), as described below. This chain will be interpolated into a continuous-time process that will be a good approximation to (2.1) in the sense to be described. For simplicity, let the approximation parameter $h > 0$ be real-valued, although a vector-valued parameter could be used as well [31]. For each h , let $\{\xi_n^h, n < \infty\}$ be a discrete-parameter Markov chain on a discrete state space $G_h \subset \mathbb{R}^r$, with finitely many points, and with transition probabilities $p^h(x, \tilde{x})$.

Local consistency: Diffusion case. Define $\Delta\xi_n^h = \xi_{n+1}^h - \xi_n^h$. Let \mathbb{E}_n^h (resp. $\mathbb{E}_{x,n}^h$) denote the conditional expectation given all data to step n

(and in addition that $\xi_n^h = x$, resp.). Define the martingale difference $\beta_n^h = \Delta\xi_n^h - \mathbb{E}_n^h \Delta\xi_n^h$. Suppose that the following “local consistency” conditions holds in G_h :²

$$\begin{aligned} \mathbb{E}_{x,n}^h \Delta\xi_n^h &\equiv b_h(x) \Delta t^h(x) = b(x) \Delta t^h(x) + o(\Delta t^h(x)), \\ \text{covar}_{x,n}^h [\Delta\xi_n^h - \mathbb{E}_{x,n}^h \Delta\xi_n^h] &\equiv a_h(x) \Delta t^h(x) = a(x) \Delta t^h(x) + o(\Delta t^h(x)), \\ \sup_{n,\omega} |\xi_{n+1}^h - \xi_n^h| &\xrightarrow{h} 0, \end{aligned} \tag{5.1}$$

for some function $\Delta t^h(x) > 0$, that we call an “interpolation interval.” We assume that $\lim_{h \rightarrow 0} \sup_{x \in G_h} \Delta t^h(x) = 0$, but $\inf_{x \in G_h} \Delta t^h(x) > 0$ for each $h > 0$. Define $\Delta t_n^h = \Delta t^h(\xi_n^h)$. The local consistency (5.1) is essentially all that is required of the approximating chain, together with analogous conditions for the reflecting boundary, if any.³

The reference [31, Chapter 5] describes many convenient methods for constructing chains that satisfy the required properties. By (5.1), the chain has the local conditional drift and covariance properties of (2.1). With all of the usual methods for constructing the $p^h(x, \tilde{x})$, the interpolation intervals are obtained automatically as a byproduct [31, Chapter 5]. There is considerable flexibility; local consistency need not hold everywhere, as seen in [31, Section 5.5].

The simplest example is the one-dimensional model $dx = b(x)dt + \sigma(x)dw$ where $\sigma^2(x) \geq h|b(x)|$ for all x , and the state space is $G_h = \{0, \pm h, \pm 2h, \dots\}$. Then

$$p^h(x, x \pm h) = (\sigma^2(x) \pm hb(x))/2\sigma^2(x) \text{ and } \Delta t^h(x) = h^2/\sigma^2(x) \tag{5.2}$$

yield a locally consistent chain, where each state communicates only with its nearest neighbors.

Constant interpolation interval. For the numerical approximation of the general control problem as in [31], the possible dependence of $\Delta t^h(x)$ on x is an advantage from the point of view of computation. But it complicates the computations for the filtering problem where the approximating chain must be able to “track real time.” Having a constant $\Delta t^h(x)$ can then be useful. A chain with a constant interpolation interval is easily obtained from any locally consistent chain for which $p^h(x, x) = 0$ for all x ,⁴ and

²(5.1) defines the functions $b_h(\cdot)$ and $a_h(\cdot)$.

³If there is a jump component, then at each step, the chain is locally consistent with the diffusion (2.1), with conditional probability $(1 - \lambda \Delta t^h(x))$, and with the jump component with conditional probability $\lambda \Delta t^h(x)$. See [31, Section 5.6].

⁴This is the usual case, using the methods of [31].

the modified transition probabilities $\bar{p}^h(x, \tilde{x})$ and interpolation interval $\bar{\Delta}^h$ are readily obtained. Define $\bar{\Delta}^h = \inf_{\xi \in G_h} \Delta t^h(\xi)$. The possibility that $\bar{\Delta}^h < \Delta t^h(x)$ at some point x is compensated for by allowing the state x to communicate with itself. Conditioned on the event that a state does not communicate with itself, the transition probabilities are the $p^h(\cdot)$. Thus, the general formula for getting $\bar{p}^h(x, \tilde{x})$ from the $p^h(\cdot)$ is ([31, Section 7.7])

$$\begin{aligned} \bar{p}^h(x, \tilde{x}) &= p^h(x, \tilde{x})(1 - \bar{p}^h(x, x)), \quad \text{for } x \neq \tilde{x}, \\ \bar{p}^h(x, x) &= 1 - \frac{\bar{\Delta}^h}{\Delta t^h(x)}. \end{aligned} \tag{5.3}$$

Continuous-time interpolations and convergence proofs. The numerical algorithms use the Markov chain, but the proofs of convergence are based on continuous-time interpolations of the chain. The simplest interpolation, called $\xi^h(\cdot)$, uses the intervals $\Delta t_n^h = \Delta t^h(\xi_n^h)$ and is defined as follows. With $t_n^h = \sum_{i=0}^{n-1} \Delta t_i^h$ define $\xi^h(t) = \xi_n^h$, $t \in [t_n^h, t_{n+1}^h)$. Although ξ_n^h is a Markov chain, $\xi^h(\cdot)$ is not.

The proofs are facilitated by using an interpolation $\psi^h(\cdot)$ that is a continuous-time Markov chain, asymptotically equivalent to $\xi^h(\cdot)$, and constructed as follows. Let $\{\nu_n\}$ be random variables that are independent of $\{\xi_n^h\}$, mutually independent and identically distributed, and with ν_n being exponentially distributed with mean unity. Using $\Delta \tau_n^h = \Delta t_n^h \nu_n$ and $\tau_n^h = \sum_{i=0}^{n-1} \Delta \tau_i^h$, define $\psi^h(t) = \xi_n^h$ on $[\tau_n^h, \tau_{n+1}^h)$. $\psi^h(\cdot)$ is a continuous-time Markov chain, whose holding times $\Delta \tau_n^h$, given ξ_n^h , are exponentially distributed with mean Δt_n^h .

We can decompose $\psi^h(\cdot)$ in terms of a compensator and martingale as $\psi^h(t) = x(0) + \int_0^t b_h(\psi^h(s)) ds + B_\tau^h(t)$, where the martingale $B_\tau^h(t)$ has quadratic variation process $\int_0^t a_h(\psi^h(s)) ds$. It can be shown that ([31, Section 10.4.1]) there is a martingale $w^h(\cdot)$ (with respect to the filtration generated by the path and control processes, possibly augmented by an “independent” Wiener process) such that

$$B_\tau^h(t) = \int_0^t \sigma_h(\psi^h(s)) dw^h(s) = \int_0^t \sigma(\psi^h(s)) dw^h(s) + \epsilon^h(t),$$

where $\sigma_h(\cdot)[\sigma_h(\cdot)]' = a_h(\cdot)$, $w^h(\cdot)$ has quadratic variation process $\mathbb{I}t$ and converges weakly to a standard Wiener process. The martingale $\epsilon^h(\cdot)$ is due to the difference between $\sigma(x)$ and $\sigma_h(x)$ and

$$\lim_{h \rightarrow 0} \mathbb{E} \sup_{s \leq t} |\epsilon^h(s)|^2 = 0 \tag{5.4}$$

for each t . Thus

$$\psi^h(t) = x(0) + \int_0^t \int_U b_h(\psi^h(s)) ds + \int_0^t \sigma(\psi^h(s)) dw^h(s) + \epsilon^h(t). \quad (5.5)$$

By Theorem 3.1, $\psi^h(\cdot)$ is tight. The weak-sense limit is (2.1). The same result holds if there is a jump term. See [31, Chapter 10] for more detail.

