

# A REGULARITY CRITERION FOR THE DISSIPATIVE QUASI-GEOSTROPHIC EQUATIONS

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ABSTRACT. We establish a regularity criterion for weak solutions of the dissipative quasi-geostrophic equations in mixed time-space Besov spaces.

## 1. INTRODUCTION

In this paper we obtain a regularity criterion for weak solutions of the 2D dissipative quasi-geostrophic equations. We consider the following initial value problem

$$(1.1) \quad \begin{cases} \theta_t + u \cdot \nabla \theta + (-\Delta)^{\gamma/2} \theta = 0, & x \in \mathbb{R}^2, t \in (0, \infty), \\ \theta(0, x) = \theta_0(x), \end{cases}$$

where  $\gamma \in (0, 2]$  is a fixed parameter and the velocity  $u = (u_1, u_2)$  is divergence free and determined by the Riesz transforms of the potential temperature  $\theta$ :

$$u = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta) = (-\partial_{x_2} (-\Delta)^{-1/2} \theta, \partial_{x_1} (-\Delta)^{-1/2} \theta).$$

The 2D quasi-geostrophic equation is an important model in geophysical fluid dynamics used in meteorology and oceanography (see, for example, Pedlosky [26]). It is derived from general quasi-geostrophic equations in the special case of constant potential vorticity and buoyancy frequency.

The main mathematical question concerning the initial value problem (1.1) is whether there exists a global in time smooth solution to (1.1) evolving from any given smooth initial data. Before we recall the known results in this direction we note that cases  $\gamma > 1$ ,  $\gamma = 1$  and  $\gamma < 1$  are called subcritical, critical and supercritical, respectively. Existence of a global weak solution was established by Resnick [28]. Furthermore, in the subcritical case, Constantin and Wu [10] proved that every sufficiently smooth initial data give a rise to a unique global smooth solution. In the critical case,  $\gamma = 1$ , Constantin, Cordoba and Wu [8] established existence of a unique global classical solution corresponding to any initial data that are small in  $L^\infty$ . The assumption requiring smallness in  $L^\infty$  was removed by Caffarelli and Vasseur [1], Kiselev, Nazarov and Volberg [21] and Dong and Du [18]. In [21] the authors proved persistence of a global solution in  $C^\infty$  corresponding to any  $C^\infty$  periodic initial data. Dong and Du in [18] adapted the method of [21] and obtained global well-posedness for the critical 2D dissipative quasi-geostrophic equations with  $H^1$  initial data in the whole space. On the other hand, Caffarelli

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1991 *Mathematics Subject Classification.* 35Q35.

*Key words and phrases.* regularity criteria, quasi-geostrophic equations.

H.D. was partially supported by a start-up funding from the Division of Applied Mathematics of Brown University. N.P. was partially supported by a start-up funding from the College of Natural Sciences of the University of Texas at Austin.

and Vasseur established regularity of Leray-Hopf solution by proving the following three claims:

- (1) Every Leray-Hopf weak solution corresponding to initial data  $\theta_0 \in L^2$  is in  $L_{\text{loc}}^\infty(\mathbb{R}^2 \times (0, \infty))$
- (2) The  $L^\infty$  solutions are Hölder regular i.e. they are in  $C^\gamma$  for some  $\gamma > 0$
- (3) Every Hölder regular solution is a classical solution in  $C^{1,\beta}$ .

While the main question addressing global in time existence is settled in the critical case, it still remains open in the supercritical case,  $\gamma < 1$ . In this case Chae and Lee [4], Wu [30] and Chen, Miao and Zhang [6] established existence of a global solution in Besov spaces evolving from small initial data (see also [25, 23]). Recently, Constantin and Wu in [11] implemented the approach of [1] in the supercritical case. They proved that every Leray-Hopf weak solution corresponding to initial data  $\theta_0 \in L^2$  is in  $L_{\text{loc}}^\infty(\mathbb{R}^n \times (0, \infty))$  and hence the claim (1) is valid in the supercritical case. Concerning an analogue of the claim (2), Constantin and Wu in [11] proved that  $L^\infty$  solutions are Hölder continuous under the additional assumption that the velocity  $u \in C^{1-\gamma}$ . In a separate paper [12] Constantin and Wu considered the step (3) of the above approach and established a conditional regularity result of the type: if a Leray-Hopf solution is in the sub-critical space  $L^\infty((t_0, t_1); C^\delta(\mathbb{R}^2))$  for some  $\delta > 1 - \gamma$  on the time interval  $[t_0, t_1]$ , then such a solution is a classical solution on  $(t_0, t_1]$ .

In this paper we extend the conditional regularity result of [12] to scaling invariant mixed time-space Besov spaces  $L^{r_0}((0, T); B_{p,\infty}^\alpha)$  with

$$(1.2) \quad \alpha = \frac{2}{p} + 1 - \gamma + \frac{\gamma}{r_0}.$$

More precisely, we show that if

$$\theta \in L_t^{r_0}((0, T); B_{p,\infty}^\alpha(\mathbb{R}^2))$$

is a weak solution of the 2D quasi-geostrophic equation (1.1), then  $\theta$  is a classical solution of (1.1) in  $(0, T] \times \mathbb{R}^2$ . Significance of this space is that it is a critical space, by which we mean scaling invariant under the scaling transformation

$$\theta_\lambda = \lambda^{\gamma-1} \theta(\lambda x, \lambda^\gamma t).$$

Since the following embedding relations

$$L_t^\infty L_x^2 \cap L_t^\infty C_x^\delta \hookrightarrow L_t^\infty L_x^2 \cap L_t^\infty \dot{B}_{p,\infty}^{\delta(1-\frac{2}{p})} \hookrightarrow L_t^{r_0} B_{p,\infty}^\alpha,$$

hold for sufficiently large  $p$  and  $r_0$ , our regularity result can be understood as an extension of the regularity result of Constantin and Wu [12] to critical spaces.

