

# SECOND-ORDER ELLIPTIC AND PARABOLIC EQUATIONS WITH $B(\mathbb{R}^2, VMO)$ COEFFICIENTS

HONGJIE DONG AND N.V. KRYLOV

ABSTRACT. The solvability in Sobolev spaces  $W_p^{1,2}$  is proved for nondivergence form second order parabolic equations for  $p > 2$  close to 2. The leading coefficients are assumed to be measurable in the time variable and two coordinates of space variables, and almost VMO (vanishing mean oscillation) with respect to the other coordinates. This implies the  $W_p^2$ -solvability for the same  $p$  of nondivergence form elliptic equations with leading coefficients measurable in two coordinates and VMO in the others. Under slightly different assumptions, we also obtain the solvability results when  $p = 2$ .

## 1. INTRODUCTION

In this paper, we consider the  $W_p^{1,2}$ -solvability of parabolic equations in nondivergence form:

$$Lu - \lambda u = f, \quad (1.1)$$

where  $\lambda \geq 0$  is a constant,  $f \in L_p$ , and

$$Lu = -u_t + a^{jk} D_{jk} u + b^j D_j u + cu. \quad (1.2)$$

We assume that all the coefficients are bounded and measurable, and  $a^{jk}$  are symmetric and uniformly elliptic, i.e.

$$|b^j| + |c| \leq K, \quad a^{jk} = a^{kj}, \quad \delta |\xi|^2 \leq a^{jk} \xi^j \xi^k \leq \delta^{-1} |\xi|^2.$$

If all the coefficients are time-independent, we also consider the  $W_p^2$ -solvability of elliptic equations in nondivergence form:

$$Mu - \lambda u = f, \quad (1.3)$$

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where

$$Mu = a^{jk} D_{jk}u + b^j D_j u + cu.$$

We concentrate on rather irregular coefficients. The Sobolev space theory of second-order parabolic and elliptic equations with discontinuous coefficients was studied extensively in the last thirty years. One important class of discontinuous coefficients contains functions with vanishing mean oscillation (VMO), the study of which was initiated in [3] and continued in [4] and [1] (see also the references in [15]). The proofs in these references are based on the Calderón-Zygmund theorem and the Coifman-Rochberg-Weiss commutator theorem. Before that the Sobolev space theory had been established for some other types of discontinuous coefficients; see, for instance, [19, 20, 5].

In [14], the second author gave a unified approach to investigating the  $L_p$  solvability of both divergence and nondivergence form parabolic and elliptic equations with leading coefficients that are in VMO in the spatial variables (and measurable in the time variable in the parabolic case). Unlike the arguments in [3, 4, 1], the proofs in [14] rely mainly on pointwise estimates of sharp functions of spatial derivatives of solutions. By doing this, VMO coefficients are treated in a rather straightforward manner. This method was later improved and generalized in a series of papers [15, 10, 11, 7, 8, 9, 16, 6].

The theory of elliptic and parabolic equations with partially VMO coefficients was originated in [10]. In [10] the  $W_p^2$ -solvability for any  $p > 2$  was established for nondivergence form elliptic equations with leading coefficients measurable in one variable and VMO in the others. This result was extended in [11] to parabolic equations with leading coefficients measurable in a spatial variable and VMO in the others. For nondivergence form parabolic equations, more general solvability results were obtained later in [7, 8, 9], in which most leading coefficients are measurable in the time variable as well as one spatial variable, and VMO in the other variables.

A natural question to ask is whether we still have the  $W_p^2$ -solvability for elliptic equations if the leading coefficients are measurable in *two* spatial variables and, say, VMO in the others. Unfortunately, the answer is negative for general  $p > 2$ . Indeed, an example by Ural'tseva (see [18]) tells us that even with leading coefficients depending only on the first two coordinates, there is no unique solvability in  $W_p^2$  for any fixed  $p > 2$  if the ellipticity constant is sufficiently small. Nevertheless, Ural'tseva's example does not rule out the possibility of  $W_p^2$ -solvability for  $p$  sufficiently close to 2 depending on the ellipticity constant. This is the main motivation of our article.

In this article, we establish the solvability in Sobolev spaces  $W_p^{1,2}$  for nondivergence form second-order parabolic equations for  $p > 2$  close to 2 (Theorem 2.1). The leading coefficients are assumed to be measurable in the time variable  $t$  and two space variables  $(x^1, x^2)$ , and VMO with respect to the others. Additionally, we assume  $a^{11} + a^{22}$  is uniformly continuous with respect to  $(x^1, x^2)$ . This result implies, in particular, the  $W_p^2$ -solvability for the same  $p$  of nondivergence form elliptic equations with leading coefficients measurable in two coordinates and VMO in the others (Theorem 2.2). Thus we give a positive answer to the aforementioned question for those  $p$  in a restricted range. An interesting application of Theorem 2.2, shown at the end of Section 2, is the  $W_p^2$ -solvability of elliptic equations in domains with rough coefficients. We also investigate the case when  $p = 2$ , which is of independent interest. For elliptic equations, the  $W_2^2$ -solvability is established when the leading coefficients are measurable function of  $(x^1, x^2)$  only. This extends a previously known result proved in [5] and [10] where the coefficients only depend on  $x^1$ . For parabolic equations, we obtain the  $W_2^{1,2}$ -solvability under the condition that  $a^{jk}$  depend only on  $(t, x^1, x^2)$  and  $a^{11} + a^{22}$  is uniformly continuous in  $(x^1, x^2)$ .

Next we give a brief description of our arguments. The proofs are based on the aforementioned method from [14]. However, since  $a^{ij}$  are merely measurable in  $(x^1, x^2)$ , we are only able to estimate the sharp function of a portion of the Hessian matrix  $D^2u$  (Theorem 5.1), more specifically,  $D_{x''}^2u$  (see the beginning of the next section for the notation). To bound the  $W_p^{1,2}$  norm of the solution by the  $L_p$  norms of the right-hand side of the equation and  $D_{x''}^2u$ , we use a result in [12] proved for 2D parabolic equations with measurable coefficients. These together with the  $W_2^{1,2}$ -solvability obtained in Section 4 enable us to establish the  $W_p^{1,2}$  estimate.

An outline of the paper: in the next section, we introduce the notation and state the main results, Theorem 2.1, 2.2, 2.6, and 2.8. Section 3 contains a few preliminary estimates. In Section 4 we establish the  $W_2^{1,2}$ -solvability and estimate the sharp function of  $D_{x''}^2u$ . We finish the proof of  $W_p^{1,2}$ -solvability in the last section by combining the results in the previous sections.

## 2. MAIN RESULTS

First we introduce some notation. Let  $d \geq 2$ . A typical point in  $\mathbb{R}^d$  is denoted by  $x = (x^1, \dots, x^d)$ . If  $d \geq 3$  we write  $x = (x', x'')$ , where  $x' = (x^1, x^2)$  and  $x'' = (x^3, \dots, x^d)$ .

We set

$$D_j u = u_{x^j}, \quad D_{jk} u = u_{x^j x^k}, \quad D_t u = u_t.$$

By  $Du$  and  $D^2 u$  we mean the gradient and the Hessian matrix of  $u$ . On many occasions we need to take these objects relative to only part of variables. The reader understands the meaning of the following notation which we use if  $d \geq 3$ :

$$\begin{aligned} D_{x'} u &= u_{x'}, & D_{x''} u &= u_{x''}, & D_{x'}^2 u &= D_{x' x'} u = u_{x' x'}, \\ D_{x' x''} u &= u_{x' x''}, & D_{x''}^2 u &= D_{x'' x''} u = u_{x'' x''}. \end{aligned}$$

For  $-\infty \leq S < T \leq \infty$ , we set

$$\begin{aligned} W_p^{1,2}((S, T) \times \mathbb{R}^d) &= \{u : u, u_t, Du, D^2 u \in L_p((S, T) \times \mathbb{R}^d)\}, \\ W_p^2(\mathbb{R}^d) &= \{u : u, Du, D^2 u \in L_p(\mathbb{R}^d)\}, \\ \mathbb{R}_T &= (-\infty, T), \quad \mathbb{R}_T^{d+1} = \mathbb{R}_T \times \mathbb{R}^d. \end{aligned}$$

We also use the abbreviations

$$C_0^\infty = C_0^\infty(\mathbb{R}^{d+1}), \quad L_p = L_p(\mathbb{R}^{d+1}), \quad W_p^{1,2} = W_p^{1,2}(\mathbb{R}^{d+1}), \dots$$

For real- or complex- or matrix-valued functions  $A(t, x)$  on  $\mathbb{R}^{d+1}$  we understand  $\|A\|_{L_p}^p$  as

$$\int_{\mathbb{R}^{d+1}} |\text{trace } A \bar{A}^*|^{p/2} dx dt.$$

Accordingly are introduced the norms in  $W$  spaces.