Define $t^h(\cdot)$ and $\tau^h(\cdot)$ by $t^h(s) = t_n^h$ on $[t_n^h, t_{n+1}^h)$ and $\tau^h(s) = \tau_n^h$ on $[\tau_n^h, \tau_{n+1}^h)$. By Theorem 5.1, both converge to the process with value t at time t . Hence the interpolations $\xi^h(\cdot)$ and $\psi^h(\cdot)$ are asymptotically equivalent.

Discretizing time. A modification that allows us to keep track of the elapsed real time when the $\Delta t^h(x)$ are not constant augments the chain $\{\xi_n^h\}$ by adding a (random) discretization of time. This is a simplified version of the “implicit” approximation in [31] and will allow us to use the original intervals $\Delta t^h(x)$. Let $\delta \geq \sup_x \Delta t^h(x)$ denote the discretization level for the time variable, whose value at step n we denote by $\phi_n^{h,\delta}$. The spatial components satisfy (5.1). So one needs only determine the probability that the time variable advances, conditioned on the current values $\xi_n^h = x$, $\phi_n^{h,\delta} = i\delta$. This is obtained by a “local consistency” argument and no matter how the $p^h(\cdot)$ were derived, the conditional probability that the time variable advances is

$$p^{h,\delta}(i\delta, i\delta + \delta | x) = \Delta t^h(x) / \delta. \quad (5.6)$$

Define the martingale difference $\beta_{0,n}^{h,\delta} = (\phi_{n+1}^{h,\delta} - \phi_n^{h,\delta}) - \Delta t_n^h$, whose conditional covariance is $\Delta t_n^h(\delta - \Delta t_n^h)$. We can write

$$\phi_{n+1}^{h,\delta} = \phi_n^{h,\delta} + \Delta t_n^{h,\delta} + \beta_{0,n}^{h,\delta}. \quad (5.7)$$

An alternative approximating chain. There is an alternative way of interpolating the $\xi_n^{h,\delta}$ which looks at the process only at those times that the time variable $\phi_n^{h,\delta}$ advances. Define $v_0^{h,\delta} = 0$ and for $n > 0$ define

$$v_n^{h,\delta} = \min\{i > v_{n-1}^{h,\delta} : \phi_i^{h,\delta} - \phi_{i-1}^{h,\delta} = \delta\}. \quad (5.8)$$

Then define $\hat{\xi}_n^{h,\delta} = \xi_{v_n^{h,\delta}}^{h,\delta}$. By Theorem 5.1, the interpolation $\hat{\xi}^{h,\delta}(\cdot)$ (intervals δ) and $\xi^h(\cdot)$ are asymptotically (as $h \rightarrow 0$, then $\delta \rightarrow 0$) equivalent.

Asymptotic equivalence of the time scales. Define the continuous-parameter interpolation $\phi^{h,\delta}(t) = \phi_n^h$ for $t \in [t_n^h, t_{n+1}^h)$. Define the stopping

times

$$\tilde{d}^h(t) = \min \left\{ n : \sum_{i=0}^{n-1} \Delta t_i^h = t_n^h \geq t \right\}, \quad \tilde{d}_\tau^h(t) = \min \left\{ n : \tau_n^h \geq t \right\}. \quad (5.9)$$

Theorem 5.1. *For each $t > 0$,*

$$\lim_{h \rightarrow 0} \mathbb{E} \left[\sup_{s \leq t} \left(\sum_{i=0}^{\tilde{d}^h(s)} (\Delta \tau_i^h - \Delta t_i^h) \right)^2 \right] = 0. \quad (5.10)$$

As $h \rightarrow 0$ and $\delta \rightarrow 0$, $\phi^{h,\delta}(\cdot)$ converges weakly and in mean square (uniformly on any finite time interval) to the process with value t at time t .

Proof. By the mutual independence of the exponentially distributed random variables $\{\nu_n\}$ and their independence of $\{\xi_m^h, \phi_n^{h,\delta}\}$, the process $A_n^h = \sum_{i=0}^n (\Delta \tau_i^h - \Delta t_i^h)$ is a martingale. By Doob's inequality for martingales, the expectation of the bracketed term in (5.10), conditioned on $\{\Delta t_i^h\}$, is bounded by

$$\begin{aligned} & \mathbb{E} \sup_{s \leq t} \left[\sum_{i=0}^{\tilde{d}^h(s)} \left[\Delta \tau_i^h - \Delta t_i^h \right]^2 \middle| \Delta t_i^h, i < \infty \right] \\ & \leq 4 \mathbb{E} \left[\sum_{i=0}^{\tilde{d}^h(t)} \left[\Delta \tau_i^h - \Delta t_i^h \right]^2 \middle| \Delta t_i^h, i < \infty \right] \\ & = 4 \sum_{i=0}^{\tilde{d}^h(t)} [\Delta t_i^h]^2 \leq 4(t + \sup_n \Delta t_n^h) \sup_n \Delta t_n^h \xrightarrow{h} 0, \end{aligned}$$

which yields (5.10). To prove the assertions concerning the asymptotic behavior of $\phi^{h,\delta}(\cdot)$ define $d^h(t) = \max \left\{ n : \sum_{i=0}^{n-1} \Delta t_i^h = t_n^h \leq t \right\}$ and write

$$\phi^{h,\delta}(t) = \sum_{i=0}^{d^{h,\delta}(t)-1} \Delta t_i^{h,\delta} + \sum_{i=0}^{d^{h,\delta}(t)-1} \beta_{0,i}^{h,\delta}.$$

The first sum equals t , modulo $\sup_n \Delta t_n^{h,\delta}$. The variance of the martingale term is bounded by δt , modulo $\delta + \sup_n \Delta t_n^{h,\delta}$, and the term converges weakly to the zero process. This yields the last assertion of the theorem. ■

6 Approximating Filters

6.1 A Filter Based on the Chain $\{\xi_n^h\}$

Let us first define the optimal filter for the process $\xi^h(\cdot)$ with $\Delta t^h(x) = \bar{\Delta}^h$, a constant. Define the observation process $y^h(t) = \int_0^t g(\xi^h(s))ds + w_0(t)$ and $y_n^h = y^h(n\bar{\Delta}^h) - y^h(n\bar{\Delta}^h - \bar{\Delta}^h)$ where $g(\cdot)$ and $w_0(\cdot)$ are as in (2.2). Set $\mathcal{Y}_t^h = \{y^h(s), s \leq t\}$, and $\tilde{Q}_n^h(x) = P\{\xi_n^h = x | \mathcal{Y}_{n\bar{\Delta}^h}^h\}$. Define

$$R^h(x, y_n^h) = \exp \left[g(x)'y_n^h - \frac{1}{2}|g(x)|^2\bar{\Delta}^h \right].$$

Then (2.10) becomes

$$\tilde{Q}_n^h(x) = \frac{\sum_{\tilde{x}} R^h(x, y_n^h)p^h(\tilde{x}, x)\tilde{Q}_{n-1}^h(\tilde{x})}{\text{normalization}}, \quad (6.1)$$

where $\tilde{Q}_0^{h,\delta}(x)$ is the probability that $\xi_0^h = x$. Let $\tilde{\xi}^h(\cdot)$ have the law of $\xi^h(\cdot)$ but be independent of it and $w_0(\cdot)$. Iterating (6.1) yields, for $t = n\bar{\Delta}^h$,

$$\mathbb{E}_{\mathcal{Y}_t^h} \phi(\xi_n^h) = \frac{\mathbb{E} \phi(\tilde{\xi}^h(t)) \exp[\int_0^t g'(\tilde{\xi}^h(s))dy^h(s) - \int_0^t |g(\tilde{\xi}^h(s))|^2 ds/2]}{\text{normalization}}. \quad (6.2)$$

The unnormalized form of (6.1) is

$$\tilde{q}_n^h(x) = \sum_{\tilde{x}} R^h(x, y_n^h)p^h(\tilde{x}, x)\tilde{q}_{n-1}^h(\tilde{x}). \quad (6.3)$$

(6.3) can be split into two steps: First update the effects of the dynamics as

$$\sum_{\tilde{x}} p^h(\tilde{x}, x)\tilde{q}_{n-1}^h(\tilde{x}), \quad (6.4)$$

and then incorporate the observation by multiplying by $R^h(x, y_n^h)$.