In order to prove the regularity result we first establish local existence and uniqueness of weak solutions to (1.1) in certain mixed time-space Besov spaces of Chemin type  $\tilde{L}^r B_{p,q}^\alpha$  (for a definition of this space, see Section 2). We prove such existence and uniqueness results following the approach of Q. Chen et al [6]. We choose  $\alpha$  according to (1.2) which in turn implies that the space  $B_{p,q}^\alpha$  itself is subcritical. Therefore the time of existence depends only on the norm of the initial data and not on the profile. We combine the local existence (stated in Proposition 3.1) and uniqueness of weak solutions (stated in Proposition 3.3) to prove regularity by using a contradiction argument in the spirit of the work of Giga [20] in the context of the Navier-Stokes equations.

We recall that the first conditional regularity result for solutions to (1.1) was obtained by Constantin, Majda and Tabak [9]. Recently Chae established a conditional regularity result in Sobolev spaces in [3] and in Triebel-Lizorkin spaces in [2], while B.-Q. Dong and Chen in [15] extended the regularity criterion of Chae [3] to Besov spaces by proving that a solution to (1.1) is regular on the time interval  $(0, T]$  if

$$\nabla\theta \in L^r((0, T); \dot{B}_{p, \infty}^0) \text{ with } \frac{2}{p} + \frac{\gamma}{r} = \gamma, \quad \frac{4}{\gamma} \leq p \leq \infty.$$

In comparison with [15] we require less regularity for  $\theta$ . We note that these conditional regularity results are in the spirit of the conditional regularity results available for the 3D Navier-Stokes equations e.g. [22, 27, 29, 19, 7].

**Organization of the paper.** The paper is organized as follows. In Section 2 we introduce the notation that shall be used throughout the paper and we review known estimates on the nonlinear term. In Section 3 we state the main results of the paper. Then in Section 4 we give proof of the existence and regularity results, while in the appendix Section 5 we fill out details of the existence result stated in Section 3.

## 2. NOTATION AND PRELIMINARIES

**2.1. Notation and spaces.** We recall that for any  $\beta \in \mathbb{R}$  the fractional Laplacian  $(-\Delta)^\beta$  is defined via its Fourier transform:

$$\widehat{(-\Delta)^\beta f}(\xi) = |\xi|^{2\beta} \hat{f}(\xi).$$

We note that by a weak solution to (1.1) we mean  $\theta(t, x)$  in  $(0, \infty) \times \mathbb{R}^2$  such that for any smooth function  $\phi(t, x)$  satisfying  $\phi(t, \cdot) \in \mathcal{S}$  for each  $t$ , the identity

$$\begin{aligned} & \int_{\mathbb{R}^2} \theta(T, \cdot) \phi(T, \cdot) dx - \int_{\mathbb{R}^2} \theta(0, \cdot) \phi(0, \cdot) dx - \int_0^T \int_{\mathbb{R}^2} \theta \phi_t dx dt \\ & - \int_0^T \int_{\mathbb{R}^2} u \theta \nabla \phi dx dt + \int_0^T \int_{\mathbb{R}^2} \theta \Lambda^\gamma \phi dx dt = 0 \end{aligned}$$

holds for any  $T > 0$ .

Before we recall the definition of the spaces that will be used throughout the paper, we shall review the Littlewood-Paley decomposition. For any integer  $j$ , define  $\Delta_j$  to be the Littlewood-Paley projection operator with  $\Delta_j v = \phi_j * v$ , where

$$\hat{\phi}_j(\xi) = \hat{\phi}(2^{-j}\xi), \quad \hat{\phi} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}), \quad \hat{\phi} \geq 0,$$

$$\text{supp } \hat{\phi} \subset \{\xi \in \mathbb{R}^2 \mid 1/2 \leq |\xi| \leq 2\}, \quad \sum_{j \in \mathbb{Z}} \hat{\phi}_j(\xi) = 1 \text{ for } \xi \neq 0.$$

Formally, we have the Littlewood-Paley decomposition

$$v(\cdot, t) = \sum_{j \in \mathbb{Z}} \Delta_j v(\cdot, t).$$

Also denote

$$\Lambda = (-\Delta)^{1/2}, \quad \bar{\Delta}_{-1} = \sum_{j < 0} \Delta_j.$$

As usual, for any  $p \in [1, \infty)$  and  $s \geq 0$ , we denote by  $\dot{W}_p^s$  and  $W_p^s$ , respectively the homogeneous and inhomogeneous Sobolev spaces with norms

$$\begin{aligned} \|v\|_{\dot{W}_p^s} &:= \left\| \left( \sum_{k \in \mathbb{Z}} |2^{ks} \Delta_k v|^2 \right)^{1/2} \right\|_{L^p} \sim \|\Lambda^s v\|_{L^p}, \\ \|v\|_{W_p^s} &:= \|v\|_{\dot{W}_p^s} + \|v\|_{L^p}. \end{aligned}$$

When  $p = 2$ , we use  $\dot{H}^s$  and  $H^s$  instead of  $\dot{W}_p^s$  and  $W_p^s$ . For any  $p, q \in [1, \infty]$  and  $s \in \mathbb{R}$ , we denote by  $\dot{B}_{p,q}^s$  and  $B_{p,q}^s$ , respectively the homogeneous and inhomogeneous Besov spaces equipped with norms

$$\begin{aligned} \|v\|_{\dot{B}_{p,q}^s} &:= \begin{cases} \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \|\Delta_j v\|_{L^p}^q \right)^{1/q}, & \text{for } q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\Delta_j v\|_{L^p}, & \text{for } q = \infty, \end{cases} \\ \|v\|_{B_{p,q}^s} &:= \begin{cases} \left( \sum_{j \geq 0} 2^{jsq} \|\Delta_j v\|_{L^p}^q \right)^{1/q} + \|\bar{\Delta}_{-1} v\|_{L^p}, & \text{for } q < \infty, \\ \sup_{j \geq 0} 2^{js} \|\Delta_j v\|_{L^p} + \|\bar{\Delta}_{-1} v\|_{L^p}. & \text{for } q = \infty, \end{cases} \end{aligned}$$

If  $s > 0$ , we have

$$B_{p,q}^s = \dot{B}_{p,q}^s \cap L^p, \quad \|v\|_{B_{p,q}^s} \sim \|v\|_{\dot{B}_{p,q}^s} + \|v\|_{L^p}.$$