Our first two results concern the  $W_2^{1,2}$ - and  $W_2^2$ -solvability of equations (1.1) and (1.3) with measurable leading coefficients independent of  $x''$ . It seems to the authors that even these results are new if  $d \geq 3$ . Set

$$\text{tr}_2 a = a^{11} + a^{22}.$$

**Theorem 2.1.** *Let  $T \in (-\infty, +\infty]$ . Assume that  $a^{jk}$  depend only on  $(t, x')$  and there exists an increasing function  $\omega(r)$ ,  $r \geq 0$ , such that  $\omega(0+) = 0$  and*

$$|\text{tr}_2 a(t, x') - \text{tr}_2 a(t, y')| \leq \omega(|x' - y'|)$$

for all  $t, x', y'$ . Then

i) *There are constants  $N = N(d, \delta, K, \omega)$  and  $\lambda_0 = \lambda_0(d, \delta, K, \omega) \geq 0$  such that for any  $u \in W_2^{1,2}(\mathbb{R}_T^{d+1})$  and  $\lambda \geq \lambda_0$  we have*

$$\begin{aligned} \lambda \|u\|_{L_2(\mathbb{R}_T^{d+1})} + \sqrt{\lambda} \|Du\|_{L_2(\mathbb{R}_T^{d+1})} + \|D^2 u\|_{L_2(\mathbb{R}_T^{d+1})} + \|u_t\|_{L_2(\mathbb{R}_T^{d+1})} \\ \leq N \|Lu - \lambda u\|_{L_2(\mathbb{R}_T^{d+1})}. \end{aligned} \quad (2.1)$$

ii) *For any  $\lambda > \lambda_0$  and  $f \in L_2(\mathbb{R}_T^{d+1})$ , there exists a unique solution  $u \in W_2^{1,2}(\mathbb{R}_T^{d+1})$  of equation (1.1) in  $\mathbb{R}_T^{d+1}$ .*

iii) In case  $b^j \equiv c \equiv 0$  and  $\text{tr}_2 a$  depends only on  $t$ , we can take  $\lambda_0 = 0$  in i) and ii).

Here is a similar result for elliptic equations.

**Theorem 2.2.** Assume  $a^{jk} = a^{jk}(x')$ . Then

i) There are constants  $N = N(d, \delta, K)$  and  $\lambda_0 = \lambda_0(d, \delta, K) \geq 0$  such that for any  $u \in W_2^2(\mathbb{R}^d)$  and  $\lambda \geq \lambda_0$  we have

$$\lambda \|u\|_{L_2(\mathbb{R}^d)} + \sqrt{\lambda} \|Du\|_{L_2(\mathbb{R}^d)} + \|D^2u\|_{L_2(\mathbb{R}^d)} \leq N \|Mu - \lambda u\|_{L_2(\mathbb{R}^d)}.$$

ii) For any  $\lambda > \lambda_0$  and  $f \in L_2(\mathbb{R}^d)$ , there exists a unique solution  $u \in W_2^2(\mathbb{R}^d)$  of equation (1.3) in  $\mathbb{R}^d$ .

iii) In case  $b^j \equiv c \equiv 0$ , we can take  $\lambda_0 = 0$  in i) and ii).

Theorem 2.1 is proved in Section 4 and Theorem 2.2 is derived from it below in the present section.

**Remark 2.3.** Theorem 2.2 generalizes Theorem 2.5 of [10] and the main result of [5], where the coefficients are independent of  $(x^2, \dots, x^d)$ . From Theorem 2.1 one can get Theorem 3.2 of [11] where again the coefficients are independent of  $(x^2, \dots, x^d)$  but there is no restriction on  $\text{tr}_2 a$ . To show this we introduce a new coordinate  $y \in \mathbb{R}$ , define

$$\tilde{L} = L + (2\delta^{-1} - a^{11})D_y^2,$$

and let  $u(t, x, y) = u(t, x)\eta(y)$ , where  $\eta \in C_0^\infty(-2, 2)$  is a nonnegative function and  $\eta \equiv 1$  on  $[-1, 1]$ . It is clear that

$$\tilde{L}u(t, x, y) - \lambda u(t, x, y) = \tilde{f},$$

where

$$\tilde{f}(t, x, y) = (Lu - \lambda u)(t, x)\eta(y) + (2\delta^{-1} - a^{11})u(t, x)\eta''(y).$$

We now apply Theorem 2.2 i) with  $\tilde{L}$  and  $u(t, x, y)$  in place of  $L$  and  $u(t, x)$ . With a sufficiently large  $\lambda$ , we will arrive at (2.1) for function  $u(t, x)$ . The remaining assertions of Theorem 2.1 in case the coefficients are independent of  $(x^2, \dots, x^d)$  are obtained as in its proof given in Section 4. This argument also obviously applies if  $d = 1$ .

**Remark 2.4.** The conditions on  $a^{jk}$  in Assertion i) and ii) of Theorem 2.2 can be relaxed. By using a partition of unity and the method of freezing the coefficients, we can allow  $a^{jk}$  to be measurable in  $x'$  and uniformly continuous in  $x''$ . In this case, the constants  $\lambda_0$  and  $N$  also depend on the modulus of continuity of  $a^{jk}$  with respect to  $x''$ . Similarly, in Theorem 2.1 we can allow  $a^{jk}$  to be measurable in  $(t, x')$ , uniformly continuous in  $x''$  and  $\text{tr}_2 a$  to be measurable in  $t$  and uniformly continuous in  $x$ .

To state two more results we need some new notation. If  $B$  is a Borel subset of a hyperplane  $\Gamma$  in a Euclidean space, we denote by  $|B|$  its volume relative to  $\Gamma$ . This notation is somewhat ambiguous because  $B$  also belongs to the ambient space, where its volume can be zero. However, we hope that from the context it will be clear relative to which hyperplane we take the volume in each instance. If there is a measurable function  $f$  on  $B$  which is integrable with respect to Lebesgue measure  $\ell$  on  $\Gamma$  we set

$$(f)_B = \int_B f(x) \ell(dx) := \frac{1}{|B|} \int_B f(x) \ell(dx).$$

If  $d \geq 3$ , let

$$\begin{aligned} B'_r(x') &= \{y' \in \mathbb{R}^2 : |x' - y'| < r\}, \\ B''_r(x'') &= \{y'' \in \mathbb{R}^{d-2} : |x'' - y''| < r\}, \quad B_r(x) = B'_r(x') \times B''_r(x''), \\ Q_r(t, x) &= (t - r^2, t) \times B_r(x), \end{aligned}$$

and let  $\mathbb{Q}$  be the collection of all  $Q_r(t, x)$ . We call  $r$  the radius of  $Q = Q_r(t, x)$ . Set  $B''_r = B''_r(0)$ ,  $B_r = B_r(0)$ ,  $Q_r = Q_r(0, 0)$ . If  $d = 2$ , we denote  $B_r(x)$  and  $Q_r(t, x) = (t - r^2, t) \times B_r(x)$  to be the usual balls and parabolic cylinders. For a function  $g$  defined on  $\mathbb{R}^{d+1}$ , we denote its (parabolic) maximal and sharp function, respectively, by

$$\begin{aligned} \mathbb{M}g(t, x) &= \sup_{Q \in \mathbb{Q}: (t, x) \in Q} \int_Q |g(s, y)| dy ds, \\ g^\#(t, x) &= \sup_{Q \in \mathbb{Q}: (t, x) \in Q} \int_Q |g(s, y) - (g)_Q| dy ds. \end{aligned}$$

In the next theorem we require a quite mild regularity assumption on  $a^{jk}$ . They are assumed to be measurable in  $t$  and  $x'$ , and almost VMO with respect to  $x''$ . More precisely, we impose the following assumption in which  $\gamma > 0$  will be specified later and  $R_0 > 0$  is a fixed number.