The approximation to the optimal filter for $x(\cdot)$ and continuous-time observations. The most direct numerical approximation to the optimal filter (2.5) is either (6.1), (6.2), or (6.3) with the *actual physical observations* $y(\cdot)$ used in place of $y^h(\cdot)$. Both (6.1) and (6.3) are recursive formulas. The initial condition $\tilde{Q}_0^h(\cdot)$ is any approximation to the weak-sense density of $x(0)$ and which converges weakly to that density as $h \rightarrow 0$.

In summary, redefine

$$R^h(t) = \exp \left[\int_0^t g'(\xi_i^h(s))dy(s) - \int_0^t |g(\xi^h(s))|^2 ds/2 \right].$$

Then

$$E_t^h \phi(x(t)) = \frac{\mathbb{E}[\phi(\xi^h(t))R^h(t)|\mathcal{Y}_t]}{\mathbb{E}[R^h(t)|\mathcal{Y}_t]}. \quad (6.5)$$

Theorem 4.1 yields the convergence to $\mathbb{E}[\phi(x(t))|\mathcal{Y}_t]$. One can use the observation $y_n^h = g(x(n\bar{\Delta}^h))\Delta + [w_0(n\bar{\Delta}^h) - w_0(n\bar{\Delta}^h - \bar{\Delta}^h)]$ with the same result. To simplify the computation, one need not introduce the observation at each step, but treat them as discrete-time observations with small interval Δ , as below.

Discrete-time observations. The continuous-time filter was dealt with since it has been of great theoretical interest, although discrete-time observations are of greater practical interest. Suppose that the observations are taken in discrete time as in (2.3c). Then between observations, we can get a weak-sense approximation to the solution of the Kolmogorov forward equation for the weak-sense density by iterating (6.4). This computation can be simplified considerably. If the intervals between observations are constant and the state space does not change, then one can precompute the multi-step transition probability between those times. In general, one would not iterate at each step, but use increasing powers of the one-step transition probability. For $\Delta t^h(x) = \bar{\Delta}^h$, and Δ the interval between observations, the complexity is less than $N^3 \log(\Delta/\bar{\Delta}^h)$, and depends on the sparseness of the matrices. If the state space is too large or changes in time as the system evolves, then for practical algorithms, one must control its size. The best approach is heavily dependent on the problem. The dominant effects might be diffusion or a strong drift, or the observations might skew the shape significantly. One looks for regions with good geometry (for convenience in programming) and whose conditional probability is close to unity. For example, one might use a 3σ ellipse centered at the conditional mean, or an analogous rectangle.

Comments. If the same system is to be used frequently, then one is tempted to optimize the grid that defines the state space of the approximating chain. Some such results are in [35], which were derived for the control problem. The various functions $b(\cdot), \sigma(\cdot), g(\cdot)$ need not be continuous, provided that the time (on any finite interval) that the limit process spends in an ϵ -neighborhood of the discontinuities goes to zero as $\epsilon \rightarrow 0$ [31, pages 275, 295]. The robustness of the behavior of the approximate filters over a very long time interval to uncertainties in the dynamics, signal, and observation noise processes is of interest, and is shown to be the case in [5, 6, 7].

6.2 Alternative Filter Approximations

6.2.1 Using The Process $\hat{\xi}_n^{h,\delta}$

Recall the process $\{\hat{\xi}_n^{h,\delta}\}$ with interpolation $\hat{\xi}^{h,\delta}(\cdot)$ (intervals δ) defined below (5.8). Let $\delta \geq \max_x \Delta t^h(x)$. Then $\hat{\xi}^{h,\delta}(\cdot)$ can be used in (4.1) and (4.2) whether or not $\Delta t^h(x)$ is constant. When $\xi_n^h = x$, the probability of an advance in the time variable is just $\Delta t^h(x)/\delta$, with the mean increment being $\Delta t^h(x)$. The main computational problem is getting the one-step transition probability $P^{h,\delta}(x, \tilde{x})$ of $\{\hat{\xi}_n^{h,\delta}\}$.

Define the matrix $P_1^{h,\delta} = \{p^h(x, \tilde{x})\Delta t^h(x)/\delta; x, \tilde{x}\}$, the matrix of probabilities of going from x to \tilde{x} in one step and with time advancing. Define the matrix $P_0^{h,\delta} = \{p^h(x, \tilde{x})(1 - \Delta t^h(x)/\delta); x, \tilde{x}\}$, the set of probabilities of going from x to \tilde{x} in one step, with time not advancing. Then the transition probability for the chain $\{\hat{\xi}_n^{h,\delta}\}$ is

$$P^{h,\delta} = \sum_{n=0}^{\infty} [P_0^{h,\delta}]^n P_1^{h,\delta} = [I - P_0^{h,\delta}]^{-1} P_1^{h,\delta}. \quad (6.6)$$

If $\delta = \max_x \Delta t^h(x)$, then the number of steps that are required until the jump in the time variable will be small and the sum will converge rapidly. If this form is used for continuous-time observations, where the observations are taken to be either (2.3a) or (2.3b), with $V = \mathbb{I}$, then Δ would be a small multiple of δ .

Suppose that the form is used for discrete-time observations with interval Δ between observations, where Δ/δ is a large integer. Then the transition probability between observations is $[P^{h,\delta}]^{\Delta/\delta}$. The times at which the observations are incorporated are random, after each successive Δ/δ increases in the time variable. By Theorem 5.1, as $h \rightarrow 0$ and $\delta \rightarrow 0$, these intervals converge to the constant Δ . A computational advantage of this procedure is that the original intervals $\Delta t^h(x)$ can be used. The computation of (6.6) should be helped by the tendency (in many problems) of the path to move from points with small $\Delta t^h(x)$ (fast dynamics or large diffusion) to those with a larger interval (small dynamics and small diffusion).

6.2.2 Discrete Time and Other Forms

The general approach covers a large family of approximations, but for numerical purposes we need to restrict the domain of any approximation to a finite set of points. For example, one could base the Markov chain approximation on (4.11), and we describe one possible approach to getting the

desired transition probabilities for the dynamical update step. First, for $x \in G_h$, let $S^h(x)$ be the set of closest points on the grid to $x + \delta b(x)$ in whose convex hull $x + \delta b(x)$ lies. Then randomize among the points in $S^h(x)$, so that the correct mean increment $\delta b(x)$ is attained. The assigned weights are the transition probabilities for this first step. Now, add the noise. For each $z \in S^h(x)$, construct an approximation that is locally consistent with $\sigma(z)[w(n\delta + \delta) - w(n\delta)]$. This two-step process yields the transition probabilities for the point x . A two-step procedure was described for simplicity of explanation. If it is not too difficult, it would be preferable to approximate the transitions in one step. If the observations are taken continuously, then the form based on (4.11) can still be used, where the observations are incorporated at intervals that go to zero as $\delta \rightarrow 0$.

The simplest Markov chain approximation is that where the transitions are local, such as that given by (5.2), which were designed for the control problem. If $\Delta t^h(\cdot)$ is the constant $\bar{\Delta}^h$, then iterating the one-step transition probability $\delta/\bar{\Delta}^h$ times yields an approximation to that of (4.11). A more direct use of (4.11) could simplify the computations, depending on how it is carried out. One would try to precompute as much as possible, keeping in mind the tradeoff between accuracy of the approximation and the computational and coding requirements. The use of smaller values of δ and h would yield a better approximation to the solution of the Kolmogorov forward equation, but owing to the “corrective” effects of the observations, the filtering algorithms are often forgiving of the use of cruder models for the dynamics.

7 Robustness of the Approximating Filters

The mathematical theory of nonlinear filtering with continuous-time observations depends on the fact that the observation noise is “white-Gaussian.” Even if the physical observation noise is a wide-bandwidth process, due to its simplicity and to the fact that the probability law of the true observation noise process might not be known, one is tempted to use a form of the filter that is derived under the white noise assumption. Such a filter is appealing since it yields a reasonable algorithm, even if it is not optimal. The interpretation of the result and whether “correction” terms are needed depends on the type of approximation that is used, and many of the important issues are discussed in [29, 32]. The form (6.5) is continuous in $y(\cdot)$ (uniformly in each bounded set, in the sup norm sense on any bounded interval), since for each h , $\xi^h(\cdot)$ is piecewise constant. But the uniformity of the continuity in

h is not a priori evident. To be of genuine value, the continuity should be uniform in h .