For  $s \in \mathbb{R}$ ,  $1 \leq p, q, r \leq \infty$ ,  $I$  an interval in  $\mathbb{R}$ , the homogeneous mixed time-space Besov space  $\tilde{L}^r(I; \dot{B}_{p,q}^s)$  is the space of distributions in  $\mathcal{D}'(I; \mathcal{S}'_0(\mathbb{R}^d))$  such that

$$\|f\|_{\tilde{L}^r(I; \dot{B}_{p,q}^s)} := \left\| 2^{sj} \left( \int_I \|\Delta_j f(t)\|_{L^p(\mathbb{R}^d)}^r dt \right)^{1/r} \right\|_{l^q(\mathbb{Z})} < \infty,$$

(usual modification applied if  $r = \infty$  or  $q = \infty$ ). Also the inhomogeneous time-space Besov norm is given by

$$\|f\|_{\tilde{L}^r(I; B_{p,q}^s)} := \|f\|_{L^r(I; L^p(\mathbb{R}^d))} + \|f\|_{\tilde{L}^r(I; \dot{B}_{p,q}^s)}.$$

These spaces were introduced by J.-Y. Chemin [5].

**2.2. Preliminaries.** The following Bernstein's inequality is well-known.

**Lemma 2.1.** *i) Let  $p \in [1, \infty]$  and  $s \in \mathbb{R}$ . Then for any  $j \in \mathbb{Z}$ , we have*

$$(2.1) \quad \lambda 2^{js} \|\Delta_j v\|_{L^p} \leq \|\Lambda^s \Delta_j v\|_{L^p} \leq \lambda' 2^{js} \|\Delta_j v\|_{L^p}$$

*with some constants  $\lambda$  and  $\lambda'$  depending only on  $p$  and  $s$ .*

*ii) Moreover, for  $1 \leq p \leq q \leq \infty$ , there exists a positive constant  $C$  depending only on  $p$  and  $q$  such that*

$$(2.2) \quad \|\Delta_j v\|_{L^q} \leq C 2^{(1/p-1/q)dj} \|\Delta_j v\|_{L^p}.$$

Now we recall the generalized Bernstein's inequality and a lower bound for an integral involving fractional Laplacian which will be used in the paper. They can be found in [31], [23] and [6].

**Lemma 2.2.** *i) Let  $p \in [2, \infty)$  and  $\gamma \in [0, 2]$ . Then for any  $j \in \mathbb{Z}$ , we have*

$$(2.3) \quad \lambda 2^{\gamma j/p} \|\Delta_j v\|_{L^p} \leq \|\Lambda^{\gamma/2} (|\Delta_j v|^{p/2})\|_{L^2}^{2/p} \leq \lambda' 2^{\gamma j/p} \|\Delta_j v\|_{L^p}$$

*with some positive constants  $\lambda$  and  $\lambda'$  depending only on  $p$  and  $\gamma$ .*

ii) Moreover, we have

$$(2.4) \quad \int_{\mathbb{R}^2} (\Lambda^\gamma v) |v|^{p-2} v \geq c \|\Lambda^{\gamma/2} |v|^{p/2}\|_{L^2}^2,$$

and

$$(2.5) \quad \int_{\mathbb{R}^2} (\Lambda^\gamma \Delta_j v) |\Delta_j v|^{p-2} \Delta_j v \geq c 2^{\gamma j} \|\Delta_j v\|_{L^p}^p,$$

with some positive constant  $c$  depending only on  $p$  and  $\gamma$ .

Next we recall the commutator estimate that shall be used throughout the paper.

**Lemma 2.3.** *Let  $d \geq 1$  be an integer,  $p, q \in [1, \infty]$ ,  $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1$ ,  $\rho_1 < 1$ ,  $\rho_2 < 1$  and  $u$  be a divergence free vector field. Assume in addition that*

$$\rho_1 + \rho_2 + d \min(1, \frac{2}{p}) > 0, \quad \rho_1 + \frac{d}{p} > 0$$

Then for any  $j \in \mathbb{Z}$  we have

$$(2.6) \quad \begin{aligned} & \| [u, \Delta_j] \cdot \nabla v \|_{L_t^r(L^p(\mathbb{R}^d))} \\ & \leq C c_j 2^{-j(\frac{d}{p} + \rho_1 + \rho_2 - 1)} \|\nabla u\|_{\tilde{L}_t^{r_1}(\dot{B}_{p,q}^{\frac{d}{p} + \rho_1 - 1}(\mathbb{R}^d))} \|\nabla v\|_{\tilde{L}_t^{r_2}(\dot{B}_{p,q}^{\frac{d}{p} + \rho_2 - 1}(\mathbb{R}^d))}, \end{aligned}$$

where  $C$  is a positive constant independent of  $j$  and  $\{c_j\} \in l^q$  satisfying  $\|c_j\|_{l^q} \leq 1$ . Here

$$[u, \Delta_j] \cdot \nabla v = u \cdot \Delta_j(\nabla v) - \Delta_j(u \cdot \nabla v).$$

*Proof.* See [6] and [14].  $\square$

Also we state the following result about a product of two functions in Besov spaces. For a proof, see, for example, [6].

**Lemma 2.4.** *Let  $s > -\frac{d}{p} - 1$ ,  $s < s_1 < \frac{d}{p}$ ,  $2 \leq p \leq \infty$ ,  $1 \leq q \leq \infty$ ,  $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1$  and  $u$  be a divergence free vector field. Then*

$$\|u \cdot \nabla v\|_{\tilde{L}_t^r(\dot{B}_{p,q}^s)} \lesssim \|u\|_{\tilde{L}_t^{r_1}(\dot{B}_{p,q}^{s_1})} \|\nabla v\|_{\tilde{L}_t^{r_2}(\dot{B}_{p,q}^{s + \frac{d}{p} - s_1})}.$$

If  $s_1 = \frac{d}{p}$  or  $s_1 = s$ , then  $q$  has to be taken to be 1.

### 3. FORMULATION OF RESULTS

In this section we formulate existence and uniqueness results that shall be used in the proof of our main regularity result. Also we formulate the main regularity result.

First we state the local well-posedness result for (1.1).

