ON UNIQUENESS OF BOUNDARY BLOW-UP SOLUTIONS OF A CLASS OF NONLINEAR ELLIPTIC EQUATIONS

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ABSTRACT. We study boundary blow-up solutions of semilinear elliptic equations $Lu = u_+^p$ with $p > 1$, or $Lu = e^{au}$ with $a > 0$, where L is a second order elliptic operator with measurable coefficients. Several uniqueness theorems and an existence theorem are obtained.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, where $n \geq 2$, and let $\partial\Omega$ denote its boundary. We consider operators L of the form

$$L = a^{ij} D_{ij} + b^i D_i - c = a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + b^i(x) \frac{\partial}{\partial x_i} - c(x)$$

whose coefficients a^{ij}, b^i, c are assumed to be measurable functions on $\mathbb{R}^n$ and satisfy

$$a^{ij} = a^{ji}, \quad \sum_i (b^i)^2 \leq K, \quad 0 \leq c \leq K \tag{1}$$

for some fixed constant $K > 0$. We also assume that the principal coefficients a^{ij} satisfy the uniform ellipticity condition; i.e., there are constants $0 < \lambda \leq \Lambda < +\infty$ such that for all $x \in \Omega$, we have

$$\lambda |\xi|^2 \leq a^{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n. \tag{2}$$

Here and throughout the article, the summation convention over repeated indices is enforced.

In this article, we study the problem

$$Lu(x) = f(u(x)) \quad \text{for} \quad x \in \Omega, \tag{3}$$

$$u(x) \to +\infty \quad \text{as} \quad d(x) := \operatorname{dist}(x, \partial\Omega) \to 0, \tag{4}$$

where $f(t) = t_+^p := \{\max(t, 0)\}^p$ with $p > 1$, or $f(t) = e^{at}$. Solutions of the problem (3), (4) are called boundary blow-up solutions, or large solutions.

Problems of this type have been studied by many authors. Bieberbach (1916) considered the equation $\Delta u = e^u$ when $n = 2$, in connection to a problem in Riemannian geometry. Later, Loewner and Nirenberg (1974) studied the equation $\Delta u = u_+^{(n+2)/(n-2)}$ $(n > 2)$, which arises in conformal differential geometry. The problem (3), (4) is also related to probability theory. The equation $Lu = u_+^p$, $1 < p \leq 2$, appears in the analytical theory of a Markov processes called superdiffusions; see e.g. Dynkin (2002). By using the potential theory, Labutin (2003) recently gave a necessary and sufficient Wiener type condition for the existence of boundary blow-up solutions to $\Delta u = u_+^p$ with $p > 1$.

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If the domain Ω is regular enough (e.g. Ω satisfies an exterior cone condition), and if the coefficients a^{ij} are Hölder continuous in Ω, then existence of classical solutions of the problem (3), (4) can be established by the method of supersolutions and subsolutions together with uniform upper bound estimates of Keller (1957) and Osserman (1957). In fact, Keller and Osserman proved existence of boundary blow-up solutions of $\Delta u = f(u)$ for a much larger class of functions f including $f(t) = e^t$ and $f(t) = t_+^p$ with $p > 1$; see, e.g., the above references for the details.

The question of uniqueness of boundary blow-up solutions has been studied by many authors. In the case when the domain Ω is smooth (e.g. Ω is of C^2), Bandle and Marcus (1992, 1995) and Lazer and McKenna (1994) proved uniqueness of solutions of the problem of (3), (4) for a class of functions f including $f(t) = t_+^p$ with $p > 1$ by analyzing the asymptotic behavior of boundary blow-up solutions near the boundary.

Uniqueness of boundary blow-up solution in non-smooth domains was also studied by several other authors. Le Gall (1994) investigated the uniqueness of boundary blow-up solution of $\Delta u = u^2$ in non-smooth domains by means of a probabilistic representation. Marcus and Véron (1993) proved the uniqueness of boundary blow-up solution of $\Delta u = u^p$ in very general domains for all $p > 1$, using purely analytical methods. Quite recently, Marcus and Véron (2006) also proved the uniqueness of blow-up solutions for equation $\Delta u = f(u)$ in bounded domains Ω such that $\partial\Omega$ is a locally continuous graph, with convex f satisfying the standard Keller-Osserman condition.