Let us return to (2.5) and suppose that the first and second order partial derivatives of $g(\cdot)$ are bounded and continuous. Then, with probability one, (2.5) can be rewritten as

$$\begin{aligned} & \mathbb{E}_t \phi(x(t)) \\ &= \frac{\mathbb{E}_t \phi(\tilde{x}(t)) \exp[y'(t)g(\tilde{x}(t)) - \int_0^t y'(u)dg(\tilde{x}(u)) - \frac{1}{2} \int_0^t |g(\tilde{x}(u))|^2 du]}{\mathbb{E}_t \exp[y'(t)g(\tilde{x}(t)) - \int_0^t y'(u)dg(\tilde{x}(u)) - \frac{1}{2} \int_0^t |g(\tilde{x}(u))|^2 du]}. \end{aligned} \quad (7.1)$$

Clark [11] showed that (7.1) is locally Lipschitz continuous in $y(\cdot)$ at each $y(\cdot) \in C^m[0, T]$, for each $T < \infty$, and gave a PDE whose solution is the right side of (7.1) for any continuous $y(\cdot)$. Since this solution is (2.5) (with probability one) if $y(\cdot)$ were defined by (2.2), we would then have a continuous function of $y(\cdot)$ which can be said to be an approximation to the optimal filter, even if $w_0(\cdot)$ is only an ‘‘approximation’’ to a Wiener process, and the filter might be far from optimal under the actual observation noise. It is of greater interest to know whether the numerical approximations are robust in the observation noise. This will be seen to be true and the continuity will be uniform in the approximation parameter.

Return to the form (4.2), suppose that $g(x^h(\cdot))$ has a well-defined differential, and that that by a partial integration, we can write (4.2) as

$$\begin{aligned} & F_t^h(\phi, y(\cdot)) \equiv E_t^h \phi(x^h(t)) \\ &= \frac{\mathbb{E}_t \phi(x^h(t)) \exp[y'(t)g(x^h(t)) - \int_0^t y'(u)dg(x^h(u)) - \frac{1}{2} \int_0^t |g(x^h(u))|^2 du]}{\mathbb{E}_t \exp[y'(t)g(x^h(t)) - \int_0^t y'(u)dg(x^h(u)) - \frac{1}{2} \int_0^t |g(x^h(u))|^2 du]}. \end{aligned} \quad (7.2)$$

The partial integration can be done if $x^h(\cdot)$ is piecewise constant for each h .

Decompose the bounded point process $g(x^h(\cdot))$ into the sum of a bounded martingale and predictable projection processes as $g(x^h(t)) = M^h(t) + \Gamma^h(t)$. Assume the following conditions, which will be seen to hold for $g(\xi^h(\cdot))$. Suppose that, for each $T < \infty$, there are $C_i(T) < \infty$ such that for all $h > 0$

$$\mathbb{E} \left| M^h(T) \right|^2 \leq C_1(T), \quad \mathbb{E} \left[\int_0^t |d\Gamma^h(s)| \right]^2 \leq C_2(T). \quad (7.3a)$$

Further, suppose that for each bounded set S of functions $q(\cdot) \in C^m[0, T]$, there are $C_i(S, T) < \infty$ such that, for all h and $t \leq T$,

$$\mathbb{E} \exp \left[\int_0^t q'(s) dM^h(s) \right] + \mathbb{E} \exp \left[\int_0^t q'(s) d\Gamma^h(s) \right] \leq C_3(S, T). \quad (7.3b)$$

Instead of (7.3b) we could use, for all h and $t \leq T$,

$$\mathbb{E} \exp \left[\int_0^t q'(s) dg(x^h(s)) \right] \leq C_4(S, T). \quad (7.3c)$$

Define $\|y\|_T = \sup_{t \leq T} |y(t)|$. The theorem implies that the convergence in Theorem 4.1 is w.p.1.

Theorem 7.1. *For each $T > 0$ and bounded set $S \subset C^m[0, T]$ there is $K(S, T) < \infty$ which does not depend on h or $\phi(\cdot)$ (for all $\phi(\cdot)$ bounded by the same constant) such that, for $f(\cdot)$ and $\tilde{f}(\cdot) \in S$ and $t \leq T$,*

$$\left| F_t^h(\phi, f(\cdot)) - F_t^h(\phi, \tilde{f}(\cdot)) \right| \leq K(S, T) \left\| f(\cdot) - \tilde{f}(\cdot) \right\|_T. \quad (7.4)$$

Proof. We only need to show that there is $K_1(S, T) < \infty$, depending only on S and T , such that, for all $h > 0$ and $t \leq T$,

$$\begin{aligned} & \mathbb{E} \left| \exp \left[f'(t)g(x^h(t)) - \int_0^t f'(u)dg(x^h(u)) - \frac{1}{2} \int_0^t |g(x^h(u))|^2 du \right] \right. \\ & \quad \left. - \exp \left[\tilde{f}'(t)g(x^h(t)) - \int_0^t \tilde{f}'(u)dg(x^h(u)) - \frac{1}{2} \int_0^t |g(x^h(u))|^2 du \right] \right| \\ & \leq K_1(S, T) \left\| f(\cdot) - \tilde{f}(\cdot) \right\|_T. \end{aligned} \quad (7.5)$$

We can drop the $-\frac{1}{2} \int_0^t |g(x^h(u))|^2 du$ term, since $g(\cdot)$ is bounded. Then, using the inequality (4.5), we have the following upper bound for (7.5):

$$\begin{aligned} & \mathbb{E} \left| \left[f(t) - \tilde{f}(t) \right]' g(x^h(t)) - \int_0^t \left[f(u) - \tilde{f}(u) \right]' dg(x^h(u)) \right| \\ & \quad \times \left[e^{\left[f'(t)g(x^h(t)) - \int_0^t f'(u)dg(x^h(u)) \right]} + e^{\left[\tilde{f}'(t)g(x^h(t)) - \int_0^t \tilde{f}'(u)dg(x^h(u)) \right]} \right]. \end{aligned}$$

To prove (7.5) using the above bound, we need only show (7.6) and (7.7), where h is small and $q(\cdot) \in S'$, an arbitrary bounded set in $C^m[0, T]$:

$$\sup_{t \leq T} \mathbb{E} \left| \int_0^t q'(u) dg(x^h(u)) \right|^2 \leq K_2(T) \|q(\cdot)\|_T^2, \quad K_2(T) < \infty, \quad (7.6)$$

$$\sup_{q(\cdot) \in S'} \sup_{t \leq T} \mathbb{E} \exp \left[\int_0^t q'(u) dg(x^h(u)) \right] \leq K_3(T, S') < \infty. \quad (7.7)$$

It is sufficient to let $q(\cdot)$ be real-valued. (7.7) follows from either (7.3b) or (7.3c). The inequality (7.6) will be verified by using

$$\mathbb{E} \left| \int_0^t q(u) dg(x^h(u)) \right|^2 \leq 2 \|q(\cdot)\|_T^2 \mathbb{E} \left[\int_0^T |d\Gamma^h(u)| \right]^2 + 2 \mathbb{E} \left| \int_0^t q(u) dM^h(u) \right|^2. \quad (7.8)$$

By the martingale calculus [18, 22], the second term on the right is bounded by $8 \|q(\cdot)\|_T^2 \mathbb{E}[M^h(T)]^2$. Then use (7.3a) to complete the proof. ■

Verification of the conditions for the Markov chain approximations. Theorem 7.1 holds for all of the forms that were discussed in the previous section. We will work with $\xi^h(\cdot)$ with (5.1) holding and where $\Delta t^h(x) = \bar{\Delta}^h$, a constant, which will be seen to assure (7.3). For simplicity, let $g(\cdot)$ be real-valued. The proof in the other cases is similar, including the case with jumps. Suppose that $g(\cdot)$ is bounded and continuous, together with its partial derivatives up to second order.