**Proposition 3.1.** *Let  $\gamma \in (0, 1]$ ,  $p \in [2, \infty)$ ,  $q \in [1, \infty]$  and  $r_0 \in [2, \infty)$ . Denote by  $\alpha = \frac{2}{p} + 1 - \gamma + \frac{\gamma}{r_0}$ . Assume  $\theta_0 \in B_{p,q}^\alpha(\mathbb{R}^2)$ . Then there exists  $T \geq c \|\theta_0\|_{\dot{B}_{p,q}^\alpha}^{-r_0}$  for some constant  $c > 0$  such that the initial value problem for (1.1) has a unique weak solution*

$$\theta(t, x) \in \tilde{L}^2((0, T); B_{p,q}^{\alpha + \frac{\gamma}{2}}) \cap \tilde{L}^\infty((0, T); B_{p,q}^\alpha).$$

For any  $r \in [2, \infty]$ ,

$$(3.1) \quad \|\theta\|_{\tilde{L}_t^r B_{p,q}^{\alpha + \frac{\gamma}{r}}((0, T) \times \mathbb{R}^2)} \leq C \|\theta_0\|_{B_{p,q}^\alpha}$$

with a positive constant  $C$  independent of  $r$ , and  $\theta$  is smooth in  $(0, T) \times \mathbb{R}^2$ . Moreover, if  $q < \infty$ , we also have

$$\theta(t, x) \in C([0, T]; B_{p,q}^\alpha).$$

*Remark 3.2.* From the proof, it is clear that if  $r_0 > 2$  then the unique solution  $\theta$  is actually in

$$\tilde{L}^1((0, T); B_{p,q}^{\alpha+\gamma}) \cap \tilde{L}^\infty((0, T); B_{p,q}^\alpha).$$

Moreover, for any  $r \in [1, \infty]$  estimate (3.1) holds. However, we will not use this in our main theorem.

An analogous local well-posedness result in the critical space  $B_{p,q}^{\frac{2}{p}+1-\gamma}$  was established in [6] (see also [25, 23] for local well-posedness results in Sobolev spaces). However, we remark that with  $\theta_0$  in the critical space the time of existence  $T$  depends on the profile of  $\theta_0$  instead of the norm.

The next proposition is about the uniqueness of weak solutions in mixed time-space Besov spaces.

**Proposition 3.3.** *Let  $\gamma \in (0, 1]$ ,  $p \in [2, \infty)$ ,  $T \in (0, \infty)$  and  $r_0 \in [2, \infty)$ . Denote by  $\alpha = \frac{2}{p} + 1 - \gamma + \frac{\gamma}{r_0}$ .*

- (a) *Let  $q \in [1, \infty)$ . If  $\theta, \theta' \in \tilde{L}_t^{r_0} B_{p,q}^\alpha((0, T) \times \mathbb{R}^2)$  are two weak solutions of (1.1) with the same initial data, then  $\theta = \theta'$  in  $[0, T) \times \mathbb{R}^2$ .*
- (b) *Let  $q = \infty$ . If  $\theta, \theta' \in L_t^{r_0} B_{p,q}^\alpha((0, T) \times \mathbb{R}^2)$  are two weak solutions of (1.1) with the same initial data, then  $\theta = \theta'$  in  $[0, T) \times \mathbb{R}^2$ .*

The following regularity criteria is our main result. Roughly speaking, it says weak solutions in certain critical time-space Besov spaces are regular.

**Theorem 3.4.** *Let  $\gamma \in (0, 1]$ ,  $p \in [2, \infty)$ ,  $T \in (0, \infty)$  and  $r_0 \in [2, \infty)$ . Denote by  $\alpha = \frac{2}{p} + 1 - \gamma + \frac{\gamma}{r_0}$ . If*

$$\theta \in L_t^{r_0}((0, T); B_{p,\infty}^\alpha(\mathbb{R}^2))$$

*is a weak solution of (1.1), then  $\theta$  is in  $C^\infty((0, T) \times \mathbb{R}^2)$ , and thus it is a classical solution of (1.1) in the region  $(0, T) \times \mathbb{R}^2$ .*

#### 4. PROOFS OF EXISTENCE, UNIQUENESS AND REGULARITY

In this section we present proofs of the above stated results. In order to prove Proposition 3.1 and Proposition 3.3 we modify accordingly the approach used by Q. Chen et al [6].

##### 4.1. Proof of Proposition 3.1.

4.1.1. *A priori estimate.* We apply the operator  $\Delta_j$  to the first equation in (1.1) to obtain

$$(4.1) \quad \partial_t \Delta_j \theta + \Delta_j(u \cdot \nabla \theta) + \Lambda^\gamma \Delta_j \theta = 0,$$

which is equivalent to

$$(4.2) \quad \partial_t \Delta_j \theta + u \cdot \nabla \Delta_j \theta + \Lambda^\gamma \Delta_j \theta = [u, \Delta_j] \cdot \nabla \theta.$$

Now we multiply (4.2) by  $|\Delta_j \theta|^{p-2} \Delta_j \theta$  and integrate in  $x$ . Since  $u$  is divergence free, the integration by parts yields

$$\int_{\mathbb{R}^2} u \cdot \nabla \Delta_j \theta |\Delta_j \theta|^{p-2} \Delta_j \theta \, dx = 0.$$

Hence we have

$$(4.3) \quad \frac{1}{p} \frac{d}{dt} \|\Delta_j \theta\|_{L^p}^p + \int_{\mathbb{R}^2} (\Lambda^\gamma \Delta_j \theta) |\Delta_j \theta|^{p-2} \Delta_j \theta \, dx = \int_{\mathbb{R}^2} [u, \Delta_j] \cdot \nabla \theta |\Delta_j \theta|^{p-2} \Delta_j \theta \, dx.$$