In the main body of this article, we do not impose any regularity assumptions on the coefficients of operators L. In Theorem 3.1, we prove that if Ω satisfies "the uniform exterior ball condition" (see below for its definition), then the problem (3), (4) with $f(t) = t_+^p$ has at most one classical (or strong) solution. A similar result holds true for $f(t) = e^{at}$ in a special case when Ω is convex. Also, in Theorem 3.4 we show that if $f(t) = t_+^p$ with $p \in (1, 1 + \frac{2}{\mu(n-1)-1})$, where $\mu = \Lambda/\lambda \geq 1$, and if Ω satisfies $\partial\Omega = \partial\overline{\Omega}$, then the problem (3), (4) has at most one classical (or strong) solution. For the same $f(t)$, by assuming certain regularity of a^{ij}, in Theorem 3.5 we prove an existence and uniqueness result with $p \in (1, 1 + \frac{2}{n-2})$. In a special case $L = \Delta$, the results of Theorems 3.4 and 3.5 are contained in Marcus and Véron (1997), Véron (2001).

Our uniqueness results are based on the iteration technique, which appears in the proof of Theorem 4.1. For operators L with "good enough" (e.g. continuous) coefficients, one can also use another iteration method, introduced by Marcus and Véron (1998), with further development in Marcus and Véron (2004). In particular, they proved (Theorem 3.2 in Marcus and Véron, 2004), that there exists one and only one solution of the problem (3), (4) in the case $n = 2$, $L = \Delta$, $f(t) = e^t$, $\partial\Omega = \partial\overline{\Omega}$. We could not get this result by our method. Roughly speaking, we need the estimate $e^{u_1} \leq N e^{u_2}$ near $\partial\Omega$ for any blow-up solutions u_1 and u_2, while the method in Marcus and Véron (2004) uses a weaker estimate $u_1 \leq N u_2$ near $\partial\Omega$.

The remaining sections are organized in the following way. In Section 2, we give definitions and state some preliminary lemmas. We state the main results in Section 3 and prove them in Sections 4 and 5.

2. Preliminaries

Definition 2.1. We say that Ω satisfies the uniform exterior ball condition with constants $\delta_1 \in (0,1)$ and $r_1 > 0$, if for arbitrary $x \in \partial\Omega$ and $0 < r < r_1$, there exists a ball $B_\rho(y) \subset B_r(x) \setminus \overline{\Omega}$ with $\rho = \delta_1 r$.

Definition 2.2. We say that $u \in C^2(\Omega)$ if u is twice continuously differentiable in Ω. We write $u \in W^{2,p}_{loc}(\Omega)$ $(p \geq 1)$ if u is twice weakly differentiable and $\sum_{|\beta|\leq 2} \int_{\Omega'} \left|D^\beta u\right|^p < +\infty$ for all $\Omega' \Subset \Omega$. Here $\Omega' \Subset \Omega$ means that Ω' is a bounded open set such that its closure $\overline{\Omega'}$ is a subset of Ω.

Definition 2.3. We say that $u \in W^{2,n}_{loc}(\Omega)$ is a solution of $Lu = g$ if $Lu = g$ a.e. in Ω. Similarly, if $u \in W^{2,n}_{loc}(\Omega)$, then $Lu \geq g$ $(Lu \leq g)$ in Ω means $Lu \geq g$ $(Lu \leq g)$ a.e. in Ω.

By Sobolev imbedding theorem, we can always assume that functions in $W^{2,n}_{loc}(\Omega)$ are continuous in Ω.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let f be an increasing function. Assume that $u, v \in C^2(\Omega)$ (or $u, v \in W^{2,n}_{loc}(\Omega)$) satisfy $Lu \geq f(u)$ and $Lv \leq f(v)$ in Ω. If $\liminf_{x\to\partial\Omega}(v-u)(x) \geq 0$, then $v \geq u$ in Ω.*

Proof. Suppose, to the contrary, that there exists $x_0 \in \Omega$ such that $u(x_0) > v(x_0)$. Then for sufficiently small $\epsilon > 0$, $\Omega_\epsilon := \{u - v > \epsilon\} \neq \emptyset$ and $\overline{\Omega}_\epsilon \subset \Omega$. The function $w := u - v - \epsilon > 0$ in Ω_ϵ, and $w = 0$ on $\partial\Omega_\epsilon$. Since f is increasing, $Lw \geq f(u) - f(v) \geq 0$ in Ω_ϵ. Then, the classical maximum principle (or Aleksandrov maximum principle) implies $w \leq 0$ in Ω_ϵ; see e.g. Gilbarg and Trudinger (1983). This contradiction proves the lemma. □

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If $w \in C^2(\Omega)$ (or $w \in W^{2,n}_{loc}(\Omega)$) satisfies $Lw \geq 0$ in a non-empty subset $\Omega' = \{x \in \Omega : w(x) > 0\}$, then $\partial\Omega' \cap \partial\Omega \neq \emptyset$.*

Proof. Otherwise, $\partial\Omega' \subset \Omega$, so that $\Omega' \Subset \Omega$ and $w = 0$ on $\partial\Omega'$. Proceeding as in the proof of Lemma 2.4, we get a contradiction. □