The process $\xi^h(\cdot)$ is right-continuous and we have the decomposition into a martingale and predictable projection: $g(\xi^h(n\bar{\Delta}^h)) = M^h(n\bar{\Delta}^h) + \Gamma^h(n\bar{\Delta}^h)$. By the definition of the predictable projection,

$$\Gamma^h(n\bar{\Delta}^h + \bar{\Delta}^h) - \Gamma^h(n\bar{\Delta}^h) = \sum_{\tilde{x}} \left[g(\tilde{x}) - g(\xi_n^h) \right] p^h(\xi_n^h, \tilde{x}). \quad (7.9)$$

A Taylor series expansion and (5.1) shows that (7.9) is $O(\bar{\Delta}^h)$. Hence (7.3a) holds for the predictable projection. We have

$$\begin{aligned} & M^h(n\bar{\Delta}^h + \bar{\Delta}^h) - M^h(n\bar{\Delta}^h) \\ &= \left[g(\xi_{n+1}^h) - g(\xi_n^h) \right] - \sum_{\tilde{x}} \left[g(\tilde{x}) - g(\xi_n^h) \right] p^h(\xi_n^h, \tilde{x}). \end{aligned} \quad (7.10)$$

That (7.3a) holds for the martingale term follows from this form and (5.1).

To prove (7.3c), it is sufficient to work with real-valued $q(\cdot)$ and

$$\mathbb{E} \prod_{n=0}^{\lceil t/\bar{\Delta}^h \rceil - 1} e^{q(n\bar{\Delta}^h)[g(\xi_{n+1}^h) - g(\xi_n^h)]}. \quad (7.11)$$

Let \mathbb{E}_n^h = expectation given the data up to and including step n . We have

$$\begin{aligned} & \mathbb{E}_n^h e^{q(n\bar{\Delta}^h)[g(\xi_{n+1}^h) - g(\xi_n^h)]} \leq 1 + q(n\bar{\Delta}^h) \mathbb{E}_n^h [g(\xi_{n+1}^h) - g(\xi_n^h)] \\ & + q^2(n\bar{\Delta}^h) \mathbb{E}_n^h [g(\xi_{n+1}^h) - g(\xi_n^h)]^2 \sum_{l=2}^{\infty} \frac{[2\|q(\cdot)\|_T \|g(\cdot)\|]^l}{l!}. \end{aligned} \quad (7.12)$$

A second-order Taylor expansion of $[g(\xi_{n+1}^h) - g(\xi_n^h)]$ and (5.1) yields that (7.12) is bounded by $e^{\bar{\Delta}^h C}$ for some constant C which is bounded on each bounded $q(\cdot)$ -set. Using this estimate recursively in (7.11) yields (7.3c).

8 Assumed Form of the Conditional Density

In this and in the next section, a very different approach to the approximate nonlinear filter is taken. We are no longer interested in convergent approximations to the conditional density. Interest is confined to heuristic approximations of the first few conditional moments only. This is done by assuming a parametrized form for the conditional density and, with this form, evaluating the evolution of the parameters. Most typically one supposes that the conditional density is Gaussian, and under this assumption the equations of evolution of the conditional means and covariances are obtained. The approach originated in [25] and has been in common use since then.⁵ See, for example, the references in [1, 8, 20]. The procedure is intuitively reasonable, although here is no mathematical justification for it. But numerous examples have shown that it can be a powerful tool for obtaining good nonlinear filters.

Observations taken continuously in time. Let us first consider the model (2.1), (2.2), so that we are observing continuously in time, and wish to estimate the moments continuously in time. Define $m_i(t) = \mathbb{E}_t x_i(t)$, $m_{ij}(t) = \mathbb{E}_t(x_i(t) - m_i(t))(x_j(t) - m_j(t))$, and let $m(\cdot)$ and $M(\cdot)$ denote the vector and matrix, resp, of the components. Equation (2.7) yields

$$\begin{aligned} dm_i(t) &= \mathbb{E}_t b_i(x(t)) dt \\ &+ (\mathbb{E}_t x_i(t) g(x(t)) - \mathbb{E}_t g(x(t)) m_i(t))' (dy(t) - \mathbb{E}_t g(x(t)) dt). \end{aligned} \quad (8.1)$$

Letting \mathcal{L} now denote the differential operator of the pair $(x(\cdot), m(\cdot))$, the equations for the $m_{ij}(\cdot)$ are

$$\begin{aligned} dm_{ij}(t) &= \mathbb{E}_t \mathcal{L}(x_i(t) - m_i(t))(x_j(t) - m_j(t)) dt \\ &+ (G_{ij}^0(t) - \mathbb{E}_t g(x(t)) m_{ij}(t))' (dy(t) - \mathbb{E}_t g(x(t)) dt), \end{aligned} \quad (8.2)$$

where $G_{ij}^0(t) = \mathbb{E}_t(x_i(t) - m_i(t))(x_j(t) - m_j(t))g(x(t))$.

⁵The main motivations for the use of Gaussian approximating densities are the relative numerical simplicity, that it captures the main effects in many problems, the guidance supplied by the linear problem, and that it “works” in applications.

An assumed Gaussian form for the conditional density. The (conditional expectations in (8.1) and (8.2) are with respect to the true conditional distribution, which we do not know and the computation of which we wish to avoid. So we make the purely heuristic assumption that it has a particular parametrized form, and then estimate the parameters under this assumption. In principle any density can be used, provided that it is determined by a finite number of parameters. The simplest and most widely used form is that of a Gaussian distribution, even though it is not in general the form of the true conditional density,

Let $N(m, M)(\cdot)$ denote the Gaussian density with mean m and covariance M . Define the operator E_t by $E_t f(x(t)) = \int f(x)N(m(t), M(t))(x)dx$. Then (8.1) and (8.2) are replaced by the following heuristic approximations:

$$dm_i(t) = E_t b_i(x(t))dt + (G_i(t) - G(t)m_i(t))' (dy(t) - G(t)dt), \quad (8.3)$$

where

$$G(t) = E_t g(x(t)), \quad G_i(t) = E_t x_i(t)g(x(t)). \quad (8.4)$$

Recalling that \mathcal{L} now denotes the differential operator of the pair $(x(\cdot), m(\cdot))$, the approximating equations for the $m_{ij}(\cdot)$ are

$$dm_{ij}(t) = E_t \mathcal{L}(x_i(t) - m_i(t))(x_j(t) - m_j(t))dt + (G_{ij}(t) - G(t)m_{ij}(t))' (dy(t) - G(t)dt). \quad (8.5)$$

$$G_{ij}(t) = E_t (x_i(t) - m_i(t))(x_j(t) - m_j(t))g(x(t)). \quad (8.6)$$

An alternative to the use of the Gaussian distribution is to assume that the conditional density has the form of a Gaussian mixture of fixed finite order such as

$$\sum_{i=1}^K \alpha_i \frac{1}{(2\pi)^{r/2} \det^{1/2} M_i} \exp[-(x - m_i)' M_i^{-1} (x - m_i)], \quad \sum_{i=1}^K \alpha_i = 1, \alpha_i \geq 0.$$

Updating the $\{m_i, M_i\}$ is straightforward since one works with each component at a time. Updating the weights $\{\alpha_i\}$ is more complicated. In principle it would seem preferable to use such an assumed density that allows for greater flexibility, although at this time the class of problems for which the greater numerical complexity is justified by clearly better results is not clear. The Gaussian mixture form is discussed in [1, 20] and in some of their references, where methods for updating the weights are given. The reader is referred to these references for further information.