**Assumption 2.5** ( $\gamma$ ). For any  $t, x, y$  satisfying  $x'' = y''$  and  $|x' - y'| \leq R_0$  we have

$$|\mathrm{tr}_2 a(t, x) - \mathrm{tr}_2 a(t, y)| \leq \gamma. \quad (2.2)$$

Additionally if  $d \geq 3$ , for any  $Q = (s, t) \times B' \times B'' \in \mathbb{Q}$  with radius  $\rho \leq R_0$

$$\max_{j,k} \int_Q |a^{jk}(r, x) - \bar{a}^{jk}(r, x')| dx dr \leq \gamma$$

where

$$\bar{a}^{jk}(r, x') = \int_{B''} a^{jk}(r, x) dx''.$$

**Theorem 2.6.** *One can find a  $\theta_0 = \theta_0(\delta) > 0$  such that for any  $p \in (2, 2 + \theta_0)$  there exists a  $\gamma = \gamma(d, \delta, p) > 0$  such that under Assumption 2.5 ( $\gamma$ ) for any  $T \in (-\infty, +\infty]$  the following holds.*

i) For any  $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$ ,

$$\begin{aligned} \lambda \|u\|_{L_p(\mathbb{R}_T^{d+1})} + \sqrt{\lambda} \|Du\|_{L_p(\mathbb{R}_T^{d+1})} + \|D^2u\|_{L_p(\mathbb{R}_T^{d+1})} + \|u_t\|_{L_p(\mathbb{R}_T^{d+1})} \\ \leq N \|Lu - \lambda u\|_{L_p(\mathbb{R}_T^{d+1})}, \end{aligned} \quad (2.3)$$

provided that  $\lambda \geq \lambda_0$ , where  $\lambda_0 \geq 0$  and  $N$  depend only on  $d, \delta, p, K$ , and  $R_0$ .

ii) For any  $\lambda > \lambda_0$  and  $f \in L_p(\mathbb{R}_T^{d+1})$ , there exists a unique solution  $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$  of equation (1.1) in  $\mathbb{R}_T^{d+1}$ .

iii) In the case that  $a^{jk} = a^{jk}(t, x')$  and  $b^j \equiv c \equiv 0$  and  $\text{tr}_2 a$  depends only on  $t$ , we can take  $\lambda_0 = 0$  in i) and ii).

Theorem 2.6 implies the solvability of the Cauchy problem as in [14]. We prove Theorem 2.6 in Section 5 and now we state one more result for elliptic equations in nondivergence form.

**Assumption 2.7** ( $\gamma$ ). Either  $d = 2$  or  $d \geq 3$  and for any balls  $B' \subset \mathbb{R}^2, B'' \subset \mathbb{R}^{d-2}$  of the same radius  $r \leq R_0$ ,

$$\sup_{j,k} \int_B |a^{jk}(x) - \bar{a}^{jk}(x')| dx \leq \gamma,$$

where  $B = B' \times B''$  and

$$\bar{a}^{jk}(x') = \int_{B''} a^{jk}(x) dx''.$$

**Theorem 2.8.** *Let  $\theta_0$  be the constant in Theorem 2.6. Then for any  $p \in (2, 2 + \theta_0)$ , there exists a  $\gamma = \gamma(d, \delta, p) > 0$  such that under Assumption 2.7 ( $\gamma$ ) the following holds.*

i) For any  $u \in W_p^2(\mathbb{R}^d)$ ,

$$\lambda \|u\|_{L_p(\mathbb{R}^d)} + \sqrt{\lambda} \|Du\|_{L_p(\mathbb{R}^d)} + \|D^2u\|_{L_p(\mathbb{R}^d)} \leq N \|Mu - \lambda u\|_{L_p(\mathbb{R}^d)},$$

provided that  $\lambda \geq \lambda_0$ , where  $\lambda_0 \geq 0$  and  $N$  depend only on  $d, \delta, p, K$ , and  $R_0$  ( $R_0$  is excluded if  $d = 2$ ).

ii) For any  $\lambda > \lambda_0$  and  $f \in L_p(\mathbb{R}^d)$ , there exists a unique solution  $u \in W_p^2(\mathbb{R}^d)$  of equation (1.3) in  $\mathbb{R}^d$ .

iii) In the case that  $a^{jk} = a^{jk}(x')$  and  $b^j \equiv c \equiv 0$ , we can take  $\lambda_0 = 0$  in i) and ii).

*Proof of Theorem 2.2 and 2.8.* First we assume that  $\text{tr}_2 a$  is a constant. In this case, Theorem 2.2 and 2.8 follow from Theorem 2.1 and 2.6 respectively by using the idea that solutions to elliptic equations can

be viewed as steady state solutions to parabolic equations. We omit the details and refer the reader to the proof of Theorem 2.6 [14].

We now concentrate on proving Theorem 2.8 in the general case. Owing to mollifications, a density argument, and the method of continuity it suffices to prove assertion i) assuming that the coefficients are smooth and  $u \in C_0^\infty(\mathbb{R}^d)$  so that  $f \in C_0^\infty(\mathbb{R}^d)$ . Note that the standard mollification preserves Assumption 2.7 ( $\gamma$ ), the ellipticity constant  $\delta$ , and the bounds  $K$ . We introduce

$$\tilde{a}^{jk} = \frac{a^{jk}}{\text{tr}_2 a}, \quad \tilde{b}^j = \frac{b^j}{\text{tr}_2 a}, \quad \tilde{c} = \frac{c}{\text{tr}_2 a},$$

and let  $\tilde{M}$  be the elliptic operator constructed from them. It is easy to see that  $\text{tr}_2 \tilde{a} \equiv 1$ , the new coefficients satisfy the same boundedness and ellipticity conditions with possibly different ellipticity constant and bounds:  $\tilde{\delta}$  and  $\tilde{K}$ . Moreover, if  $a^{jk}$  satisfy Assumption 2.7 ( $\gamma$ ), then  $\tilde{a}^{jk}$  satisfy Assumption 2.7 ( $N(\delta)\gamma$ ). Therefore, one can find a  $\gamma > 0$  depending only on  $d, \delta, p$ , such that Assumption 2.7 ( $\gamma$ ) implies that for  $\tilde{a}^{jk}$  Assumption 2.7 ( $\gamma$ ) is satisfied with  $\gamma = \gamma(d, \tilde{\delta}, p)$  taken from Theorem 2.6. Let  $\tilde{\lambda}_0$  be the constant from Theorem 2.6 corresponding to  $\tilde{\delta}$  and  $\tilde{K}$ . Clearly (1.3) is equivalent to

$$\tilde{M}u - \lambda u / \text{tr}_2 a = f / \text{tr}_2 a.$$

For any  $\lambda > 2\delta^{-1}\tilde{\lambda}_0$ , by the first part of the proof there exists a unique  $v \in W_p^2$  solving

$$\tilde{M}v - \delta\lambda v / 2 = -|f / \text{tr}_2 a|.$$

Moreover,  $v$  is a bounded classical solution since the coefficients of  $\tilde{M}$  are smooth and  $|f / \text{tr}_2 a|$  is Lipschitz continuous. Due to the maximum principle  $v \geq 0$  and  $|u| \leq v$  in  $\mathbb{R}^d$ . Again, by the first part of the proof, for appropriate  $p$  and  $N$  we have

$$\lambda \|u\|_{L_p(\mathbb{R}^d)} \leq \lambda \|v\|_{L_p(\mathbb{R}^d)} \leq N \|f\|_{L_p(\mathbb{R}^d)}. \quad (2.4)$$

Since

$$\tilde{M}u - \delta\lambda u / 2 = f / \text{tr}_2 a + (1 / \text{tr}_2 a - \delta / 2)\lambda u,$$

we then obtain the desired estimate from the first part of the proof and (2.4). This proves Theorem 2.8.

To prove Theorem 2.2 it suffices to repeat the above argument taking  $p = 2$  and dropping mentioning Assumption 2.7 ( $\gamma$ ).  $\square$

An application of Theorem 2.8 is the  $W_p^2$ -solvability of the Dirichlet problem for the equation

$$a^{jk}(x^1)D_{jk}u = f$$

in  $\{|x| < 1\}$ . Here we assume that  $a^{jk}(x^1)$  are measurable in  $x^1$  and continuous near  $-1$  and  $1$ . This equation can be solved by following the steps in Chapter 11 of [17]. Notice that when locally flattening the boundary and using odd/even extensions, one gets an equation with leading coefficient measurable in two coordinates and continuous in the others.

**Remark 2.9.** The author of [7]-[9] presents quite general results on the solvability of parabolic equations in Sobolev spaces with or without mixed norms. Roughly speaking, the main case in [7]-[9] is when  $a^{11}$  is measurable in  $x^1$  (or  $t$ ) and VMO in  $(t, x^2, \dots, x^d)$  (or  $x$ ) and  $p$  is any number in  $(2, \infty)$  without any restriction on  $\text{tr}_2 a$ . Theorem 2.6 and the discussion in Remark 2.3 show that, restricted to Sobolev spaces without mixed norms, some of D. Kim's results admit generalizations allowing  $a^{11}$  which are measurable in  $(t, x^1)$  and VMO in  $(x^2, \dots, x^d)$  provided that  $p > 2$  is close enough to 2. We have no idea what happens in this situation if  $p > 2$  is arbitrary even if  $d = 1$ .

In the case of Theorem 2.8 an example by Ural'tseva (see [18]) shows that for any  $d \geq 2$  its assertion becomes false for any fixed  $p > 2$  if  $\delta$  is sufficiently small.

### 3. PRELIMINARY RESULTS

We first consider equations in  $\mathbb{R} \times \mathbb{R}^2$  with measurable coefficients.