Now we use Lemma 2.2 to obtain a lower bound on the second term on the left hand side of (4.3) and Hölder's inequality to get an upper bound on the right hand side of (4.3) to derive

$$(4.4) \quad \frac{d}{dt} \|\Delta_j \theta\|_{L^p} + \lambda 2^{\gamma j} \|\Delta_j \theta\|_{L^p} \leq C \|[u, \Delta_j] \cdot \nabla \theta\|_{L^p},$$

where  $\lambda = \lambda(p, \gamma) > 0$ . Gronwall's inequality applied on (4.4) implies

$$(4.5) \quad \|\Delta_j \theta\|_{L^p} \leq e^{-\lambda 2^{\gamma j} t} \|\Delta_j \theta(0)\|_{L^p} + C \int_0^t e^{-\lambda 2^{\gamma j} (t-s)} \|[u, \Delta_j] \cdot \nabla \theta(s)\|_{L^p} \, ds.$$

Fix  $r \in [2, \infty]$ . We take the  $L_t^r$  norm over the interval of time  $(0, T)$  to obtain:

$$(4.6) \quad \|\Delta_j \theta\|_{L_t^r L_x^p((0, T) \times \mathbb{R}^2)} \leq I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \|e^{-\lambda 2^{\gamma j} t} \|\Delta_j \theta(0)\|_{L_x^p} \|e^{-\lambda 2^{\gamma j} t}\|_{L_t^r(0, T)} \\ I_2 &= \left\| \int_0^t e^{-\lambda 2^{\gamma j} (t-s)} \|[u, \Delta_j] \cdot \nabla \theta(s)\|_{L_x^p} \, ds \right\|_{L_t^r(0, T)}. \end{aligned}$$

Since

$$\|e^{-\lambda 2^{\gamma j} t}\|_{L_t^r(0, T)} \lesssim \left( \frac{1 - e^{-r \lambda 2^{\gamma j} T}}{r \lambda 2^{\gamma j}} \right)^{\frac{1}{r}} \lesssim \lambda^{-\frac{1}{r}} 2^{-\frac{\gamma}{r} j},$$

we can bound  $I_1$  from above as follows

$$(4.7) \quad I_1 \lesssim \lambda^{-\frac{1}{r}} 2^{-\frac{\gamma}{r} j} \|\Delta_j \theta(0)\|_{L_x^p}.$$

In order to estimate  $I_2$  we use Young's inequality to obtain

$$(4.8) \quad I_2 \lesssim \|e^{-\lambda 2^{\gamma j} t}\|_{L_t^1(0, T)} \|[u, \Delta_j] \cdot \nabla \theta\|_{L_t^r L_x^p((0, T) \times \mathbb{R}^2)}.$$

Since

$$\frac{1 - e^{-\lambda 2^{\gamma j} T}}{\lambda 2^{\gamma j}} \lesssim 2^{-\gamma j},$$

as well as

$$\frac{1 - e^{-\lambda 2^{\gamma j} T}}{\lambda 2^{\gamma j}} \lesssim T,$$

we have

$$\frac{1 - e^{-\lambda 2^{\gamma j} T}}{\lambda 2^{\gamma j}} \leq 2^{-\frac{\gamma}{r_3} j} T^{1 - \frac{1}{r_3}},$$

where  $r_3$  is arbitrary real number such that  $r_3 > 1$  and will be chosen later. Hence (4.8) implies

$$(4.9) \quad I_2 \lesssim 2^{-\frac{\gamma}{r_3} j} T^{1 - \frac{1}{r_3}} \|[u, \Delta_j] \cdot \nabla \theta\|_{L_t^r L_x^p((0, T) \times \mathbb{R}^2)}.$$

Now (4.6) combined with (4.7) and (4.9) gives

$$(4.10) \quad \begin{aligned} &\|\Delta_j \theta\|_{L_t^r L_x^p((0, T) \times \mathbb{R}^2)} \\ &\lesssim \lambda^{-\frac{1}{r}} 2^{-\frac{\gamma}{r} j} \|\Delta_j \theta(0)\|_{L_x^p} + 2^{-\frac{\gamma}{r_3} j} T^{1 - \frac{1}{r_3}} \|[u, \Delta_j] \cdot \nabla \theta\|_{L_t^r L_x^p((0, T) \times \mathbb{R}^2)}. \end{aligned}$$

After we multiply (4.10) by  $2^{(\alpha+\frac{\gamma}{r})j}$  and take  $l^q(\mathbb{Z})$  norm we infer:

$$(4.11) \quad \|\theta\|_{\tilde{L}^r(\dot{B}_{p,q}^{\alpha+\frac{\gamma}{r}})} \lesssim \lambda^{-\frac{1}{r}} \|\theta(0)\|_{\dot{B}_{p,q}^\alpha} + T^{1-\frac{1}{r_3}} \|2^{(-\frac{\gamma}{r_3}+\alpha+\frac{\gamma}{r})j} [u, \Delta_j] \cdot \nabla \theta\|_{L_t^r L_x^p((0,T) \times \mathbb{R}^2)} \|l^q,$$

In order to estimate  $\|2^{(-\frac{\gamma}{r_3}+\alpha+\frac{\gamma}{r})j} [u, \Delta_j] \cdot \nabla \theta\|_{L_t^r L_x^p((0,T) \times \mathbb{R}^2)} \|l^q$  we apply Lemma 2.3 with

$$v = \theta, \quad d = 2, \quad r_1 = r_2 = 2r, \quad \rho_1 = \rho_2 = 1 - \gamma + \frac{\gamma}{2r} + \frac{\gamma}{r_0} < 1$$

and use the boundedness of the Riesz transforms to obtain

$$\begin{aligned} & \| [u, \Delta_j] \cdot \nabla \theta \|_{L_t^r L_x^p((0,T) \times \mathbb{R}^2)} \\ & \lesssim c_j 2^{-(\alpha+\frac{\gamma}{r_0}+\frac{\gamma}{r}-\gamma)j} \|u\|_{\tilde{L}^{r_1}(\dot{B}_{p,q}^{\alpha+\frac{\gamma}{r_1}})} \|\theta\|_{\tilde{L}^{r_2}(\dot{B}_{p,q}^{\alpha+\frac{\gamma}{r_2}})} \\ & \lesssim c_j 2^{-(\alpha+\frac{\gamma}{r_0}+\frac{\gamma}{r}-\gamma)j} \|\theta\|_{\tilde{L}^{r_1}(\dot{B}_{p,q}^{\alpha+\frac{\gamma}{r_1}})} \|\theta\|_{\tilde{L}^{r_2}(\dot{B}_{p,q}^{\alpha+\frac{\gamma}{r_2}})}, \end{aligned}$$