Example where an exact integration is possible. The integrals in (8.3)–(8.6) need to be evaluated. This can sometimes be done exactly, without approximation; for example, where $b(\cdot), a(\cdot)$ and $g(\cdot)$ are polynomials. This was the case for the example in [25], where the signal process was defined by the noiseless Van der Pol equation:

$$dx_1 = x_2 dt, \quad dx_2 = -x_1 + \epsilon x_2(1 - x_1^2), \quad \epsilon > 0,$$

$$dy(t) = x_1(t)dt + dw_0(t).$$

The example was selected since the dynamics have both fast and slow parts and provided a good illustration of the quality of the method. Since the odd central moments of a Gaussian distribution are zero, we have

$$G(t) = m_1(t), \quad G_j(t) = m_{1j}(t) + m_1(t)m_j(t), \quad G_{ij}(t) = m_1(t)m_{ij}(t),$$

$$E_t \mathcal{L}x_1(t) = b_1(x(t)) = m_2(t),$$

$$E_t \mathcal{L}x_2(t) = E_t b_2(x(t)) = -m_1(t) + \epsilon m_2(t) - \epsilon [2m_1(t)m_{12}(t) + m_2(t)(m_{11}(t) + m_1^2(t))].$$

The filter is

$$\begin{aligned} dm_1 &= m_2 dt + m_{11}(dy - m_1 dt), \\ dm_2 &= E_t b_2(x(t))dt + m_{12}(dy - m_1 dt), \\ \dot{m}_{11} &= -m_{11}^2 + 2m_{12}, \\ \dot{m}_{12} &= -m_{11}m_{12} + m_{22} - m_{11} - \epsilon(-m_{12} + m_{11}m_{12} + m_{12}m_1^2 + 2m_{11}m_1m_2), \\ \dot{m}_{22} &= -m_{12}^2 - 2m_{12} - 2\epsilon(-m_{22} + m_{11}m_{22} + m_1^2m_{22} + 2m_1m_2m_{12}). \end{aligned}$$

By the Gaussian assumption, the higher moments are

$$m_{1122}(t) = m_{11}(t)m_{22}(t) + 2m_{12}(t), \quad m_{1112}(t) = 3m_{11}(t)m_{12}(t).$$

In this example, the observation terms in the equations for the second moments turn out to be zero, due the fact that the third central moments are zero (due to the Gaussian assumption) and $g(x) = x_1$.

The differential equations for the approximations would have to be solved numerically. When there is an observation term, as in the equations for the $m_i(\cdot)$, one would have to use one of the pathwise approximation methods of [21]. Generally we can use the simplest one, the discrete-time (Euler) scheme. Numerical data is given in [25], where it is seen that the tracking is good. For such examples, the extended Kalman filter performs very poorly.

Observations at discrete times. If the observations are taken at times $t_i, i = 1, 2, \dots$, then between the observations use (8.3)–(8.6) with the observation term deleted. Methods of incorporating the observation will be discussed below.

Numerical evaluation of the integrals with respect to the Gaussian kernels. The integrals (i.e., the moments) in the above example could be evaluated exactly since the dynamical and signal functions were all polynomials. If an exact evaluation is not possible, then some form of numerical quadrature is required. Owing to the Gaussian assumption the most natural form is based on the Gauss-Hermite quadrature formulas, and a brief review of this method will be given next.

Numerical quadrature formulas. The theory of Gauss-Hermite quadrature is concerned with the numerical approximation of integrals of real-valued functions of a real variable with respect to a Gaussian kernel of the type

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-x^2/2} dx = \sum_{i=1}^{\mu} w_i f(t_i) + \text{error}. \quad (8.7)$$

For each order μ , the points $\{t_i, i \leq \mu\}$ and weights $\{w_i, i \leq \mu\}$ are chosen such that the integral is exact if $f(\cdot)$ is any polynomial of degree $\leq 2\mu - 1$. The points $\{t_i\}$ are symmetric about the origin and the weights satisfy $w_i \geq 0, \sum_i w_i = 1$. They are equal for t and $-t$.

In lieu of (8.7), the numerical analysis literature uses the form

$$\int_{-\infty}^{\infty} f(z)e^{-z^2} dz \approx \sum_{i=1}^{\mu} f(z_i)w'_i, \quad (8.8)$$

in which case the z_i are the zeros of the μ th order Hermite polynomial $H_{\mu}(\cdot)$ (with leading coefficient 2^{μ}) [12] and the weights are

$$w'_i = 2^{\mu+1} \mu! \sqrt{\pi} / [H_{\mu+1}(z_i)]^2.$$

With the form (8.8) and $\mu = 3$, the z_i and w'_i are [33, page 508]

$$\{(z_i, w'_i)\} = \{(0, 1.1816), (\pm 1.2247, .2954)\}.$$

For $\mu = 5$ they are

$$\{(z_i, w'_i)\} = \{(0, .9453), (\pm .9586, .3936), (\pm 2.02, .01954)\}.$$

Furthermore, if $f(\cdot)$ is a function with continuous derivatives up to order 2μ , then the error for the form (8.8) is bounded by

$$\frac{\mu! \sqrt{\pi}}{2^\mu (2\mu)!} f^{2\mu}(\xi), \quad \text{some } \xi \in (-\infty, \infty). \quad (8.9)$$

To adapt the published formulas and values to the form of the Gaussian kernel that is used in (8.7), we need do a linear transformation to get $t_i = \sqrt{2}z_i, w_i = w'_i/\sqrt{\pi}$. In what follows, (8.7) will be used to approximate integrals of functions with respect to Gaussian kernels.

Now suppose that x is vector-valued, of dimension $r > 1$. Then the simplest procedure is to use the product formula

$$\frac{1}{(2\pi)^{r/2}} \int_{\mathbb{R}^r} f(x) e^{-|x|^2/2} dx \approx \sum_{i_1=1}^{\mu} \dots \sum_{i_r=1}^{\mu} w_{i_1} \dots w_{i_r} f(t_{i_1}, \dots, t_{i_r}). \quad (8.10)$$

The formula is exact for polynomials that are of order at most $2\mu - 1$ in each variable. An error bound can be obtained from (8.9). More efficient approximations for functions that are well represented (in the region of most mass of the kernel function) by polynomials whose *total* order is $\leq 2\mu - 1$ and that are not based on product rules can be found in [37]. The product rule (8.10) will suffice for our purposes.

Now, for Σ positive definite and symmetric, consider the integral

$$\frac{1}{(2\pi)^{r/2} (\det \Sigma)^{1/2}} \int f(x) e^{-(x-\bar{x})' \Sigma^{-1} (x-\bar{x})/2} dx \quad (8.11)$$

Factor $\Sigma^{-1} = S' S$ and define $z = S(x - \bar{x})$, yielding the equivalent form

$$\frac{1}{(2\pi)^{r/2}} \int f(S^{-1}z + \bar{x}) e^{-|z|^2/2} dz \approx \sum_{i_1, \dots, i_r} w_{i_1} \dots w_{i_r} f(S^{-1}(t_{i_1}, \dots, t_{i_r}) + \bar{x}). \quad (8.12)$$

The choice of factorization method depends on the problem, with accuracy vs computational time being the main considerations. See [3] for a discussion of various factorization methods.

Evaluating the integrals in (8.3)-(8.6). Suppose that the observations are taken at the discrete times $t_i, i = 1, \dots$. Then, if an exact evaluation is not possible, for a discretization interval δ , between observations one could use the discrete-time approximation⁶

$$E_{l\delta} \phi(x(l\delta + \delta)) = E_{l\delta} \phi(x(l\delta)) + \delta E_{l\delta} \mathcal{L} \phi(x(l\delta)). \quad (8.13)$$

⁶This is the simplest method for approximating the solution of a differential equation. Higher order methods could be used whenever the results are worth the extra computational requirements.

If the observations are taken in continuous time, then one could partition the updating procedure by first updating the effects of the dynamics as in (8.13), and then incorporating the observation (2.3a) by one of the methods in the next section.

The general approach for updating the approximations of the moments between observations is the following. The observations taken at times $t_i, i = 1, 2, \dots$, where the t_i are assumed to be integral multiples of $\delta > 0$ and have the form

$$y_n = g(x(t_n)) + v_n \quad (8.14)$$

where the $\{v_n\}$ are iid, normal, with covariance \mathbb{I} and independent of $x(\cdot)$. Between observations, update the means and covariances by using

$$m_{l\delta+\delta} = m_{l\delta} + \delta \int b(x)\mathbb{N}(m_{l\delta}, M_{l\delta})(x)dx, \quad (8.15)$$

$$\begin{aligned} M_{l\delta+\delta} = & \int [x + b(x)\delta - m_{l\delta+\delta}][x + b(x)\delta - m_{l\delta+\delta}]'\mathbb{N}(m_{l\delta+\delta}, M_{l\delta})(x)dx \\ & + \delta \int a(x)\mathbb{N}(m_{l\delta+\delta}, M_{l\delta})(x)dx, \end{aligned} \quad (8.16)$$

with the Gauss-Hermite quadrature rule (8.12) used if the integrals cannot be evaluated exactly. This is the form used in [25] and [20] and is algebraically equivalent to the form in [1]. If some of the components of $b(\cdot)$ or $\sigma(\cdot)$ are either constant or linear in x , then the integrals can be partly precomputed.