Lemma 2.6. *Let $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$, where $0 \leq a < b \leq \infty$. If $u(x) = \varphi(|x|)$ for some C^2 function $\varphi : (a,b) \to \mathbb{R}$, then, the Hessian $D^2u(x_0)$ has eigenvalues $\varphi''(r)$ with multiplicity 1 and $\varphi'(r)/r$ with multiplicity $n-1$, where $r = |x_0|$. Therefore, if $\varphi'' \geq 0$ and $\varphi' \leq 0$, then*

$$a^{ij} D_{ij} u(x_0) = \mathrm{tr}(A \cdot D^2 u(x_0)) \geq \lambda \varphi''(r) + \frac{(n-1)\Lambda}{r}\varphi'(r) \tag{5}$$

for any symmetric matrix $A = \{a^{ij}\}$ whose eigenvalues belong to $[\lambda, \Lambda]$.

Proof. It is a straightforward computation. □

3. Main results

Our first two results are about the uniqueness of solutions under the general assumption that the coefficients a^{ij}, b^i, c are measurable functions satisfying (1), (2).

Theorem 3.1. *Let $f(t) = t_+^p$ with $p > 1$. Assume that Ω is a bounded domain satisfying the uniform exterior ball condition. Then there exists at most one $C^2(\Omega)$ (or $W^{2,n}_{loc}(\Omega)$) solution of the problem (3), (4).*

Theorem 3.2. *Let $f(t) = e^{at}$ with $a > 0$. Assume that Ω is a bounded convex domain in $\mathbb{R}^n$, $n \geq 2$. Then there exists at most one $C^2(\Omega)$ (or $W^{2,2}_{loc}(\Omega)$) solution of the problem* (3)*,* (4)*.*

Remark 3.3. Notice that if u is a solution to $Lu = e^{au}$ in Ω, then $v(x) := au(x/\sqrt{a})$ is a solution to

$$\bar{a}^{ij} D_{ij} v + \bar{b}^i D_i v - \bar{c} v = e^v \quad \text{in } \sqrt{a}\,\Omega,$$

where

$$\bar{a}^{ij}(x) = a^{ij}(x/\sqrt{a}), \quad \bar{b}^i(x) = b^i(x/\sqrt{a})/\sqrt{a}, \quad \bar{c}(x) = c(x/\sqrt{a})/a.$$

Therefore, without loss of generality, we shall always assume $a = 1$ in the sequel.

In the next two results, we treat the problem (3), (4) with $f(t) = t^p$ in more general bounded domains $\Omega \subset \mathbb{R}^n$. In the case $L = \Delta$, these results are known from Marcus and Véron (1997), Véron (2001).

Theorem 3.4. *If $p \in (1, 1 + \frac{2}{\mu(n-1)-1})$, where $\mu = \Lambda/\lambda \geq 1$, and $\partial\Omega = \partial\overline{\Omega}$, then there exists at most one $C^2(\Omega)$ (or $W^{2,n}_{loc}(\Omega)$) solution of the problem* (3)*,* (4)*.*

Theorem 3.5. *Suppose that $p \in (1, \infty)$ when $n = 2$, and $p \in (1, \frac{n}{n-2})$ when $n \geq 3$.*

i) If $a^{ij}(x)$ are uniformly continuous in a neighborhood of $\partial\Omega$, and $\partial\Omega = \partial\overline{\Omega}$, then there exists at most one solution of the problem (3)*,* (4)*.*

ii) If a^{ij} are Hölder continuous in Ω, i.e. $a^{ij} \in C^\beta(\Omega)$ for some $\beta \in (0, 1)$, then there exists at least one solution of the problem (3)*,* (4)*.*

Remark 3.6. One can see from the proofs in the following two sections, that the boundedness assumption of $b^i(x)$ and $c(x)$ can be replaced by

$$|b^i(x)| = o(d^{-1}(x)), \quad 0 \leq c(x) = o(d^{-2}(x)).$$

Also, the uniform ellipticity of the principal coefficients a^{ij} is required only near the boundary $\partial\Omega$, as long as the weak maximum principle is valid in the entire domain Ω. Furthermore, if the boundary is smooth (say C^2), it suffices to have L to be nondegenerate only in the normal direction near the boundary, i.e. there is a $\delta > 0$ such that for any $x_0 \in \partial\Omega$ we have $a^{ij}\nu_i\nu_j \geq \lambda$ in $B_\delta(x_0)$, where ν is the unit normal direction of $\partial\Omega$ at x_0.