where  $c_j \in l^q$  is such that  $\|c_j\|_{l^q} \leq 1$ . Therefore

$$(4.12) \quad \begin{aligned} & 2^{(-\frac{\gamma}{r_3}+\alpha+\frac{\gamma}{r})j} \| [u, \Delta_j] \cdot \nabla \theta \|_{L_t^r L_x^p((0,T) \times \mathbb{R}^2)} \\ & \lesssim c_j 2^{(-\frac{\gamma}{r_3}-\frac{\gamma}{r_0}+\gamma)j} \|\theta\|_{\tilde{L}^{r_1}(\dot{B}_{p,q}^{\alpha+\frac{\gamma}{r_1}})} \|\theta\|_{\tilde{L}^{r_2}(\dot{B}_{p,q}^{\alpha+\frac{\gamma}{r_2}})}. \end{aligned}$$

After we choose  $r_3$  such that

$$(4.13) \quad 1 = \frac{1}{r_3} + \frac{1}{r_0},$$

we observe that (4.12) implies

$$(4.14) \quad \begin{aligned} & \| 2^{(-\frac{\gamma}{r_3}+\alpha+\frac{\gamma}{r})j} \| [u, \Delta_j] \cdot \nabla \theta \|_{L_t^r L_x^p((0,T) \times \mathbb{R}^2)} \|l^q \\ & \lesssim \|\theta\|_{\tilde{L}^{r_1}(\dot{B}_{p,q}^{\alpha+\frac{\gamma}{r_1}})} \|\theta\|_{\tilde{L}^{r_2}(\dot{B}_{p,q}^{\alpha+\frac{\gamma}{r_2}})}. \end{aligned}$$

Now we combine (4.11) and (4.14) together with (4.13) to conclude

$$(4.15) \quad \|\theta\|_{\tilde{L}^r(\dot{B}_{p,q}^{\alpha+\frac{\gamma}{r}})} \lesssim \lambda^{-\frac{1}{r}} \|\theta(0)\|_{\dot{B}_{p,q}^\alpha} + T^{\frac{1}{r_0}} \|\theta\|_{\tilde{L}^{r_1}(\dot{B}_{p,q}^{\alpha+\frac{\gamma}{r_1}})} \|\theta\|_{\tilde{L}^{r_2}(\dot{B}_{p,q}^{\alpha+\frac{\gamma}{r_2}})},$$

which is our main a priori estimate. In particular, if we denote by

$$\Lambda(\theta, T) = \|\theta\|_{\tilde{L}^2(\dot{B}_{p,q}^{\alpha+\frac{\gamma}{2}})} + \|\theta\|_{\tilde{L}^\infty(\dot{B}_{p,q}^\alpha)},$$

we then have

$$(4.16) \quad \Lambda(\theta, T) \lesssim \|\theta(0)\|_{\dot{B}_{p,q}^\alpha} + T^{\frac{1}{r_0}} \Lambda(\theta, T)^2.$$

With a help of the a priori estimate (4.15), it is standard to construct a solution of (1.1) by using approximations (see, for example, [6]). For the sake of completeness, we give a sketch of a proof in the Appendix. We refer to [16] and [17] for the proof of the smoothness of  $\theta$  in  $(0, T] \times \mathbb{R}^2$ .

**4.1.2. Uniqueness.** The proof of the uniqueness part of Proposition 3.1 is not much different from that of Proposition 3.3. We refer the reader to the next section for details.

**4.2. Proof of Proposition 3.3.** Here we establish the uniqueness result for weak solutions to (1.1), i.e. Proposition 3.3. Suppose that  $\theta$  and  $\theta'$  are two solutions to (1.1) in  $\tilde{L}_t^{r_0} B_{p,q}^\alpha((0,T) \times \mathbb{R}^2)$  which correspond to the same initial data  $\theta_0(x)$ . We denote  $\delta\theta = \theta - \theta'$  and  $\delta u = u - u'$ , where  $u' = (-\mathcal{R}_2\theta', \mathcal{R}_1\theta')$ . Then it follows that:

$$(4.17) \quad \begin{cases} \partial_t \delta\theta + u \cdot \nabla \delta\theta + \delta u \cdot \nabla \theta' + \Lambda^\gamma \delta\theta = 0, & x \in \mathbb{R}^2, t > 0, \\ \delta u = \mathcal{R}^\perp \delta\theta, \\ \delta\theta(x, 0) = 0. \end{cases}$$

We follow the strategy used to derive (4.4) to obtain

$$(4.18) \quad \frac{d}{dt} \|\Delta_j \delta\theta\|_{L^p} + \lambda 2^{\gamma j} \|\Delta_j \delta\theta\|_{L^p} \leq C (\|[u, \Delta_j] \cdot \nabla \delta\theta\|_{L^p} + \|\Delta_j(\delta u \cdot \nabla \theta')\|_{L^p}).$$

Since  $\delta\theta(x, 0) = 0$ , Gronwall's inequality applied on (4.18) implies

$$\|\Delta_j \delta\theta\|_{L^p} \leq C \int_0^t e^{-\lambda 2^{\gamma j}(t-s)} (\|[u, \Delta_j] \cdot \nabla \delta\theta\|(s) + \|\Delta_j(\delta u \cdot \nabla \theta')\|(s)) ds.$$

We take the  $L_t^{r_0}$  norm over the interval of time  $(0, T)$  and use Young's inequality to obtain:

$$(4.19)$$

$$\begin{aligned} & \|\Delta_j \delta\theta\|_{L_t^{r_0} L_x^p((0,T) \times \mathbb{R}^2)} \\ & \leq C \|e^{-\lambda 2^{\gamma j} t}\|_{L_t^{r'}(0,T)} \left( \|[u, \Delta_j] \cdot \nabla \delta\theta\|_{L_t^{\frac{r_0}{2}} L_x^p((0,T) \times \mathbb{R}^2)} + \|\Delta_j(\delta u \cdot \nabla \theta')\|_{L_t^{\frac{r_0}{2}} L_x^p((0,T) \times \mathbb{R}^2)} \right), \end{aligned}$$

where  $\frac{1}{r'} = 1 - \frac{1}{r_0}$ .

