

Well-posedness for a transport equation with nonlocal velocity

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Abstract

We study a one-dimensional transport equation with nonlocal velocity which was recently considered in the work of Córdoba, Córdoba and Fontelos [4]. We show that in the subcritical and critical cases the problem is globally well-posed with arbitrary initial data in $H^{\max\{3/2-\gamma, 0\}}$. While in the supercritical case, the problem is locally well-posed with initial data in $H^{3/2-\gamma}$, and is globally well-posed under a smallness assumption. Some polynomial-in-time decay estimates are also discussed. These results improve some previous results in [4].

Key words: Finite-time singularities, local and global well-posedness, nonlocal velocity.

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1 Introduction and the main theorems

One-dimensional models of the 3D Navier-Stokes equations or the Euler equations in the vorticity formulation have been studied by many mathematicians in the last twenty years. The systematical study of such models was initiated by Contantin, Lax and Majda in [7]. Here we are interested in the initial value problem of the following equation:

$$\begin{cases} \theta_t + u\theta_x + \nu(-\Delta)^{\gamma/2}\theta = 0 & \text{on } \mathbb{R} \times (0, \infty), \\ \theta(0, x) = \theta_0(x) & x \in \mathbb{R}, \end{cases} \quad (1.1)$$

where $\gamma \in (0, 2]$ is a fixed parameter, ν is a nonnegative number and the velocity u is determined by the Hilbert transform of θ :

$$u = H\theta = \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{\theta(y)}{x-y} dy.$$

This model was recently studied in the work of Córdoba, Córdoba and Fontelos in [4]. The appearance of the Hilbert transform makes the transport equation non-local. The cases $\gamma > 1$, $\gamma = 1$ and $\gamma < 1$ are called subcritical, critical and supercritical, respectively. They showed that for the inviscid equation, i.e. $\nu = 0$, there is a formation of finite-time blowup for a family of initial data. While for $\nu > 0$, the equation has global regularity with *positive* H^2 initial data in the subcritical case $\gamma > 1$, and also in the critical case $\gamma = 1$ for *small* initial data. It remains open whether the positiveness assumption in the subcritical case and smallness assumption in the critical case can be dropped or not.

The formation of singularities of a similar model but in a divergence form

$$\theta_t + (u\theta)_x + \nu(-\Delta)^{\gamma/2}\theta = 0 \quad (1.2)$$

was considered by Chae, Córdoba, Córdoba and Fontelos in [6]. For other results about various 1D models, we refer the readers to Baker and Morlet [1], Kiselev, Nazarov and Shterenberg [14], Morlet [16], Sakajo [18], Schochet [19], Wegert and Vasudeva Murthy [20] and references therein.

The main objective of the present article is to answer the aforementioned open question. More specifically, we shall show that in the subcritical and critical cases the problem is globally well-posed with arbitrary H^{γ_0} initial data, where $\gamma_0 = \max\{3/2 - \gamma, 0\}$. In the supercritical case, the problem is locally well-posed with $H^{3/2-\gamma}$ initial data, and is globally well-posed under a smallness assumption. Thus, we give an affirmative answer to that question.

Let us point out the main difficulty of the problem. Since the velocity u is not divergence free and the nonlinearity is not in a divergence form, we no longer have energy-like inequality, which naturally gives control of the spatial L^2 norm of solutions. To circumvent this difficulty, the authors of [4] considered only positive initial data and thus control the L^2 norm by showing its monotonicity in time.

The novelty of this article is that we remove the positiveness assumption and relax the regularity condition of the initial data. This is accomplished by using bootstrap and *backward* bootstrap arguments with the aid of the para-differential calculus, and more specifically, several multiplicative inequalities and commutator estimates in the spirit of [15] and [9]. Roughly speaking, by backward bootstrap, we mean the following. First by using para-differential calculus we obtain an estimate of some homogeneous Sobolev norm of solutions. Then we make use of this information to control the L^2 norm.

To the best of our knowledge, the method developed here seems to be new. More importantly, it can be carried over to many other equations, like the Burger's equa-

tions and the (modified) quasi-geostrophic equations without the divergence free property of the velocity; see [11] and [12]. We remark that the divergence free property was heavily used in the derivation of various well-posedness results of equations such as the quasi-geostrophic equations. Our proofs show that in some cases this property is inessential.