Remark 3.7. Without much more work, Theorem 3.1, 3.2 and 3.4 can be extended to fully nonlinear elliptic equations $F[u] = u_+^p$ (or e^u), where $F[u] = F(x, u, Du, D^2u)$ and $F(x, u, p, q)$ is a function defined on the set

$$\Gamma := \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n.$$

Here $\mathbb{S}^n$ is the set of all symmetric $n \times n$ matrices, and F satisfies the following natural assumptions

$$\lambda|\xi|^2 \leq F(x, u, p, q + \xi\xi^T) - F(x, u, p, q) \leq \Lambda|\xi|^2,$$
$$|F(x, u, p, q) - F(x, u, p_1, q)| \leq K|p - p_1|,$$
$$-Ks \leq F(x, u + s, p, q) - F(x, u, p, q) \leq 0, \quad F(x, 0, 0, 0) = 0,$$

for any $(x, u, p, q) \in \Gamma$, $s \geq 0$, $p_1 \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$. In particular, elliptic Bellman equations $\sup_\beta\{L^\beta u\} = u_+^p$ (or e^u) belong to this class, where linear operators

$$L^\beta = a^{ij}_\beta(x) D_{ij} + b^i_\beta(x) D_i - c_\beta(x)$$

satisfy (1) and (2) with same constants K, λ, Λ for all β. Indeed, it suffices to notice that under the assumptions above for any two given C^2 functions u, v, we have

$$F[u] - F[v] = L^{u,v}(u - v),$$

for some linear operator $L^{u,v} = a^{ij}D_{ij} + b^i D_i - c$ satisfying assumptions (1) and (2) (see, for example, Lemma 1.1 in Safonov, 1988). In particular, by choosing $v \equiv 0$, we get $F[u] = L^u u$ for some linear operator L^u.

4. Proof of Theorems 3.1 and 3.2

Recall the notation $d(x) := \operatorname{dist}(x, \partial\Omega)$. The following theorem is the main tool of this article in obtaining the uniqueness results.

Theorem 4.1. *Assume $f(t) = t_+^p$ with $p > 1$, or $f(t) = e^t$. Let $\beta = 2p/(p-1)$ if $f(t) = t_+^p$ with $p > 1$, and $\beta = 2$ if $f(t) = e^t$. If u_1, u_2 are $C^2(\Omega)$ (or $W^{2,n}_{loc}(\Omega)$) solutions of the problem* (3), (4), *both satisfying ($i = 1, 2$)*

$$N_1 d^{-\beta} \le f(u_i) \le N_2 d^{-\beta} \quad in \quad \Delta_\rho := \{x \in \Omega : d(x) < \rho\} \tag{6}$$

for some constants $N_1, N_2, \rho > 0$, then $u_1 \equiv u_2$ in Ω.

Proof. We first consider the case $f(t) = t_+^p$ with $p > 1$. Set $\gamma = 2/(p-1)$ so that $\beta = \gamma p = \gamma + 2$. Let u_1, u_2 be two different $C^2(\Omega)$ (or $W^{2,n}_{loc}(\Omega)$) solutions of the problem (3), (4). By Lemma 2.4, they must be different in Δ_ρ, and we may assume that

$$u_2(x_0)/u_1(x_0) > k \quad \text{for some} \quad x_0 \in \Delta_\rho \quad \text{and} \quad k \ge k_0 > 1. \tag{7}$$

Note that

$$L(u_2 - ku_1) = f(u_2) - kf(u_1) \ge f(u_2) - f(ku_1) > 0 \quad \text{in} \quad \Omega' := \{u_2 > ku_1\}.$$

By Lemma 2.5 applied to $w =: u_2 - ku_1$, we have $\partial\Omega' \cap \partial\Omega \ne \emptyset$. Therefore, x_0 can be chosen arbitrary close to $\partial\Omega$, and we may assume that

$$B_r(x_0) \subset \Delta_\rho \quad \text{and} \quad K(2r + r^2) \le \Lambda, \quad \text{where } r := d(x_0)/2. \tag{8}$$

The set $\Omega_0 := \{u_2 - ku_1 > 0\} \cap B_r(x_0) \Subset \Omega$. In Ω_0, we have $r < d(x) < 3r$, and

$$L(u_2 - ku_1) = u_2^p - ku_1^p > (k^p - k)u_1^p \ge (k_0^{p-1} - 1)ku_1^p \ge c_0 k r^{-\beta},$$

where $c_0 := (k_0^{p-1} - 1)3^{-\beta}N_1 > 0$. On the other hand, the function

$$w(x) = c_1 k r^{-\beta}(r^2 - |x - x_0|^2), \quad \text{where } c_1 := c_0/(3n\Lambda), \tag{9}$$

satisfies

$$Lw \ge -c_1 k r^{-\beta}(2n\Lambda + 2Kr + Kr^2) \ge -c_0 k r^{-\beta} \quad \text{in} \quad B_r(x_0) \supset \Omega_0.$$