9 The Observation Step

From a numerical perspective, incorporating the observation is more difficult than the updating between observations. Two approaches will be discussed. The first uses quadrature rules directly to evaluate the Bayes' rule formula, and the second is based on a type of local linear approximation, with a least squares estimate of the coefficients. W.l.o.g, and for notational simplicity, we continue to suppose that the covariance matrix of the observation noise is the identity. Let m_n^-, M_n^- denote the values of the estimates of the conditional mean and covariance just before the n th observation y_n is taken, where y_n is defined by (8.14). Define the (unnormalized) Gaussian density function

$$I_n(x) = \exp[-|y_n - g(x)|^2/2]. \quad (9.1)$$

Given the new observation y_n , we would like to compute the updated values of the approximations to the mean and covariance, which we write as

$$\begin{aligned} m_n &= \frac{1}{c_n} \int x I_n(x) \mathbb{N}(m_n^-, M_n^-)(x) dx, \\ M_n &= \frac{1}{c_n} \int (x - m_n)(x - m_n)' I_n(x) \mathbb{N}(m_n^-, M_n^-)(x) dx, \end{aligned} \quad (9.2)$$

where c_n is the normalizing constant

$$c_n = \int I_n(x) \mathbb{N}(m_n^-, M_n^-) dx.$$

Approach 1. An iterative centering method for the Gaussian quadrature. A potential difficulty arises when the center point of the kernel $I_n(x)$ is not close to m_n^- . For example, consider a two-dimensional case where $g(x)$ has the form $g_0(x_1 - x_2)$, and which is illustrated in Figure 1. The upper ellipsoidal contours are those of $\mathbb{N}(m_n^-, M_n^-)(\cdot)$, the lower ellipsoid lines approximate the contours of $I_n(\cdot)$, so that there is not much “overlap” in the parts of greatest mass of the two densities. When applied directly, the numerical quadrature formulas might give poor results unless the degree μ is large. The problem can be partially alleviated by the following iterative centering procedure which attempts to alter the kernels to improve the centering, and which worked very well in the examples in [8].

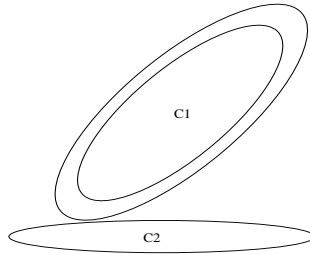


Figure 1: An example of poor alignment of the prior and observation kernels.

We seek a good centering kernel $\mathbb{N}(\rho_n, P_n)(\cdot)$, with which we rewrite (9.2) as

$$\int \left[\frac{I_n(x) f(x) \mathbb{N}(m_n^-, M_n^-)(x)}{\mathbb{N}(\rho_n, P_n)(x)} \right] \mathbb{N}(\rho_n, P_n)(x) dx, \quad (9.3)$$

for appropriate $f(\cdot)$. The form (9.3) will be used to evaluate (9.2). In (9.3) the term in brackets is the function whose integral with respect to the kernel $\mathbb{N}(\rho_n, P_n)(x)$ is to be evaluated. Since the integral will not be evaluated

exactly, but the quadrature rules will be used, the two $\mathbb{N}(\rho_n, P_n)(x)$ will not necessarily cancel each other. The analogous remark holds for the forms (9.4) below.

The values of the ρ_n, P_n in the centering kernel will be obtained by an iterative procedure. Start by defining $c_{n,0} = c_n, \rho_{n,0} = m_n^-, P_{n,0} = M_n^-$. and let $c_{n,k}, \rho_{n,k}, P_{n,k}$ denote the k th estimates, obtained from

$$c_{n,k} = \int \left[\frac{I_n(x) \mathbb{N}(m_n^-, M_n^-)(x)}{\mathbb{N}(\rho_{n,k-1}, P_{n,k-1})(x)} \right] \mathbb{N}(\rho_{n,k-1}, P_{n,k-1})(x) dx, \quad (9.4a)$$

$$\rho_{n,k} = \int \left[\frac{I_n(x) x \mathbb{N}(m_n^-, M_n^-)(x)}{\mathbb{N}(\rho_{n,k-1}, P_{n,k-1})(x)} \right] \mathbb{N}(\rho_{n,k-1}, P_{n,k-1})(x) dx, \quad (9.4b)$$

$$P_{n,k} = \int \left[\frac{I_n(x) (x - \rho_{n,k})(x - \rho_{n,k})' \mathbb{N}(m_n^-, M_n^-)(x)}{\mathbb{N}(\rho_{n,k-1}, P_{n,k-1})(x)} \right] \mathbb{N}(\rho_{n,k-1}, P_{n,k-1})(x) dx. \quad (9.4c)$$

The integrals in (9.4) are evaluated using the Gauss-Hermite quadrature rules. First center and diagonalize the kernel $\mathbb{N}(\rho_{n,k-1}, P_{n,k-1})(x)$ by the transformation $x = [P_{n,k-1}]^{1/2} t + \bar{\rho}_{n,k-1}$ to get, for example,

$$c_{n,k} = \int \left[\frac{I_n(P_{n,k-1}^{1/2} t + \bar{\rho}_{n,k-1}) \mathbb{N}(m_n^-, M_n^-)(P_{n,k-1}^{1/2} t + \bar{\rho}_{n,k-1})}{\mathbb{N}(0, I)(t)} \right] \mathbb{N}(0, I)(t) dt,$$

and so forth, and then use (8.10). For the quadrature, use order μ_0 , which might not be μ . The value of μ_0 depends on the degree to which the kernels can be “aligned.” In the four-dimensional example in Section 10, the integrations were reduced to ones in two dimensions using the precomputation method of the next paragraph, and an 8×8 grid could not be much improved on. The iterative method seems to be very stable and can improve the estimates considerably. But it is a heuristic procedure. In all cases in [8], the improvement was negligible after 4 iterations.

Simplification by precomputation. The function $g(\cdot)$ generally depends on only a few of the the state components. This can be used to reduce the dimension of the space in which the numerical integration is done, via analytic computations of conditional expectations of Gaussian random variables. To illustrate the idea, suppose that $g(\cdot)$ depends on the first $l < r$ components of x . Then $I_n(x)$ only depends on the first l components of x , and the integrations in (9.2) can be reduced to integrations over \mathbb{R}^l , which simplifies the remaining computation via Gauss-Hermite quadrature.