**Lemma 3.1.** *Let  $T \in (-\infty, \infty]$ ,  $d = 2$  and*

$$Lu = -u_t + \sum_{j,k=1}^2 a^{jk}(t, x) D_{jk} u,$$

where  $\text{tr}_2 a$  depends only on  $t$ . Then there exists a  $\theta_0 = \theta_0(\delta) > 0$  such that for any  $p \in (2 - \theta_0, 2 + \theta_0)$ ,  $u \in W_p^{1,2}(\mathbb{R}_T^3)$ , and  $\lambda \geq 0$ , we have

$$\begin{aligned} & \|D^2 u\|_{L_p(\mathbb{R}_T^3)} + \|u_t\|_{L_p(\mathbb{R}_T^3)} + \sqrt{\lambda} \|Du\|_{L_p(\mathbb{R}_T^3)} \\ & + \lambda \|u\|_{L_p(\mathbb{R}_T^3)} \leq N \|Lu - \lambda u\|_{L_p(\mathbb{R}_T^3)}, \end{aligned} \quad (3.1)$$

where  $N = N(\delta, p)$ . Moreover for  $\lambda > 0$  and  $f \in L_p(\mathbb{R}_T^3)$  there exists a unique  $u \in W_p^{1,2}(\mathbb{R}_T^3)$  solving  $Lu - \lambda u = f$  in  $\mathbb{R}_T^3$ .

*Proof.* First we consider the case that  $T = \infty$ . The change of variable

$$t \rightarrow \frac{1}{2} \int_0^t (\text{tr}_2 a)(s) ds$$

together with the argument in the proof of Theorem 2.2 and 2.8 reduce the problem to the case when  $a^{11} + a^{22} = 2$ . Moreover, by a density argument to prove (3.1) it suffices to consider  $u \in C_0^\infty$ . In case  $u \in$

$C_0^\infty(\Gamma)$  with  $\Gamma = (0, 1) \times \{|x| < 1\}$ , it follows from Theorem 3 of [12] that

$$\lambda\|u\|_{L_p} + \|u_t\|_{L_p} + \|D^2u\|_{L_p} \leq N\|Lu - \lambda u\|_{L_p}.$$

(See also [2] for a result for elliptic equations.) For general  $u \in C_0^\infty$ , we can use shifting and scaling, the fact that the above  $N$  depends only on  $\delta$  and  $p$ , and interpolation inequalities to treat  $Du$ . This proves (3.1) if  $T = \infty$ . Adding using the standard method of continuity completes the proof of the lemma when  $T = \infty$ .

For general  $T$ , we use the fact that  $u = w$  for  $t < T$ , where  $w \in W_p^{1,2}$  solves

$$Lw - \lambda w = \chi_{t < T}(Lu - \lambda u).$$

The lemma is proved.  $\square$

An immediate corollary of Lemma 3.1 is the following estimate.

**Corollary 3.2.** *Let  $T \in (-\infty, \infty]$ ,  $d \geq 3$ ,*

$$Lu = -u_t + \sum_{j,k=1}^d a^{jk}(t, x) D_{jk}u,$$

where  $\text{tr}_2 a$  depends only on  $(t, x'')$ . Then for any  $p \in (2 - \theta_0, 2 + \theta_0)$ , where  $\theta_0$  is taken from Lemma 3.1, and any  $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$  and  $\lambda \geq 0$ , we have

$$\begin{aligned} & \lambda\|u\|_{L_p(\mathbb{R}_T^{d+1})} + \sqrt{\lambda}\|Du\|_{L_p(\mathbb{R}_T^{d+1})} + \|D^2u\|_{L_p(\mathbb{R}_T^{d+1})} + \|u_t\|_{L_p(\mathbb{R}_T^{d+1})} \\ & \leq N\|Lu - \lambda u\|_{L_p(\mathbb{R}_T^{d+1})} + N\|D_{x''}^2u\|_{L_p(\mathbb{R}_T^{d+1})}, \end{aligned} \quad (3.2)$$

where  $N = N(\delta, d, p)$ .

*Proof.* We first fix  $x''$  and apply Lemma 3.1 to get

$$\begin{aligned} & \lambda\|u(\cdot, \cdot, x'')\|_{L_p(\mathbb{R}_T^3)}^p + \|D_{x'}^2u(\cdot, \cdot, x'')\|_{L_p(\mathbb{R}_T^3)}^p + \|u_t(\cdot, \cdot, x'')\|_{L_p(\mathbb{R}_T^3)}^p \\ & \leq N\left\| \sum_{j,k=1}^2 a^{jk} D_{jk}u(\cdot, \cdot, x'') - u_t(\cdot, \cdot, x'') - \lambda u(\cdot, \cdot, x'') \right\|_{L_p(\mathbb{R}_T^3)}^p. \end{aligned} \quad (3.3)$$

Upon integrating (3.3) with respect to  $x''$  we arrive at

$$\begin{aligned} & \lambda\|u\|_{L_p(\mathbb{R}_T^{d+1})} + \|D_{x'}^2u\|_{L_p(\mathbb{R}_T^{d+1})} + \|u_t\|_{L_p(\mathbb{R}_T^{d+1})} \\ & \leq N\|Lu - \lambda u\|_{L_p(\mathbb{R}_T^{d+1})} + \|D_{x''}u\|_{L_p(\mathbb{R}_T^{d+1})}. \end{aligned} \quad (3.4)$$

Observe that for any  $\varepsilon > 0$

$$\|D_{x'x''}u\|_{L_p} \leq \varepsilon\|D_{x'x'}u\|_{L_p} + N(d, p)\varepsilon^{-1}\|D_{x''x''}u\|_{L_p}, \quad (3.5)$$

which is deduced from

$$\|D_{x'x''}u\|_{L_p} \leq N\|\Delta u\|_{L_p} \leq N\|D_{x'x'}u\|_{L_p} + N\|D_{x''x''}u\|_{L_p}$$

by scaling in  $x'$ . By using (3.5), we get from (3.4)

$$\begin{aligned} & \lambda \|u\|_{L_p(\mathbb{R}_T^{d+1})} + \|D_{x'x'}u\|_{L_p(\mathbb{R}_T^{d+1})} + \|u_t\|_{L_p(\mathbb{R}_T^{d+1})} \\ & \leq N \|Lu - \lambda u\|_{L_p(\mathbb{R}_T^{d+1})} + \|D_{x''x''}u\|_{L_p(\mathbb{R}_T^{d+1})}. \end{aligned}$$

To estimate  $\|D_{x'x''}u\|_{L_p(\mathbb{R}_T^{d+1})}$  and  $\|Du\|_{L_p(\mathbb{R}_T^{d+1})}$ , we use (3.5) again and the interpolation inequality

$$\sqrt{\lambda} \|Du\|_{L_p} \leq N\lambda \|u\|_{L_p} + N \|D^2u\|_{L_p}.$$

The corollary is proved.  $\square$

In the following theorem as in Corollary 3.2 the constant  $\theta_0$  is taken from Lemma 3.1.

**Theorem 3.3.** *In case  $d \geq 3$  and  $T \in (-\infty, \infty]$  for any  $p \in (2 - \theta_0, 2 + \theta_0)$  there exists a  $\gamma(d, p, \delta) > 0$  such that, if for any  $t, x, y$ , satisfying  $x'' = y''$  and  $|x' - y'| \leq R_0$ , condition (2.2) holds, then estimate (3.2) is valid for any  $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$  and  $\lambda \geq \lambda_0$  with general  $L$  as in (1.2) and  $N$  and  $\lambda_0 \geq 0$  depending only on  $d, p, \delta$ , and  $R_0$ . Furthermore, if  $u(t, x) = 0$  for  $|x| \geq R_0$  and  $b^j = c = 0$ , then we can take  $\lambda_0 = 0$  and  $N$  to be independent of  $R_0$ .*

*Proof.* The idea is to use Corollary 3.2 in combination with a standard method based on freezing the coefficients and partitions of unity. We will show only the first step. Assume that  $u$  is of class  $W_p^{1,2}$  and  $u(t, x) = 0$  for  $|x| \geq R_0$ . Define  $\mathbf{a}(t, x) = \mathbf{a}(t, x'') = \mathbf{a}(t, 0, x'')$  and

$$L_0v = \text{tr}_2 \mathbf{a}(\text{tr}_2 a)^{-1} a^{jk} D_{jk}v - v_t.$$

Then (3.2) holds with  $L_0$  in place of  $L$ . However, on the support of  $u$

$$|\text{tr}_2 \mathbf{a}(\text{tr}_2 a)^{-1} - 1| \leq N(\delta) |\text{tr}_2 \mathbf{a} - \text{tr}_2 a| \leq N(\delta)\gamma,$$

so that

$$\|L_0u - (a^{jk} D_{jk}u - u_t)\|_{L_p} \leq N(\delta)\gamma \|D^2u\|_{L_p},$$

which shows how to choose  $\gamma > 0$  in order for this error term times the  $N$  from (3.2) to be absorbed into the left-hand side of (3.2).  $\square$

#### 4. EQUATIONS WITH COEFFICIENTS MEASURABLE IN $(t, x')$ AND PROOF OF THEOREM 2.1

In this section we consider the operator

$$Lu(t, x) = -u_t(t, x) + a^{jk}(t, x') D_{jk}u(t, x) \quad (4.1)$$

assuming that  $\text{tr}_2 a$  depends only on  $t$ .