Now let us pick  $\eta$  such that

$$(4.20) \quad 1 - \frac{\gamma}{r'} - \eta + \frac{4}{p} > 0.$$

We bound  $\|e^{-\lambda 2^{\gamma j} t}\|_{L_t^{r'}(0,T)}$  from above by  $2^{-\frac{\gamma}{r'} j}$ , then multiply (4.19) by  $2^{(\frac{2}{p}-\eta)j}$  and take  $l^q$  norm with respect to  $j$  to infer:

$$(4.21) \quad \|\delta\theta\|_{\tilde{L}^{r_0}(\dot{B}_{p,q}^{\frac{2}{p}-\eta})} \lesssim C(I_3 + I_4),$$

where

$$\begin{aligned} I_3 &= \left\| 2^{(\frac{2}{p}-\eta-\frac{\gamma}{r'})j} \|[u, \Delta_j] \cdot \nabla \delta\theta\|_{L_t^{\frac{r_0}{2}} L_x^p((0,T) \times \mathbb{R}^2)} \right\|_{l^q(\mathbb{Z})}, \\ I_4 &= \|\delta u \cdot \nabla \theta'\|_{\tilde{L}_t^{\frac{r_0}{2}} \dot{B}_{p,q}^{\frac{2}{p}-\eta-\frac{\gamma}{r'}}((0,T) \times \mathbb{R}^2)}. \end{aligned}$$

In order to estimate  $I_3$  we apply Lemma 2.3 with

$$v = \delta\theta, \quad d = 2, \quad (r_1, r_2) = (r_0, r_0), \quad (\rho_1, \rho_2) = (1 - \frac{\gamma}{r'}, -\eta)$$

and the boundedness of the Riesz transforms as follows

$$\begin{aligned} & \|[u, \Delta_j] \cdot \nabla \delta\theta\|_{L_t^{\frac{r_0}{2}} L_x^p((0,T) \times \mathbb{R}^2)} \\ & \lesssim c_j 2^{-(\frac{2}{p}-\frac{\gamma}{r'}-\eta)j} \|u\|_{\tilde{L}^{r_0}(\dot{B}_{p,q}^{\frac{2}{p}-\frac{\gamma}{r'}+1})} \|\delta\theta\|_{\tilde{L}^{r_0}(\dot{B}_{p,q}^{\frac{2}{p}-\eta})} \\ & \lesssim c_j 2^{-(\frac{2}{p}-\frac{\gamma}{r'}-\eta)j} \|\theta\|_{\tilde{L}^{r_0}(\dot{B}_{p,q}^{\frac{2}{p}-\frac{\gamma}{r'}+1})} \|\delta\theta\|_{\tilde{L}^{r_0}(\dot{B}_{p,q}^{\frac{2}{p}-\eta})}, \end{aligned}$$

where  $c_j \in l^q$  is such that  $\|c_j\|_{l^q} \leq 1$ . Since

$$\frac{2}{p} - \frac{\gamma}{r'} + 1 = \alpha,$$

we obtain

$$(4.22) \quad I_3 \lesssim \|\theta\|_{\tilde{L}^{r_0}(\dot{B}_{p,q}^\alpha)} \|\delta\theta\|_{\tilde{L}^{r_0}(\dot{B}_{p,q}^{\frac{2}{p}-\eta})}.$$

On the other hand to estimate  $I_4$  we use Lemma 2.4 with

$$s = \frac{2}{p} - \frac{\gamma}{r'} - \eta, \quad s_1 = \frac{2}{p} - \eta$$

and the boundedness of the Riesz transforms to obtain

$$(4.23) \quad I_4 \lesssim \|\delta\theta\|_{\tilde{L}^{r_0}(\dot{B}_{p,q}^{\frac{2}{p}-\eta})} \|\theta'\|_{\tilde{L}^{r_0}(\dot{B}_{p,q}^\alpha)}.$$

Now we combine (4.21), (4.22) and (4.23) to conclude

$$(4.24) \quad \|\delta\theta\|_{\tilde{L}^{r_0}(\dot{B}_{p,q}^{\frac{2}{p}-\eta})} \lesssim \|\delta\theta\|_{\tilde{L}^{r_0}(\dot{B}_{p,q}^{\frac{2}{p}-\eta})} \left( \|\theta\|_{\tilde{L}^{r_0}(\dot{B}_{p,q}^\alpha)} + \|\theta'\|_{\tilde{L}^{r_0}(\dot{B}_{p,q}^\alpha)} \right).$$

We first look at part (a) of the proposition, i.e. the case  $q < \infty$ . As  $T \rightarrow 0$ , the terms in the parenthesis on the right hand side of (4.24) go to 0. For part (b), i.e.  $q = \infty$ , from (4.24) and the Minkowski's inequality we get

$$(4.25) \quad \|\delta\theta\|_{\tilde{L}^{r_0}(\dot{B}_{p,q}^{\frac{2}{p}-\eta})} \lesssim \|\delta\theta\|_{\tilde{L}^{r_0}(\dot{B}_{p,q}^{\frac{2}{p}-\eta})} \left( \|\theta\|_{L^{r_0}(\dot{B}_{p,q}^\alpha)} + \|\theta'\|_{L^{r_0}(\dot{B}_{p,q}^\alpha)} \right).$$

As  $T \rightarrow 0$ , the terms in the parenthesis on the right hand side of (4.25) go to 0. Thus in both cases if  $T$  is chosen small enough, then  $\|\delta\theta\|_{\tilde{L}^{r_0}(\dot{B}_{p,q}^{\frac{2}{p}-\eta})((0,T) \times \mathbb{R}^2)} = 0$ , which in turn implies  $\delta\theta = 0$ . Now the standard continuity argument can be employed to show that  $\delta\theta(x, t) = 0$  for all  $x \in \mathbb{R}^2$  and  $t \geq 0$ .