Next we state our main results, which, for readers' convenience, will be stated again in the following sections.

For all the three cases, we have the following local well-posedness result.

Theorem 1.1 (Local well-posedness). Let $\gamma \in (0, 3/2]$ and $\theta_0 \in H^{\gamma_0}(\mathbb{R})$, where $\gamma_0 = 3/2 - \gamma$. Then there exists $T_1 > 0$ such that the initial value problem of (1.1) has a unique solution

$$\theta(t, x) \in C([0, T_1]; H^{\gamma_0}(\mathbb{R})) \cap L^2(0, T_1; H^{\gamma_0 + \gamma/2}(\mathbb{R})).$$

Moreover the solution θ satisfies

$$\sup_{0 < t < T_1} t^{\beta/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^{\gamma_0 + \beta}} < \infty,$$

for any $\beta \geq 0$ and

$$\lim_{t \rightarrow 0} t^{\beta/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^{\gamma_0 + \beta}} = 0,$$

for any $\beta > 0$.

Theorem 1.2 (Local well-posedness). For $\gamma \in (3/2, 2]$, the conclusion of Theorem 1.1 remains true with $\gamma_0 = 0$.

Remark 1.3. Note that for $\gamma \in (0, 3/2]$, $H^{\gamma_0}(\mathbb{R})$ is the critical Sobolev space and $\dot{H}^{\gamma_0}(\mathbb{R})$ is a scaling invariant space of (1.1).

Our next theorem is the global well-posedness of (1.1) in the critical and subcritical

cases.

Theorem 1.4 (Global well-posedness). Let $\gamma \in [1, 2]$ and $\theta_0 \in H^{\gamma_0}(\mathbb{R})$, where $\gamma_0 = \max\{0, 3/2 - \gamma\}$. Then the initial value problem (1.1) has a unique (smooth) global solution

$$\theta(t, x) \in C([0, \infty); H^{\gamma_0}(\mathbb{R})) \cap L_{loc}^2(0, \infty; H^{\gamma_0 + \gamma/2}(\mathbb{R})). \quad (1.3)$$

This theorem is highly in contrast with a result in [6], in which a finite-time blowup was observed for (1.2) with certain initial data when $\nu \in [0, 1]$ and $\gamma = 1$. We note that, furthermore, the solution in Theorem 1.4 decay in time polynomially in \dot{H}^β and C^k for any $\beta > 0$ and integer $k \geq 0$ provided that θ_0 is nonpositive; see Section 7.3.

In contrary, for the inviscid equation, i.e. $\nu = 0$, we recall the following interesting result by Córdoba, Córdoba and Fontelos.

Theorem 1.5 (Córdoba, Córdoba and Fontelo [4], [5]). Assume $\nu = 0$, $\theta_0 \in C_0^{1+\delta}$ be negative and compactly supported. Then there is no global in time, locally bounded solutions to (1.1).

The remaining part of the article is organized as follows: we define some notations and recall some preliminary estimates in the next section. Section 4 and 5 are devoted to the proofs of the local well-posedness results (Theorem 1.1 and 1.2). After deriving a Beale-Kato-Majda type blow-up criterion, the global regularity result (Theorem 1.4) is proved in Section 6 by adapting the method in [13] with suitable modifications. We end this article by making a few remarks in Section 7.

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3 Preliminaries

Throughout this paper, we employ the letter $C(\alpha, \beta, \dots)$ to denote a positive constant depending on parameters α, β, \dots . Sometimes we omit α, β, \dots if there is no confusion. We also denote $c(T)$ to be a positive constant satisfying $c(T) \rightarrow 0$ as $T \rightarrow 0$. It should be understood that C and c may vary from line to line.