Then the function $w_1 := u_2 - ku_1 + w$ satisfies $Lw \ge 0$ in Ω_0, and by the maximum principle, it attains its maximum on $\overline{\Omega_0}$ at some point $x_1 \in \partial\Omega_0$. Note that x_1 cannot belong to $B_r(x_0)$, because on the set $(\partial\Omega_0) \cap B_r(x_0)$, we must have $u_2 = ku_1$, which in turn implies $w_1 = w \le w(x_0) < w_1(x_0) \le w_1(x_1)$. Therefore, $x_1 \in \partial B_r(x_0)$, so that $w(x_1) = 0$, and

$$(u_2 - ku_1)(x_1) = w_1(x_1) \ge w_1(x_0) > w(x_0) = c_1 k r^{2-\beta} = c_1 k r^{-\gamma}. \tag{10}$$

Since $d(x_1) \ge r$, from (6) it follows

$$f(u_1(x_1)) = u_1^p(x_1) \le N_2 d^{-\beta}(x_1) \le N_2 r^{-\beta} = N_2 r^{-\gamma p}.$$

This estimate together with (9), (10) implies

$$u_2(x_1) > (1+c_2)ku(x_1), \quad \text{where } c_2 := c_1 N_2^{-1/p} > 0.$$

Again, replacing x_1 by another point near $\partial\Omega$ if necessary, we may assume that (7), (8) hold with $(1+c_1)k, x_1, r_1 := d(x_1)/2$ in place of k, x_0, r respectively. By iterating, we obtain a sequence $\{x_j\}_{j=0}^{\infty} \subset \Delta_\rho$ such that $u_2(x_j)/u_1(x_j) > (1+c_2)^j k_0$, which tends to infinity. However, (6) implies $u_2/u_1 \le (N_2/N_1)^{1/p}$ in Δ_ρ. This contradiction proves that $u_1 \equiv u_2$ in Ω.

Now we consider the case $f(t) = e^t$. We proceed similarly as above. Let u_1, u_2 be two different solutions of the problem (3), (4). We may assume that

$$u_2(x_0) - u_1(x_0) > k \quad \text{for some} \quad x_0 \in \Delta_\rho \quad \text{and} \quad k \ge k_0 > 0.$$

By Lemma 2.5 applied to $w := u_2 - u_1 - k$, we may also assume that x_0 is chosen such that (8) holds. Then

$$L(u_2 - u_1 - k) \ge e^{u_2} - e^{u_1} > (e^k - 1)e^{u_1} \ge c_3 k r^{-2}$$

on the set $\Omega_0 := \{u_2 - u_1 > k\} \cap B_r(x_0)$, where $c_3 := N_1/9$. On the other hand, the function

$$w(x) := c_4 k r^{-2}(r^2 - |x - x_0|^2), \quad \text{where} \quad c_4 := c_3/(3n\Lambda) > 0,$$

satisfies $Lw \ge -c_3 k r^{-2}$ in Ω_0. Then the function $w_1 := u_2 - u_1 - k + w$ satisfies $Lw_1 \ge 0$ in Ω_0, hence it attains its maximum on $\overline{\Omega_0}$ at some point $x_1 \in \partial\Omega_0$, which cannot belong to $B_r(x_0)$. Therefore, $x_1 \in \partial B_r(x_0)$, $w(x_1) = 0$, and

$$u_2(x_1) - u_1(x_1) - k = w_1(x_1) \ge w_1(x_0) > w(x_0) = c_4 k.$$

As before, by iterating this process, we obtain a sequence $\{x_j\}_{j=0}^{\infty} \subset \Delta_\rho$ such that $u_2(x_j) - u_1(x_j) > (1+c_4)^j k_0$, which tends to infinity. However, (6) implies that $u_2 - u_1 \le \ln(N_2/N_1)$ in Δ_ρ. Again, this contradiction leads to the conclusion that $u_1 \equiv u_2$ in Ω. The theorem is proved. □

Remark 4.2. By easy modifications of the proof above, one can see that Theorem 4.1 can be extended to any locally Lipschitz, increasing function f, which is equal to e^t in (N_1, ∞), or f which is equal to t^p in (N_1, ∞), for some $N_1 > 0$, and satisfies the additional condition $f(\mu t) \ge \mu f(t)$ for any $\mu \ge 1$ and $t \in \mathbb{R}$.

We derive a lower and upper bounds in the following two lemmas.

Lemma 4.3. *Let $f(t)$, β, and Δ_ρ be as in Theorem 4.1. If u is a $C^2(\Omega)$ (or $W^{2,n}_{loc}(\Omega)$) solution of the problem* (3), (4), *then we have*

(11) $$1/N_1 \le f(u) \text{ in } \Omega, \quad f(u) \le N_2 d^{-\beta} \text{ in } \Delta_1 := \{x \in \Omega : d(x) < 1\}.$$

Here N_1, N_2 are positive constants depending only on n, λ, Λ, K, and p if $f(t) = t_+^p$ with $p > 1$; N_1 may also depend on $\operatorname{diam}\Omega$.