First we generalize Theorem 2.5 of [10] (see also [5]) proved for elliptic equations with  $a^{jk}$  depending only on one coordinate of  $x$ .

**Theorem 4.1.** *There is a constant  $N = N(\delta)$  such that for any  $u \in C_0^\infty$  and  $\lambda \geq 0$  we have*

$$\lambda \|u\|_{L_2} + \sqrt{\lambda} \|Du\|_{L_2} + \|D^2u\|_{L_2} + \|u_t\|_{L_2} \leq N \|Lu - \lambda u\|_{L_2}.$$

The case that  $d = 2$  is taken care of by Lemma 3.1. To prove the theorem in case  $d \geq 3$ , we need some preparations. To start with, we assume without loss of generality that the coefficients  $a^{jk}$  are infinitely differentiable and have bounded derivatives.

Set  $f = Lu - \lambda u$  and let  $\tilde{g}(t, x', \xi'')$  be the Fourier transform of a function  $g(t, x)$  with respect to  $x''$ . Then

$$\begin{aligned} -\tilde{u}_t(t, x', \xi'') + \sum_{j,k=1}^2 a^{jk}(t, x') D_{jk} \tilde{u}(t, x', \xi'') + i \sum_{j=1}^2 B^j(t, x', \xi'') D_j \tilde{u}(t, x', \xi'') \\ - C(t, x', \xi'') \tilde{u}(t, x', \xi'') = \tilde{f}(t, x', \xi''), \end{aligned} \quad (4.2)$$

where

$$B^j(t, x', \xi'') = 2 \sum_{k>2} a^{jk}(t, x') \xi^k, \quad C(t, x', \xi'') = \lambda + \sum_{j,k>2} a^{jk}(t, x') \xi^j \xi^k.$$

In the following lemma  $\xi$  is considered as a parameter.

**Lemma 4.2.** *Let  $d \geq 3$  and  $|\xi''|^2 + \lambda > 0$ . Then we have*

$$|\tilde{u}(t, x', \xi'')| \leq \hat{u}(t, x', \xi''), \quad (4.3)$$

where, for each  $\xi'' \in \mathbb{R}^{d-2}$ ,  $\hat{u}(t, x', \xi'')$  is the unique bounded classical solution of

$$\begin{aligned} -\hat{u}_t(t, x', \xi'') + \sum_{j,k=1}^2 a^{jk}(t, x') D_{jk} \hat{u}(t, x', \xi'') \\ - (\lambda + \delta |\xi''|^2) \hat{u}(t, x', \xi'') = -|\tilde{f}(t, x', \xi'')|. \end{aligned} \quad (4.4)$$

Furthermore,

$$(|\xi''|^2 + \lambda) \|\tilde{u}(\cdot, \cdot, \xi'')\|_{L_2(\mathbb{R} \times \mathbb{R}^2)} \leq N(\delta) \|\tilde{f}(\cdot, \cdot, \xi'')\|_{L_2(\mathbb{R} \times \mathbb{R}^2)}. \quad (4.5)$$

*Proof.* The idea of the proof is to eliminate the first-order terms in (4.2) by using probability theory and Girsanov's transformation. Let  $a'$  be the  $2 \times 2$  matrix, which stands at the upper left corner of  $a$ . Set  $\sigma = \sqrt{2a'}$ . Fix a point  $(t_0, x')$  and let  $x'_t$  be the solution of the following Itô's equation

$$x'_t = x' + \int_0^t \sigma(t_0 - s, x'_s) dw_s$$

on a probability space carrying a two-dimensional Wiener process  $w_t$ . Also set

$$B = (B^1, B^2), \quad \hat{B} = B\sigma^{-1},$$

$$\begin{aligned} \rho_t(\xi'') &= \exp\left(i \int_0^t \hat{B}(t_0 - s, x'_s, \xi'') dw_s\right. \\ &\quad \left. - \int_0^t (C(t_0 - s, x'_s, \xi'') - (1/2)|\hat{B}(t_0 - s, x'_s, \xi'')|^2) ds\right). \end{aligned}$$

As is easy to check by using Itô's formula and (4.2)

$$\begin{aligned} d(\rho_t(\xi'')\tilde{u}(t_0 - t, x'_t, \xi'')) &= \rho_t(\xi'')\tilde{f}(t_0 - t, x'_t, \xi'') dt \\ &\quad + [i\rho_t(\xi'')\tilde{u}(t_0 - t, x'_t, \xi'')\hat{B}(t_0 - t, x'_t, \xi'') \\ &\quad + \rho_t(\xi'')D\tilde{u}(t_0 - t, x'_t, \xi'')\sigma(t_0 - t, x'_t)] dw_t. \end{aligned}$$

We integrate this relation between 0 and  $T \in (0, \infty)$  and take expectations of the result. Then, since  $\tilde{u}$  and  $\hat{B}$  are bounded ( $\xi''$  is fixed), the expectation of the stochastic integral disappears and we obtain

$$\tilde{u}(t_0, x', \xi'') = E\rho_T(\xi'')\tilde{u}(t_0 - T, x'_T, \xi'') - E \int_0^T \rho_t(\xi'')\tilde{f}(t_0 - t, x'_t, \xi'') dt. \quad (4.6)$$

Next observe that

$$\begin{aligned} \delta|\xi''|^2 &\leq \delta|\xi|^2 \leq a^{jk}(t, x')\xi^j\xi^k \\ &= (1/2)|\sigma(t, x')\xi'|^2 + \sum_{j=1}^2 \xi^j B^j(t, x', \xi'') + C(t, x', \xi'') - \lambda. \end{aligned}$$

Substituting  $\xi' \rightarrow \sigma^{-1}(t, x')\xi'$  we see that

$$0 \leq (1/2)|\xi'|^2 + \sum_{j=1}^2 \xi^j \hat{B}^j(t, x', \xi'') + C(t, x', \xi'') - \lambda - \delta|\xi''|^2.$$

Since this is true for any  $\xi'$ , we have

$$\begin{aligned} |(1/2) \sum_{j=1}^2 \xi^j \hat{B}^j(t, x', \xi'')|^2 &\leq (1/2)|\xi'|^2(C(t, x', \xi'') - \lambda - \delta|\xi''|^2), \\ (1/2)|\hat{B}(t, x', \xi'')|^2 &\leq C(t, x', \xi'') - \lambda - \delta|\xi''|^2, \end{aligned}$$

implying that

$$|\rho_t(\xi'')| \leq e^{-(\lambda + \delta|\xi''|^2)t}.$$

Therefore, passing to the limit as  $T \rightarrow \infty$  in (4.6) we obtain

$$\begin{aligned} \tilde{u}(t_0, x', \xi'') &= -E \int_0^\infty \rho_t(\xi'')\tilde{f}(t_0 - t, x'_t, \xi'') dt, \\ |\tilde{u}(t_0, x', \xi'')| &\leq E \int_0^\infty |\tilde{f}(t_0 - t, x'_t, \xi'')| e^{-(\lambda + \delta|\xi''|^2)t} dt =: \hat{u}(t_0, x', \xi''). \end{aligned}$$

Next notice that equation (4.4) has a unique solution in  $W_2^{1,2}(\mathbb{R} \times \mathbb{R}^2)$  by Lemma 3.1. It is a bounded classical solution since  $u \in C_0^\infty$ ,  $a$  is smooth, and  $\tilde{f}$  is Lipschitz continuous in  $(t, x')$ . This solution is the above  $\hat{u}$  which is proved by using Itô's formula in the same way as above. Estimate (4.5) for  $\hat{u}$  in place of  $\tilde{u}$  follows from Lemma 3.1. Having (4.5) for  $\hat{u}$  in place of  $\tilde{u}$  gives us (4.5) as is. The lemma is proved.  $\square$

**Remark 4.3.** Inequality (4.3) can also be proved without using probability theory along the following lines. By the maximum principle, we have  $\hat{u} \geq 0$ . Fix  $\xi''$  and let

$$\Omega = \{(t, x') \in \mathbb{R} \times \mathbb{R}^2 : \tilde{u}(t, x', \xi'') \neq 0\},$$

which is open and bounded. For any  $(t, x') \in \Omega$ ,  $|\tilde{u}|$  has continuous first derivatives in  $(t, x')$  and second derivatives in  $x'$  and we have

$$\begin{aligned} D_t|\tilde{u}| &= \frac{1}{2|\tilde{u}|}(\tilde{u}D_t\tilde{u} + \tilde{u}D_t\tilde{u}), \\ D_j|\tilde{u}| &= \frac{1}{2|\tilde{u}|}(\tilde{u}D_j\tilde{u} + \tilde{u}D_j\tilde{u}), \\ D_{jk}|\tilde{u}| &= \frac{1}{2|\tilde{u}|}(\tilde{u}D_{jk}\tilde{u} + \tilde{u}D_{jk}\tilde{u} + (D_j\tilde{u})D_k\tilde{u} + (D_k\tilde{u})D_j\tilde{u}) \\ &\quad - \frac{1}{4|\tilde{u}|^3}(\tilde{u}D_j\tilde{u} + \tilde{u}D_j\tilde{u})(\tilde{u}D_k\tilde{u} + \tilde{u}D_k\tilde{u}). \end{aligned}$$