In more detail, let $X = (X_1, \dots, X_r) \in \mathbb{R}^r$ have the distribution $\mathcal{N}(\mu, P)(\cdot)$, and let z be the generic point in \mathbb{R}^l . Consider the integral

$$\int_{\mathbb{R}^r} x_{l+1} I_n(x) \mathcal{N}(\mu, P)(x) dx.$$

By taking conditional expectations it can be rewritten as

$$\begin{aligned} E [I_n(X_1, \dots, X_l) X_{l+1}] &= E [I_n(X_1, \dots, X_l) E (X_{l+1} | X_1, \dots, X_l)] \\ &= E I_n(X_1, \dots, X_l) [a_1 X_1 + \dots + a_l X_l] \\ &= \int_{\mathbb{R}^l} I_n(z) (a_1 z_1 + \dots + a_l z_l) \mathcal{N}(\mu_{n,r}, P_{n,r})(z) dz, \end{aligned} \tag{9.5}$$

where $\mathcal{N}(\mu_{n,r}, P_{n,r})(\cdot)$ is the marginal distribution of (X_1, \dots, X_l) . The value $(a_1 z_1 + \dots + a_l z_l)$ is the conditional expectation of X_{l+1} given $(X_1, \dots, X_l) = (z_1, \dots, z_l)$, and a_1, \dots, a_r are functions of (μ_n, P_n) . In a similar manner, all the integrals involved in (9.2) can be easily rewritten as integrals of functions of x_1, \dots, x_l ,

Approach 2. Least squares approximations. The approach taken by [1] yields the same formulas as that of [20], although they are derived slightly differently. The approximation of the updated mean and covariance is based on the minimization of linear least squares errors, given the new observation. Let us start by recalling a result for a linear and Gaussian problem. Define $y = Hx + v$, where v and x are mutually independent and normally distributed, with v having mean zero and nondegenerate covariance R , and x having mean ρ^- and covariance $P^- = P_{xx}$. Define the mean $\bar{y} = Ey = H\rho^-$. Define $\rho = \mathbb{E}[x|y]$, $P = \text{cov}[x|y]$ and the covariances $P_{yy} = \text{cov}(y) = HP_{xx}H' + R$, $P_{xy} = \text{cov}(x, y) = E(x - \rho^-)(y - \bar{y})' = P_{xy}H'$. Then

$$\begin{aligned} \rho &= \rho^- + W(y - \bar{y}), \\ W &= P_{xy} [P_{yy}]^{-1}, \\ P &= P_{xx} - P_{xy} [P_{yy}]^{-1} P'_{xy}. \end{aligned} \tag{9.6}$$

This is also a minimal linear least squares estimator, given the initial mean and covariance.

Now return to the nonlinear filtering problem and, since we are only interested in the final formulas, consider the procedure of [20]. Let the n th observation be $y_n = g(x_n) + v_n$, where $x_n = x(t_n)$, be taken at time t_n , and (before the observation is taken) let x_n have conditional mean m_n^- and covariance M_n^- . For notational simplicity and consistency, let v_n have covariance \mathbb{I} . The procedure for incorporating the observation works as

follows. Write $g_n = g(x_n)$. First approximate the conditional mean (given the data to just before the n th observation is taken) of g_n by the Gaussian approximation

$$\bar{g}_n = \int g(x) \mathbb{N}(m_n^-, M_n^-)(x) dx. \quad (9.7)$$

Then approximate the covariance of g_n by

$$P_{g_n, g_n} = \int (g(x) - \bar{g}_n)(g(x) - \bar{g}_n)' \mathbb{N}(m_n^-, M_n^-)(x) dx, \quad (9.8)$$

and the conditional cross covariance between x_n and \bar{g}_n as:

$$P_{x_n, g_n} = \int (x - m_n^-)(g(x) - \bar{g}_n)' \mathbb{N}(m_n^-, M_n^-)(x) dx. \quad (9.9)$$

Use the Gauss-Hermite numerical integration formula if necessary. Now use the form (9.6) to update the conditional mean and covariance as:

$$\begin{aligned} m_n &= m_n^- + W_n(y_n - \bar{g}_n), \\ W_n &= P_{x_n, g_n} [P_{g_n, g_n} + \mathbb{I}]^{-1}, \\ M_n &= M_n^- - P_{x_n, g_n} [P_{g_n, g_n} + \mathbb{I}]^{-1} P_{x_n, g_n}'. \end{aligned} \quad (9.10)$$

Now, continue by computing the estimate m_{n+1}^-, M_{n+1}^- , incorporate y_{n+1} , etc. See the references for numerical examples.

10 A Numerical Example

An example and data from [8] will be given to illustrate Approach 1. The data is for the four-dimensional model

$$\begin{aligned} dx_1(t) &= x_3(t)dt + edw_1(t), & dx_2(t) &= x_4(t)dt + edw_2(t), \\ dx_3(t) &= f_1(x(t))dt + edw_3(t), & dx_4(t) &= f_2(x(t))dt + edw_4(t), \end{aligned}$$

where the $\{w_i\}$ are mutually independent Wiener processes and

$$f_i(x) = \frac{-50x_i}{\sqrt{x_1^2 + x_2^2}} I_{\{\sqrt{x_1^2 + x_2^2} \geq 9\}} I_{\{x_1 x_3 + x_2 x_4 \geq 0\}}.$$

The model is supposed to represent a ship that is confined to move in a constrained area. The position vector (x_1, x_2) and the associated velocities are (x_3, x_4) , resp. Whenever the ship's position exceeds a radius of nine from the center (gets close to a shoreline), there is a large force (acceleration)

that turns it away from the shore. The observations were taken at intervals $\Delta = .05$. The role of the indicator function $I_{\{x_1x_3+x_2x_4\geq 0\}}$ is to assure that the corrective force is applied only when the distance is increasing, since (in the absence of noise) $x_1x_3 + x_2x_4 = d[x_1^2 + x_2^2]/dt$.

First, suppose the “bearing only ” observation, a standard test case, where

$$y_{1,n} = \arctan [x_2(n\Delta)/x_1(n\Delta)] + sv_{1,n},$$

where the $v_{1,n}$ are mutually independent, normally distributed, with mean zero and variance unity, and independent of $x(\cdot)$. The values $e = .32, s = .4$ are used in the plotted data. Since the angles are measured in radians, the noise is quite large.

Since the observations depend only on the two position coordinates, the computation for the incorporation of the observation could be reduced to one in two dimensions. The procedure of Approach 1 was used, with four iterations and an 8×8 grid used for the numerical quadrature. The sample path $(x_1(\cdot), x_2(\cdot))$ and the estimates $(m_1(\cdot), m_2(\cdot))$ for the position coordinates for a typical run are plotted in Figure 2. The initial point is in the lower left hand quadrant and the path is that for 30 units of time. The tracking is very good. The average sample values of the mean square path errors were close to the values of the mean of the $M_{ii}(\cdot), i = 1, 2$, a further indicator of the quality of the approach.

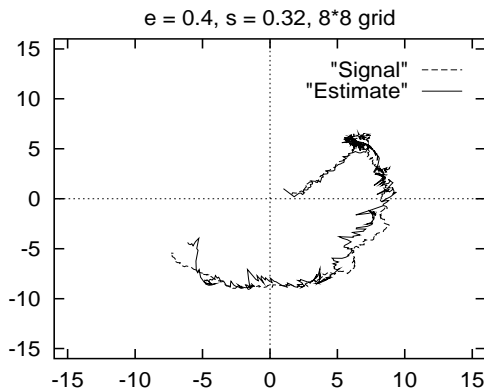


Figure 2: NLF, Angle observation.

Now let us add an observation of the range, $y_{2,n} = \sqrt{x_1^2(n\Delta) + x_2^2(n\Delta)} + sv_{2,n}$, where the $\{v_{2,n}\}$ are mutually independent, normally distributed with mean zero and variance unity, and independent of the other processes. As seen in the typical Figure 3, the tracking of the position coordinates is

excellent. Comparing (via simulations) the results with those for all of the current forms of the particle method, showed that the latter method required so many points to get reasonable results, that it was not competitive as a real-time procedure.

Numerical examples based on Approach 2 can be found in [1, 20]. Approach 1 uses a more direct approximation, without the intermediate “mean square” approximation step, so that one might initially expect that it would be better, although it involves more computation. However, to date, there are no definitive numerical or mathematical comparisons of the two approaches. Each does well on the class of problems to which it has been applied in the references. The versions of each of the approaches that were discussed are only one form of the possibilities. It is important to keep in mind that there are numerous variations of the general ideas. For all of the examples in the references, the extended Kalman filter was unstable and gave very poor results.

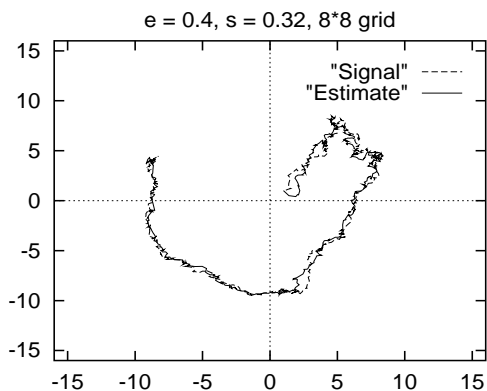


Figure 3: NLF, Angle-Range observation.

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