Proof. First, we consider the case $f(t) = t_+^p$ with $p > 1$. Without loss of generality, we assume that Ω contains the origin. The lower bound follows from an observation that $\varepsilon e^{\eta x_1}$ is a bounded subsolution if we first choose η sufficiently large, and then $\varepsilon > 0$ sufficiently small. It remains to get the upper bound. Fix $x_0 \in \Delta_1$ and $r < d(x_0) < 1$. Denote

$$w_0(x) := N_0(1 - |x|^2)^{-\gamma}, \quad w(x) := r^{-\gamma} w_0((x - x_0)/r),$$

where $\gamma := 2/(p-1)$ as before. If we set $N_0 := (2\gamma(n+2\gamma)\Lambda+2\gamma K)^{\gamma/2}$, then we have $Lw \le w^p$ in $B_r(x_0)$ for any elliptic operator L whose coefficients satisfy (1), (2). By Lemma 2.4, $u(x) \le w(x)$ in $B_r(x_0)$. In particular, we have $u(x_0) \le w(x_0) = N_0 r^{-\gamma}$. Therefore, we get the desired bound (11) with $N_2 := N_0^p$ by letting $r \to d(x_0)$.

The case $f(t) = e^t$ is treated similarly. Without loss of generality, we may assume that Ω lies in the half-space $\{x_1 > 0\}$. Fix positive constants η_1 and η_2, such that

$$\lambda\eta_1^2 - K\eta_1 - K \ge 1, \quad \eta_2 \ge \sup_{\Omega} e^{\eta_1 x_1}.$$

Then the function $v := e^{\eta_1 x_1} - \eta_2$ satisfies $v \le 0$ and

$$Lv = (a^{11}\eta_1^2 + b^1\eta_1 - c)e^{\eta_1 x_1} + c\eta_2 \ge (\lambda\eta_1^2 - K\eta_1 - K)e^{\eta_1 x_1} \ge 1 \ge e^v$$

in Ω. Hence $u \ge v$ in Ω, and the lower bound follows. For the upper bound, we fix $x_0 \in \Delta_1$ and set

$$w_0(x) := \ln N_2 - 2\ln(1-|x|^2), \quad w(x) := w_0((x-x_0)/r) - 2\ln r,$$

where $N_2 := 4n(\Lambda + K)$. Then $Lw \le e^w$ in $B_r(x_0)$. Again, Lemma 2.4 implies that $u(x_0) \le w(x_0) = \ln(N_2/r^2)$. By letting $r \to d(x_0)$, we obtain the bound (11). The lemma is proved. □

Remark 4.4. In the previous lemma, the assumption (4) was used only for the proof of the lower bound in (11). Note that the upper bound

$$f(u(x)) \le N_2 d^{-\beta}(x) \qquad \forall x \in \Omega \tag{12}$$

is valid for any $C^2(\Omega)$ (or $W^{2,n}_{loc}(\Omega)$) solution u of (3).

Lemma 4.5. *Let Ω be a bounded domain satisfying the uniform exterior ball condition with constants r_1 and δ_1 (see Definition 2.1). Assume $f(t) = t_+^p$, where $p > 1$, and set $\beta := 2p/(p-1)$. If u is a $C^2(\Omega)$ (or $W^{2,n}_{loc}(\Omega)$) solution of the problem* (3), (4)*, then*

$$f(u) \ge N d^{-\beta} \quad \text{in} \quad \Delta_\rho := \{x \in \Omega : d(x) < \rho\}, \tag{13}$$

where $\rho := \min(r_1, 1/2)$, and $N > 0$ is a constant depending only on $n, \lambda, \Lambda, K, p, \delta_1$, and r_1.

Proof. For a fixed point $x_0 \in \Delta_\rho$, choose $z_0 \in \partial\Omega$ such that $|x_0 - z_0| = r_0 := d(x_0)$, and then y_0 such that $B_{\delta_1 r_0}(y_0) \subset B_{r_0}(z_0) \setminus \overline{\Omega}$.