Therefore, by (4.2)

$$\begin{aligned} -D_t|\tilde{u}| + \sum_{j,k=1}^2 a^{jk}D_{jk}|\tilde{u}| &= \operatorname{Re}(-D_t|\tilde{u}| + a^{jk}D_{jk}|\tilde{u}|) \\ &= \frac{1}{|\tilde{u}|}\operatorname{Re}(\tilde{u}\tilde{f} + C|\tilde{u}|^2) - \frac{1}{|\tilde{u}|}\sum_{j=1}^2 B^j\operatorname{Im}(\tilde{u}D_j\tilde{u}) \\ &\quad + \sum_{j,k=1}^2 \frac{a^{jk}}{4|\tilde{u}|^3}[2|\tilde{u}|^2((D_j\tilde{u})D_k\tilde{u} + (D_k\tilde{u})D_j\tilde{u}) \\ &\quad - (\tilde{u}D_j\tilde{u} + \tilde{u}D_j\tilde{u})(\tilde{u}D_k\tilde{u} + \tilde{u}D_k\tilde{u})]. \end{aligned}$$

The last sum on the right-hand side above is equal to

$$\sum_{j,k=1}^2 \frac{a^{jk}}{4|\tilde{u}|^3}[|\tilde{u}|^2((D_j\tilde{u})D_k\tilde{u} + (D_k\tilde{u})D_j\tilde{u}) - \tilde{u}^2(D_j\tilde{u})D_k\tilde{u} - \tilde{u}^2(D_j\tilde{u})D_k\tilde{u}]$$

$$\begin{aligned}
 &= \sum_{j,k=1}^2 -\frac{a^{jk}}{4|\tilde{u}|^3}(\tilde{u}D_j\tilde{u} - \tilde{u}D_j\tilde{u})(\tilde{u}D_k\tilde{u} - \tilde{u}D_k\tilde{u}) \\
 &= \sum_{j,k=1}^2 \frac{a^{jk}}{|\tilde{u}|^3} \text{Im}(\tilde{u}D_j\tilde{u})\text{Im}(\tilde{u}D_k\tilde{u}).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 -D_t|\tilde{u}| + \sum_{j,k=1}^2 a^{jk}D_{jk}|\tilde{u}| &\geq -|\tilde{f}| + \left(\lambda + \sum_{j,k>2} a^{jk}(x')\xi^j\xi^k\right)|\tilde{u}| \\
 -\frac{2}{|\tilde{u}|} \sum_{j=1}^2 \sum_{k>2} a^{jk}(x')\xi^k \text{Im}(\tilde{u}D_j\tilde{u}) &+ \sum_{j,k=1}^2 \frac{a^{jk}}{|\tilde{u}|^3} \text{Im}(\tilde{u}D_j\tilde{u})\text{Im}(\tilde{u}D_k\tilde{u}) \\
 &\geq -|\tilde{f}| + \lambda|\tilde{u}| + \delta|\xi''|^2|\tilde{u}|.
 \end{aligned}$$

In the last inequality we used the uniform ellipticity condition. By the maximum principle, we obtain (4.3).

*Proof of Theorem 4.1.* Recall that we may assume  $d \geq 3$ . By squaring both sides of (4.5), integrating with respect to  $\xi''$ , and using Parseval's identity we obtain

$$\lambda^2 \|u\|_{L_2}^2 + \|u_{x''x''}\|_{L_2}^2 \leq N(\delta) \|f\|_{L_2}^2,$$

which along with Corollary 3.2 proves the theorem with a constant  $N$  perhaps depending on  $d$  and  $\delta$ .

To show that it is independent of  $d$ , we use Lemma 3.1 to get

$$\begin{aligned}
 &\|\tilde{u}_{x'x'}(\cdot, \cdot, \xi'')\|_{L_2(\mathbb{R} \times \mathbb{R}^2)}^2 + \|\tilde{u}_t(\cdot, \cdot, \xi'')\|_{L_2(\mathbb{R} \times \mathbb{R}^2)}^2 \\
 &\leq N \left\| -\tilde{u}_t + \sum_{j,k=1}^2 a^{jk}D_{jk}\tilde{u}(\cdot, \cdot, \xi'') \right\|_{L_2(\mathbb{R} \times \mathbb{R}^2)}^2.
 \end{aligned}$$

We also use that

$$|B(t, x', \xi'')| \leq N|\xi''|, \quad C(t, x', \xi'') \leq N(\lambda + |\xi''|^2).$$

Then from Lemma 4.2 and (4.2) we conclude that for  $\xi'' \neq 0$

$$\begin{aligned}
 &\|\tilde{u}_{x'x'}(\cdot, \cdot, \xi'')\|_{L_2(\mathbb{R} \times \mathbb{R}^2)}^2 + \|\tilde{u}_t(\cdot, \cdot, \xi'')\|_{L_2(\mathbb{R} \times \mathbb{R}^2)}^2 \\
 &\leq N(|\xi''|^2 \|\tilde{u}_{x'}(\cdot, \cdot, \xi'')\|_{L_2(\mathbb{R} \times \mathbb{R}^2)}^2 + \|\tilde{f}(\cdot, \cdot, \xi'')\|_{L_2(\mathbb{R} \times \mathbb{R}^2)}^2).
 \end{aligned}$$

Here for any  $\varepsilon > 0$

$$\begin{aligned}
 &|\xi''|^2 \|\tilde{u}_{x'}(\cdot, \cdot, \xi'')\|_{L_2(\mathbb{R} \times \mathbb{R}^2)}^2 \\
 &\leq \varepsilon \|\tilde{u}_{x'x'}(\cdot, \cdot, \xi'')\|_{L_2(\mathbb{R} \times \mathbb{R}^2)}^2 + N\varepsilon^{-1} |\xi''|^4 \|\tilde{u}(\cdot, \cdot, \xi'')\|_{L_2(\mathbb{R} \times \mathbb{R}^2)}^2 \\
 &\leq \varepsilon \|\tilde{u}_{x'x'}(\cdot, \cdot, \xi'')\|_{L_2(\mathbb{R} \times \mathbb{R}^2)}^2 + N \|\tilde{f}(\cdot, \cdot, \xi'')\|_{L_2(\mathbb{R} \times \mathbb{R}^2)}^2. \tag{4.7}
 \end{aligned}$$

It follows that

$$\|\tilde{u}_{x'x'}(\cdot, \cdot, \xi'')\|_{L_2(\mathbb{R} \times \mathbb{R}^2)}^2 + \|\tilde{u}_t(\cdot, \cdot, \xi'')\|_{L_2(\mathbb{R} \times \mathbb{R}^2)}^2 \leq N \|\tilde{f}(\cdot, \cdot, \xi'')\|_{L_2(\mathbb{R} \times \mathbb{R}^2)}^2.$$

Upon integrating this inequality with respect to  $\xi''$  and using Parseval's identity we arrive at

$$\|u_{x'x'}\|_{L_2}^2 + \|u_t\|_{L_2}^2 \leq N \|f\|_{L_2}^2.$$

Going back to (4.7) and integrating again we obtain

$$\|u_{x'x''}\|_{L_2}^2 \leq N \|f\|_{L_2}^2.$$

After that, to finish proving the theorem, it only remains to combine the above estimates and use the interpolation inequality

$$\lambda^2 \|Du\|_{L_2}^2 \leq \lambda^4 \|u\|_{L_2}^2 + \|\Delta u\|_{L_2}^2 = \lambda^4 \|u\|_{L_2}^2 + \|D^2 u\|_{L_2}^2.$$

The theorem is proved.  $\square$

Next we give a proof of Theorem 2.1.

*Proof of Theorem 2.1.* Part iii) follows from the first two by using a scaling argument. By the same reason as in the proof of Lemma 3.1, it suffices to prove i) and ii) for  $T = \infty$ . In case  $T = \infty$ , assertion i) is obtained from Theorem 4.1 in the way outlined in the proof of Theorem 3.3 and assertion ii) is obtained by the method of continuity.  $\square$

Next, we go back to considering the operator  $L$  introduced in (4.1) and present the key results of this section.