**4.3. Proof of Theorem 3.4.** We prove the theorem by a contradiction. Assume  $\theta$  is not a regular solution in  $(0, T) \times \mathbb{R}^2$ . Without loss of generality, one may assume  $T$  is the first blowup time. Since  $\theta \in L_t^{r_0} B_{p,\infty}^\alpha$ , for almost all  $s \in (0, T)$  we have  $\theta(s, \cdot) \in B_{p,\infty}^\alpha$ . For any such  $s$ , consider the initial value problem (1.1) with initial data  $\theta_0 = \theta(s, \cdot)$ . By applying the local well-posedness result (Proposition 3.1), (1.1) has a unique weak solution

$$\bar{\theta} \in \tilde{L}^2((0, T_s); B_{p,\infty}^{\alpha+\frac{\gamma}{2}}) \cap \tilde{L}^\infty((0, T_s); B_{p,\infty}^\alpha) \cap \tilde{L}^{r_0}((0, T_s); B_{p,\infty}^{\alpha+\frac{\gamma}{r_0}})$$

for some

$$(4.26) \quad T_s \geq c \|\theta(s, \cdot)\|_{B_{p,\infty}^\alpha}^{-r_0}$$

with a constant  $c > 0$  independent of  $s$ . Moreover, by simple embedding relations we have

$$\bar{\theta} \in \tilde{L}^{r_0}((0, T_s); B_{p,\infty}^{\alpha+\frac{\gamma}{r_0}}) \hookrightarrow \tilde{L}^{r_0}((0, T_s); B_{p,r_0}^\alpha) \hookrightarrow L^{r_0}((0, T_s); B_{p,\infty}^\alpha).$$

Now we apply the uniqueness result Proposition 3.3 and get  $\bar{\theta}(\cdot, \cdot) = \theta(s + \cdot, \cdot)$ . The last equality and (4.26) imply that

$$T - s \geq c \|\theta(s, \cdot)\|_{B_{p,\infty}^\alpha}^{-r_0}.$$

Therefore, for almost all  $s \in (0, T)$ , we have

$$\|\theta(s, \cdot)\|_{\dot{B}_{p,\infty}^\alpha} \geq c^{\frac{1}{r_0}} (T-s)^{-\frac{1}{r_0}},$$

which contradicts the condition  $\theta \in L_t^{r_0}((0, T); B_{p,\infty}^\alpha(\mathbb{R}^2))$ . The theorem is proved.

## 5. APPENDIX

The appendix is devoted to the proof of the existence part in Theorem 3.1. Consider the following successive approximations:  $\theta^0 \equiv u^0 \equiv 0$ , and for  $k = 0, 1, 2, \dots$ ,

$$(5.1) \quad \begin{cases} \theta_t^{k+1} + u^k \cdot \nabla \theta_x^{k+1} + (-\Delta)^{\gamma/2} \theta^{k+1} = 0 & x \in \mathbb{R}^2, t \in (0, \infty), \\ u^{k+1} = (-\mathcal{R}_2 \theta^{k+1}, \mathcal{R}_1 \theta^{k+1}) \\ \theta^{k+1}(0, x) = \theta_0(x) & x \in \mathbb{R}, \end{cases}$$

Similar to (4.16), we have

$$(5.2) \quad \Lambda(\theta^{k+1}, T) \lesssim \|\theta(0)\|_{\dot{B}_{p,q}^\alpha} + T^{\frac{1}{r_0}} \Lambda(\theta^k, T) \Lambda(\theta^{k+1}, T).$$

If we choose  $T = c \|\theta_0\|_{\dot{B}_{p,q}^\alpha}^{-r_0}$  for small  $c > 0$  depending on  $\lambda$  and the implicit constant in (5.2), it then holds that for any  $k = 0, 1, 2, \dots$ ,

$$(5.3) \quad \Lambda(\theta^k, T) \lesssim \|\theta_0\|_{\dot{B}_{p,q}^\alpha}.$$

Due to the  $L^p$  maximum principle for (1.1), we also have

$$(5.4) \quad \|\theta^k\|_{\tilde{L}^2(B_{p,q}^{\alpha+\frac{\gamma}{2}})} + \|\theta^k\|_{\tilde{L}^\infty(B_{p,q}^\alpha)} \lesssim \|\theta_0\|_{B_{p,q}^\alpha}.$$

Now by the first equation of (1.1) and Lemma 2.4, we have  $\theta_t^k \in L^\infty(B_{p,q}^{\frac{\gamma}{r_0}-\gamma})$  with uniformly bounded norm for  $k = 1, 2, 3, \dots$ . Since we also have  $\theta^k \in L^\infty(B_{p,q}^\alpha)$  with uniform bounded norm, due to Lion-Aubin compactness theorem, there exists a subsequence, which we still denote by  $\theta^k$ , and  $\theta = \theta(t, x)$  such that

$$\theta^k \rightarrow \theta \quad \text{in } L_{\text{loc}}^p((0, T) \times \mathbb{R}^2).$$

Moreover,  $\theta$  satisfies (1.1) in the sense of distributions and

$$(5.5) \quad \|\theta\|_{\tilde{L}^2(B_{p,q}^{\alpha+\frac{\gamma}{2}})} + \|\theta\|_{\tilde{L}^\infty(B_{p,q}^\alpha)} \lesssim \|\theta_0\|_{B_{p,q}^\alpha}.$$

As in [17],  $\theta$  is smooth in  $(0, T) \times \mathbb{R}^2$  and satisfies the first equation of (1.1) in the same region in the classical sense.

We claim  $\theta \in C([0, T]; B_{p,q}^\alpha)$  if  $q < \infty$ . Observe that from (4.1), Lemma 2.4 and Lemma 2.1 *i*) we know for  $j = 1, 2, 3, \dots$ ,

$$\partial_t \Delta_j \theta \in L^\infty((0, T); B_{p,q}^\alpha).$$

It follows immediately that

$$(5.6) \quad \Delta_j \theta \in C([0, T]; B_{p,q}^\alpha).$$

On the other hand, (5.4) implies that as  $k \rightarrow \infty$

$$\sum_{|j| \leq k} \Delta_j \theta \rightarrow \theta \quad \text{in } L^\infty((0, T); B_{p,q}^\alpha).$$

This together with (5.6) proves the claim.

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