Set $\delta := \delta_1/2$ and $r := 2r_0$. Observe that if $m = m(K, n, \delta)$ is sufficiently large, then $v_0(t) := (1-t)^m$ satisfies

$$\begin{cases} \lambda v_0''(t) + \dfrac{(n-1)\Lambda}{t} v_0'(t) + K v_0'(t) - K v_0(t) \ge v_0^p(t), & \forall t \in (\delta, 1), \\ v_0'(t) < 0, \quad \forall t \in (\delta, 1), \\ v_0(1) = 0, \quad v_0(t) > 0 \quad \forall t \in [\delta, 1). \end{cases} \tag{14}$$

Note that $\delta r = \delta_1 r_0 < r = 2r_0 < 2\rho \le 1$. Using Lemma 2.6, it is easy to check that the function $v(x) := r^{-\gamma} v_0(|x - y_0|/r)$ with $\gamma := 2/(p-1)$ satisfies $Lv \ge v^p$ in $\Omega \cap B_r(y_0)$. Thus, by Lemma 2.4, since $|x_0 - y_0| \le (2-\delta_1)r_0 = (1-\delta)r$, we have

$$u(x_0) \ge v(x_0) \ge 2^{-\gamma} d^{-\gamma}(x_0)\, v_0(1-\delta)$$

From here the desired lower bound follows with $N := 2^{-\gamma p} v_0^p(1-\delta)$. The lemma is proved. □

Now we are ready to prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1: It follows readily from Theorem 4.1, Lemma 4.3, and Lemma 4.5.

Proof of Theorem 3.2: Fix a constant $D > \operatorname{diam}(\Omega)$. We may assume that

$$\overline{\Omega} \subset \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \ 0 < x_1 < D\}.$$

Note that the function $v_0 := -2 \ln x_1$ satisfies

$$Lv_0 = 2a_{11}x_1^{-2} - 2b^1 x_1^{-1} + 2c \ln x_1 \geq 2\lambda x_1^{-2} - 2K x_1^{-1} - 2K |\ln x_1|$$

for $x_1 > 0$. Choose constants $\delta = \delta(\lambda, K) \in (0,1)$ and $N = N(\lambda, K, D) \geq K\,|\ln \lambda|$, such that

$$\begin{aligned} Lv_0 &\geq \lambda x_1^{-2} \qquad \text{for} \qquad 0 < x_1 < \delta, \\ N - c \ln \lambda + Lv_0 &\geq \lambda x_1^{-2} \qquad \text{for} \qquad \delta \leq x_1 < D. \end{aligned}$$

As in the proof of Lemma 4.3, take a function $v := e^{\eta_1 x_1} - \eta_2$ satisfying

$$v \leq 0, \quad Lv \geq 1 \quad \text{for} \quad 0 < x_1 < D.$$

Then the function $w := Nv + \ln \lambda + v_0$ satisfies

$$Lw \geq N - c \ln \lambda + Lv_0 \geq \lambda x_1^{-2} = e^{\ln \lambda + v_0} \geq e^w \quad \text{for} \quad 0 < x_1 < D.$$

By Lemma 2.4, we must have

$$u \geq w, \quad e^u \geq e^w \geq \lambda\, e^{-N\eta_2} x_1^{-2} \quad \text{in} \quad \Omega.$$

Finally, note that the conditions (1), (2) on the coefficients of L are invariant with respect to parallel translations and rotations in $\mathbb{R}^n$. Therefore, for any fixed $x = (x_1, \dots, x_n) \in \Omega$, we can always assume that $x_2 = \dots = x_n = 0$, and $x_1 > 0$ can be made arbitraryly close to $d(x) = \operatorname{dist}(x, \partial\Omega)$. This means that we have the lower bound

$$e^u \geq \lambda\, e^{-N\eta_2} d^{-2} =: N_1 d^{-2} \quad \text{in} \quad \Omega.$$

This estimate, together with Theorem 4.1 and the upper bound in Lemma 4.3, yields the uniqueness.

5. Proof of Theorems 3.4 and 3.5

In this section, we prove the uniqueness of a solution of the problem (3), (4) with $f(t) = t_+^p$, in more general domains.

Proof of Theorem 3.4: Assume that u is a $C^2(\Omega)$ (or $W^{2,n}_{loc}(\Omega)$) solution of the problem (3), (4), and set

$$\gamma := 2/(p-1), \quad \gamma_0 := \gamma((\gamma+1)\lambda + (1-n)\Lambda).$$

Note that the assumption $p \in (1, 1 + \frac{2}{\mu(n-1)-1})$ implies $\gamma_0 > 0$. Let $r_0 \in (0,1)$ be such that $2(Kr_0^2 + K\gamma r_0) \leq \gamma_0$. Fix $x_0 \in \Delta_{r_0/2}$ and choose $z_0 \in \partial\Omega$ such that $|x_0 - z_0| = r := d(x_0)$. From $\partial\Omega = \partial\overline{\Omega}$ it follows that there exists a point $y_0 \in B_{r/2}(z_0) \setminus \overline{\Omega}$. Using Lemma 2.6, it is easy to check that the function

$$v(x) := c_0\, |x - y_0|^{-\gamma} - c_0 (2r)^{-\gamma}, \quad \text{where} \quad c_0 := (\gamma_0/2)^{\gamma/2},$$

satisfies $Lv \geq v^p$ in $\Omega \cap B_{2r}(y_0)$. Moreover, $v \in C^2(\overline{\Omega})$, $v < +\infty$ on $(\partial\Omega) \cap \overline{B}_{2r}(y_0)$, and $v = 0$ on $\overline{\Omega} \cap \partial B_{2r}(y_0)$. Therefore, by Lemma 2.4, $u(x) \geq v(x)$ in Ω. In particular, we have

$$u(x_0) \geq v(x_0) \geq c_1 d^{-\gamma}(x_0), \quad \text{where} \quad c_1 := (1.5^{-\gamma} - 2^{-\gamma}) c_0.$$