**Theorem 4.4.** *Let  $d \geq 3$ ,  $\kappa \geq 2$ , and  $r > 0$ . Assume that  $u \in C_0^\infty(\mathbb{R}^d)$  and  $Lu = 0$  in  $Q_{\kappa r}$ . Then there exist constants  $N = N(d, \delta)$  and  $\alpha = \alpha(d, \delta) \in (0, 1]$  such that for any multi-index  $\gamma = (\gamma', \gamma'')$*

$$\int_{Q_r} |D^{\gamma''} u - (D^{\gamma''} u)_{Q_r}|^2 dx dt \leq N \kappa^{-2\alpha} \left( |D^{\gamma''} u|^2 \right)_{Q_{\kappa r}}.$$

*Proof.* By observing that  $LD^{\gamma''} u = 0$  we see that it suffices to concentrate on  $\gamma = 0$ . By using scaling we reduce the general situation to the one in which  $r = 1$ . By Lemma 4.2.4 of [13] and Theorem 7.21 of [21]

$$\operatorname{osc}_{Q_{1/\kappa}} u \leq N \kappa^{-\alpha} \|u\|_{L_2(Q_1)}$$

with  $\alpha$  and  $N$  as in the statement. Scaling this estimate shows that

$$\operatorname{osc}_{Q_1} u \leq N \kappa^{-\alpha} (|u|^2)_{Q_\kappa}^{1/2}.$$

It only remains to observe that

$$\int_{Q_1} |u - (u)_{Q_1}|^2 dx dt \leq N (\operatorname{osc}_{Q_1} u)^2.$$

The theorem is proved.  $\square$

**Theorem 4.5.** *Let  $d \geq 3$  and let  $\alpha$  be the constant in Theorem 4.4. Then there is a constant  $N$  depending only on  $d, \delta$  such that for any  $u \in W_{2,\text{loc}}^{1,2}$ ,  $r \in (0, \infty)$ , and  $\kappa \geq 4$*

$$\begin{aligned} & \left( |u_{x''x''}(t, x) - (u_{x''x''})_{Q_r}|^2 \right)_{Q_r} \\ & \leq N\kappa^{d+2} (|Lu|^2)_{Q_{\kappa r}} + N\kappa^{-2\alpha} (|u_{x''x''}|^2)_{Q_{\kappa r}}. \end{aligned} \quad (4.8)$$

*Proof.* Fix an  $r \in (0, \infty)$  and a  $\kappa \geq 4$ . We may certainly assume that  $a^{jk}$  are infinitely differentiable and have bounded derivatives. Also changing  $u$  for large  $|t| + |x|$  does not affect (4.8). Therefore, we may assume that  $u \in W_p^{1,2}$  and moreover  $u \in C_0^\infty$ .

Now we define  $f = Lu - \lambda u \in C_0^\infty$ . Take a  $\zeta \in C_0^\infty$  such that  $\zeta = 1$  in  $Q_{\kappa r/2}$  and  $\zeta = 0$  outside the closure of  $Q_{\kappa r} \cup (-Q_{\kappa r})$ . Define  $v$  to be the unique  $W_2^{1,2}((S, T) \times \mathbb{R}^d)$ -solution of the equation

$$Lv = (1 - \zeta)f$$

with zero initial condition at  $t = S$ , where  $S < -\kappa r$  and  $T > \kappa r$  are such that  $u(t, x) = 0$  for  $t \notin (S, T)$ . By classical theory we know that such a  $v$  indeed exists and is unique and infinitely differentiable. Since  $(1 - \zeta)f = 0$  in  $Q_{\kappa r/2}$  and  $\kappa/2 \geq 2$ , by Theorem 4.4 with  $v$  in place of  $u$  we obtain

$$\begin{aligned} & \left( |v_{x''x''} - (v_{x''x''})_{Q_r}|^2 \right)_{Q_r} \\ & \leq N\kappa^{-2\alpha} (|v_{x''x''}|^2)_{Q_{\kappa r/2}} \leq N\kappa^{-2\alpha} (|v_{x''x''}|^2)_{Q_{\kappa r}}. \end{aligned} \quad (4.9)$$

On the other hand, obviously  $\bar{v}(t, x) := v(t, x)I_{(S, T)}(t)$  is of class  $W_2^{1,2}(\mathbb{R}_T^{d+1})$  and the function  $w := u - \bar{v} \in W_2^{1,2}(\mathbb{R}_T^{d+1})$  satisfies

$$Lw = \zeta f$$

in  $\mathbb{R}_0^{d+1}$ . Therefore, by Theorem 2.1 (iii),

$$\int_{\mathbb{R}_0^{d+1}} |w_{xx}|^2 dx dt \leq N \int_{\mathbb{R}_0^{d+1}} |\zeta f|^2 dx dt \leq N \int_{Q_{\kappa r}} |f|^2 dx dt,$$

which implies

$$(|w_{xx}|^2)_{Q_{\kappa r}} \leq N (|f|^2)_{Q_{\kappa r}}, \quad (4.10)$$

and

$$(|w_{xx}|^2)_{Q_r} \leq N\kappa^{d+2} (|f|^2)_{Q_{\kappa r}}. \quad (4.11)$$

Combining (4.9)-(4.11) together, we conclude

$$\begin{aligned} & \left( |u_{x''x''} - (u_{x''x''})_{Q_r}|^2 \right)_{Q_r} \leq \left( |v_{x''x''} - (v_{x''x''})_{Q_r}|^2 \right)_{Q_r} + N (|w_{x''x''}|^2)_{Q_r} \\ & \leq N\kappa^{-2\alpha} (|v_{x''x''}|^2)_{Q_{\kappa r}} + N\kappa^{d+2} (|f|^2)_{Q_{\kappa r}} \end{aligned}$$

$$\leq N\kappa^{-2\alpha} (|u_{x''x''}|^2)_{Q_{\kappa r}} + N\kappa^{d+2} (|f|^2)_{Q_{\kappa r}}.$$

The theorem is proved.  $\square$

## 5. PROOF OF THEOREM 2.6

We shall prove Theorem 2.6 in this section. We consider the operator

$$Lu = -u_t + a^{jk} D_{jk}u + b^j D_j u + cu,$$

where  $a^{jk}$  satisfy Assumption 2.5 ( $\gamma$ ) with some  $\gamma > 0$  to be specified later. Assertion (iii) of Theorem 2.6 is obtained from (i) and (ii) by using scaling. Assertion (ii) is obtained from (i) by the method of continuity. If  $d = 2$ , assertion (i) is derived from Lemma 3.1 in a standard way alluded to a few times before. Therefore, it only remains to prove assertion (i) assuming that  $d \geq 3$ .

Set

$$L_0 u = -u_t + a^{jk} D_{jk} u.$$

First we generalize Theorem 4.5.

**Theorem 5.1.** *Let  $\alpha$  be the constant in Theorem 4.4,  $\gamma > 0$ ,  $\tau, \sigma \in (1, \infty)$ ,  $1/\tau + 1/\sigma = 1$ . Take a  $u \in W_2^{1,2}$  and set  $f = L_0 u$ . Then under Assumption 2.5 ( $\gamma$ ) there exists a positive constant  $N$  depending only on  $d$ ,  $\delta$ , and  $\tau$  such that, for any  $(t_0, x_0) \in \mathbb{R}^{d+1}$ ,  $r \in (0, \infty)$ , and  $\kappa \geq 4$ ,*

$$\begin{aligned} & (|u_{x''x''}(t, x) - (u_{x''x''})_{Q_r(t_0, x_0)}|^2)_{Q_r(t_0, x_0)} \\ & \leq N\kappa^{d+2} (|f|^2)_{Q_{\kappa r}(t_0, x_0)} + N\kappa^{d+2} \gamma^{1/\sigma} (|u_{xx}|^{2\tau})_{Q_{\kappa r}(t_0, x_0)}^{1/\tau} \\ & \quad + N\kappa^{-2\alpha} (|u_{x''x''}|^2)_{Q_{\kappa r}(t_0, x_0)}, \end{aligned} \quad (5.1)$$

provided that  $u$  vanishes outside  $Q_{R_0}$ .