Also, by Lemma 4.3, $u(x_0) \le c_2 d^{-\gamma}(x_0)$ in Δ_1, for some $c_2 > 0$ depending only on n, λ, Λ, K, and p. Since $x_0 \in \Delta_{r_0/2}$ is arbitrary, we have proved that

$$c_1 d^{-\gamma} \le u \le c_2 d^{-\gamma} \quad \text{in } \Delta_{r_0/2}.$$

Now the desired statement follows from Theorem 4.1.

Proof of Theorem 3.5: We prove part ii) first. Let $\{\Omega_m\}_{m=1}^{\infty}$ be an exhausting sequence of smooth subdomains of Ω; i.e., $\Omega_m \Subset \Omega_{m+1} \Subset \Omega$ and $\bigcup_{m=1}^{\infty} \Omega_m = \Omega$. Let u_m be the unique boundary blow-up solution of $Lu = u_+^p$ in Ω_m for each $m \ge 1$. (For existence of such solutions u_m, see, e.g. Keller, 1957; uniqueness is a consequence of Theorem 3.1.) By Lemma 2.4, $\{u_m\}_{m=1}^{\infty}$ is a decreasing sequence, and by Lemma 4.3, it is bounded below by some constant $1/N_1 > 0$. Hence, the limit function u exists in Ω and by the standard elliptic theory, it is a solution of $Lu = u^p$ in Ω.

We claim that u is indeed a boundary blow-up solution. In order to prove this, it suffices to show that for any $y_0 \in \partial\Omega$,

$$u_m(x) \ge N_0 |x - y_0|^{-\gamma} \quad \text{in} \quad \Omega_m \cap B_{r_0}(y_0), \tag{15}$$

where $\gamma = 2/(p-1)$ as before, and N_0, r_0 are positive constants independent of m.

We first do a linear transformation to make $y_0 = 0$, $a^{ij}(y_0) = \delta^{ij}$, and still use the same notations for simplicity. Due to (2), the scales in these two coordinate systems are comparable. Therefore, we only need to verify (15) in the new coordinates. Set

$$v_0(x) = c_p\, |x|^{-\gamma}, \quad \text{where} \quad c_p := \{\gamma(\gamma + 2 - n)/2\}^{\gamma/2}.$$

Since a^{ij} are uniformly continuous, one can choose $r_1 > 0$ sufficiently small, such that in $\Omega_m \cap B_{r_1}(0)$,

$$\begin{aligned} Lv_0(x) \;&=\; c_p \left\{ \Delta(|x|^{-\gamma}) + (a^{ij}(x) - a^{ij}(0)) D_{ij}(|x|^{-\gamma}) \right. \\ &+\; \left. b^i D_i(|x|^{-\gamma}) - c|x|^{-\gamma} \right\} \\ &\ge\; c_p \left\{ \gamma(\gamma + 2 - n) - K\gamma r_1 - K r_1^2 + N(n,p)\, \omega(r_1) \right\} |x|^{-\gamma-2} \\ &\ge\; (c_p/2)\gamma(\gamma + 2 - n)|x|^{-\gamma-2} = v_0^p(x), \end{aligned}$$

where $\omega(r_1) = \max_{i,j}\{\text{osc}_{\Omega \cap B_{r_1}(0)}\, a^{ij}\}$. Then the function $v(x) := c_p|x|^{-\gamma} - c_p r_1^{-\gamma}$ satisfies $Lv \ge v^p$ in $\Omega_m \cap B_{r_1}(0)$, and $v(x) = 0$ on $\partial B_{r_1}(0)$. Therefore, by Lemma 2.4, we have $u_m(x) \ge v(x)$ in $\Omega_m \cap B_{r_1}(0)$, and the desired estimate (15) follows with $N_0 := c_p(1 - 2^{-\gamma})$ and $r_0 := r_1/2$.

For the proof of i), due to Theorem 4.1, it suffices to get the estimate

$$N_1 d^{-\gamma} \le u \le N_2 d^{-\gamma}$$

in a neighborhood of $\partial\Omega$. Arguing as in the proof of Theorem 3.4, this estimate can be proved by using the barrier function $v(x)$ constructed in the proof of ii). The details are left to the reader.

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