*Proof.* We fix  $(t_0, x_0) \in \mathbb{R}^{d+1}$ ,  $\kappa \geq 4$ , and  $r \in (0, \infty)$ . Choose  $Q = (s_1, s_2) \times B' \times B''$  to be  $Q_{\kappa r}(t_0, x_0)$  if  $\kappa r < R_0$  and  $Q_{R_0}$  if  $\kappa r \geq R_0$ . Recall the definition of  $\bar{a}(t, x')$  given in Assumption 2.5, set  $y'_0$  to be the center of  $B'$ , and introduce  $\bar{a}_0(t) = \bar{a}(t, y'_0)$ ,

$$\mathbf{a}^{jk} = \frac{\bar{a}^{jk}}{\text{tr}_2 \bar{a}} \text{tr}_2 \bar{a}_0, \quad \hat{f} = \mathbf{a}^{jk} D_{jk} u - u_t.$$

Obviously,  $\mathbf{a}$  depends only on  $(t, x')$ ,  $\text{tr}_2 \mathbf{a} = \text{tr}_2 \bar{a}_0$  depends only on  $t$  and takes values between  $2\delta$  and  $2\delta^{-1}$ , and

$$\begin{aligned} & \int_Q |a^{jk} - \mathbf{a}^{jk}| dx dt \leq N \int_Q |a^{jk} \text{tr}_2 \bar{a} - \mathbf{a}^{jk} \text{tr}_2 \bar{a}| dx dt \\ & \leq N \int_Q |a^{jk} - \bar{a}^{jk}| dx dt + N \int_Q |\text{tr}_2 \bar{a} - \text{tr}_2 \bar{a}_0| dx dt \leq N\gamma. \end{aligned} \quad (5.2)$$

Also

$$\hat{f} = (\mathbf{a}^{jk} - a^{jk}) D_{jk}u + f.$$

Then by Theorem 4.5 with an appropriate translation and  $\mathbf{a}$  in place of  $a$ ,

$$\begin{aligned} & \int_{Q_r(t_0, x_0)} |u_{x''x''} - (u_{x''x''})_{Q_r(t_0, x_0)}|^2 dx \\ & \leq N\kappa^{d+2} \left( |\hat{f}|^2 \right)_{Q_{\kappa r}(t_0, x_0)} + N\kappa^{-2\alpha} \left( |u_{x''x''}|^2 \right)_{Q_{\kappa r}(t_0, x_0)}, \end{aligned} \quad (5.3)$$

where  $N$  and  $\alpha$  depend only on  $d$  and  $\delta$ . By the definition of  $\hat{f}$ ,

$$\int_{Q_{\kappa r}(t_0, x_0)} |\hat{f}|^2 dx dt \leq 2 \int_{Q_{\kappa r}(t_0, x_0)} |f|^2 dx dt + 2I, \quad (5.4)$$

where

$$\begin{aligned} I &= \int_{Q_{\kappa r}(t_0, x_0)} |(\mathbf{a}^{jk} - a^{jk}) D_{jk}u|^2 dx dt \\ &= \int_{Q_{\kappa r}(t_0, x_0) \cap Q_{R_0}} |(\mathbf{a}^{jk} - a^{jk}) D_{jk}u|^2 dx dt. \end{aligned}$$

By Hölder's inequality, we have

$$I \leq N I_1^{1/\sigma} I_2^{1/\tau}, \quad (5.5)$$

where

$$I_1 = \sum_{j,k} \int_{Q_{\kappa r}(t_0, x_0) \cap Q_{R_0}} |\mathbf{a}^{jk} - a^{jk}|^{2\sigma} dx dt, \quad I_2 = \int_{Q_{\kappa r}(t_0, x_0)} |u_{xx}|^{2\tau} dx dt.$$

According to (5.2) we have

$$I_1 \leq \sum_{j,k} \int_Q |\mathbf{a}^{jk} - a^{jk}|^{2\sigma} dx dt \leq N\gamma|Q| \leq N(\kappa r)^{d+2}\gamma.$$

This together with (5.3)-(5.5) yields (5.1). The theorem is proved.  $\square$

**Remark 5.2.** Assume that  $a^{jk}$  ( $= a^{kj}$ ) are independent of  $x''$  for  $j = 1, 2$  and  $k = 1, \dots, d$ . Also assume that  $\text{tr}_2 a$  depends only on  $t$ . Then in the above proof we have  $\mathbf{a}^{jk} = \bar{a}^{jk}$  for all  $j, k$  and  $\mathbf{a}^{jk} = a^{jk}$  for  $j = 1, 2$  and  $k = 1, \dots, d$ . Therefore, in the definition of  $I$  we only need to sum over  $j, k \geq 3$ , so that only  $u_{x''x''}$  are involved in its estimate. It follows that in this case we can replace  $|u_{xx}|^{2\tau}$  in (5.1) with  $|u_{x''x''}|^{2\tau}$ .

**Lemma 5.3.** Let  $\theta_0$  be the constant in Lemma 3.1,  $p \in (2, 2 + \theta_0)$  and  $f \in L_p$ . Then there exist strictly positive constants  $\gamma$  and  $N$  both depending only on  $d, p$ , and  $\delta$  such that under Assumption 2.5 ( $\gamma$ ), for any  $u \in W_p^{1,2}$  vanishing outside  $Q_{R_0}$  and satisfying  $L_0 u = f$ , we have

$$\|u_t\|_{L_p} + \|D^2 u\|_{L_p} \leq N \|f\|_{L_p}.$$

*Proof.* Let  $\alpha$  be the constant in Theorem 4.4. Choose  $\tau \in (1, \infty)$  such that  $p > 2\tau$ . Inequality (5.1) implies that on  $\mathbb{R}^{d+1}$

$$\begin{aligned} u_{x''x''}^\# &\leq N\kappa^{(d+2)/2}\mathbb{M}^{1/2}(|f|^2) \\ &+ N\kappa^{(d+2)/2}\gamma^{1/(2\sigma)}\mathbb{M}^{1/(2\tau)}(|u_{xx}|^{2\tau}) + N\kappa^{-\alpha}\mathbb{M}^{1/2}(|u_{xx}|^2). \end{aligned}$$

We apply the Fefferman-Stein theorem on sharp functions and the Hardy-Littlewood maximal function theorem to the above inequality to get

$$\begin{aligned} \|u_{x''x''}\|_{L_p} &\leq N\|u_{x''x''}^\#\|_{L_p} \leq N\kappa^{(d+2)/2}\|\mathbb{M}(|f|^2)\|_{L_{p/2}}^{1/2} \\ &+ N\kappa^{(d+2)/2}\gamma^{1/(2\sigma)}\|\mathbb{M}(|u_{xx}|^{2\tau})\|_{L_{p/(2\tau)}}^{1/(2\tau)} + N\kappa^{-\alpha}\|\mathbb{M}(|u_{xx}|^2)\|_{L_{p/2}}^{1/2} \\ &\leq N\kappa^{(d+2)/2}\|f\|_{L_p} + N\left(\kappa^{(d+2)/2}\gamma^{1/(2\sigma)} + \kappa^{-\alpha}\right)\|u_{xx}\|_{L_p}, \end{aligned}$$

where in the last inequality we use the fact that  $p > 2\tau$ . From this estimate and the last assertion of Theorem 3.3 we have

$$\begin{aligned} \|u_{xx}\|_{L_p} + \|u_t\|_{L_p} &\leq N\kappa^{(d+2)/2}\|f\|_{L_p} \\ &+ N\left(\kappa^{(d+2)/2}\gamma^{1/(2\sigma)} + \kappa^{-\alpha}\right)\|u_{xx}\|_{L_p}. \end{aligned}$$

To finish the proof of the lemma, it suffices to choose a big enough  $\kappa$  and then a small  $\gamma$  so that

$$N\left(\kappa^{(d+2)/2}\gamma^{1/(2\sigma)} + \kappa^{-\alpha}\right) \leq 1/2.$$

□

Now we are in the position to prove Theorem 2.6.

*Proof of Theorem 2.6.* As is pointed out in the beginning of the section, it suffices to prove assertion i) for  $d \geq 3$ . Similarly to the proof of assertions i) and ii) of Theorem 2.1, we only need to prove (2.3) for  $T = \infty$  and  $u \in C_0^\infty$ . This in turn is obtained from Lemma 5.3 by using a partition of unity and an idea of Agmon (see, for instance, Section 6.3 of [17]). □

We finish the paper with a statement valid for any  $p > 2$  which partially generalizes Lemma 5.3. Its proof is an immediate consequence of Remark 5.2 and the argument from the proof of Lemma 5.3.

**Theorem 5.4.** *Assume that  $d \geq 3$  and  $a^{jk}$  ( $= a^{kj}$ ) are independent of  $x''$  for  $j = 1, 2$  and  $k = 1, \dots, d$ . Also assume that  $\text{tr}_2 a$  depends only on  $t$ . Let  $p > 2$  and  $u \in W_p^{1,2}$  be such that  $u$  vanishes outside  $Q_{R_0}$ . Then there are a strictly positive constant  $\gamma$  and  $N$  both depending only on  $d, p$ , and  $\delta$  such that under Assumption 2.5 ( $\gamma$ ) we have*

$$\|D_{x''}^2 u\|_{L_p} \leq N\|L_0 u\|_{L_p}.$$

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DIVISION OF APPLIED MATHEMATICS, BROWN UNIVERSITY, 182 GEORGE STREET, PROVIDENCE, RI 02912

*E-mail address:* Hongjie\_Dong@brown.edu

127 VINCENT HALL, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455

*E-mail address:* krylov@math.umn.edu