First we recall the Littlewood-Paley decomposition. For any integer j , define Δ_j to be the Littlewood-Paley projection operator with $\Delta_j v = \phi_j * v$, where

$$\begin{aligned} \hat{\phi}_j(\xi) &= \hat{\phi}(2^{-j}\xi), \quad \hat{\phi} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}), \quad \hat{\phi} \geq 0, \\ \text{supp} \hat{\phi} &\subset \{\xi \in \mathbb{R}^2 \mid 1/2 \leq |\xi| \leq 2\}, \quad \sum_{j \in \mathbb{Z}} \hat{\phi}_j(\xi) = 1 \text{ for } \xi \neq 0. \end{aligned}$$

We have the Littlewood-Paley decomposition

$$v(\cdot, t) = \sum_{j \in \mathbb{Z}} \Delta_j v(\cdot, t).$$

For any $p \in [1, \infty)$ and $s \geq 0$, as usual we denote $\dot{W}^{s,p}$ and $W^{s,p}$ to be the homogeneous and inhomogeneous Sobolev spaces with norms

$$\|v\|_{\dot{W}^{s,p}} := \left\| \left(\sum_{k \in \mathbb{Z}} |2^{ks} \Delta_k v|^2 \right)^{1/2} \right\|_{L^p} \sim \|\Lambda^s v\|_{L^p},$$

$$\|v\|_{W^{s,p}} := \|v\|_{\dot{W}^{s,p}} + \|v\|_{L^p}.$$

When $p = 2$, we use \dot{H}^s and H^s instead of $\dot{W}^{s,p}$ and $W^{s,p}$. Also denote $\Lambda = (-\Delta)^{1/2}$ and $\tilde{\Delta}_j = \sum_{|k-j| \leq 1} \Delta_k$.

Define $G(t, x) = G_\gamma(t, x)$ by its Fourier transform $\widehat{G}_\gamma(t, \xi) = e^{-t|\xi|^\gamma}$ for $t > 0$. Then $G_\gamma(t, x)$ is the fundamental solution of the linear operator $\partial_t + (-\Delta)^{\gamma/2}$. It also has the scaling property

$$G_\gamma(t, x) = t^{-\frac{1}{\gamma}} G_\gamma(1, xt^{-\frac{1}{\gamma}}).$$

We shall use the next two standard linear estimates, the proofs of which can be found, for example, in [15].

Lemma 3.1. For any $\gamma > 0$ and any function $v \in L^2$, we have

$$e^{-2^{\gamma j+1}\lambda' t} \|\Delta_j v\|_{L^2} \leq \|G(t, \cdot) * \Delta_j v\|_{L^2} \leq e^{-2^{\gamma j+1}\lambda t} \|\Delta_j v\|_{L^2}, \quad (3.1)$$

where λ and λ' are some positive constants depending only on γ .

Lemma 3.2. For any $\gamma > 0$ and $s \geq 0$, there exists a positive constant C depending only on s and γ such that for any $v \in L^2$, we have

$$\sup_{t \in (0, \infty)} t^{s/\gamma} \|G(t, \cdot) * v\|_{\dot{H}_x^s} \leq C \|v\|_{L^2}, \quad (3.2)$$

$$\lim_{t \rightarrow 0} t^{s/\gamma} \|G(t, \cdot) * v\|_{\dot{H}_x^s} = 0. \quad (3.3)$$

Moreover, for $s \in [0, \gamma/2]$ we have

$$\|G(t, \cdot) * v\|_{L_t^{\gamma/s} \dot{H}_x^s} \leq C \|v\|_{L^2} \quad (3.4)$$

4 Local well-posedness and smoothness when $\gamma \in (0, 3/2]$

This section is devoted to the proof of the local well-posedness of (1.1) when the range of γ is $(0, 3/2]$. Here is the main result of this section.

Theorem 4.1. Let $\gamma \in (0, 3/2]$ and $\theta_0 \in H^{\gamma_0}(\mathbb{R})$, where $\gamma_0 = 3/2 - \gamma$. Then there

exists $T_1 > 0$ such that the initial value problem of (1.1) has a unique solution

$$\theta(t, x) \in C([0, T_1]; H^{\gamma_0}(\mathbb{R})) \cap L^2(0, T_1; H^{\gamma_0 + \gamma/2}(\mathbb{R})).$$

Moreover the solution θ satisfies

$$\sup_{0 < t < T_1} t^{\beta/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^{\gamma_0 + \beta}} < \infty, \quad (4.1)$$

for any $\beta \geq 0$ and

$$\lim_{t \rightarrow 0} t^{\beta/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^{\gamma_0 + \beta}} = 0, \quad (4.2)$$

for any $\beta > 0$.

We follow pretty much the strategies in [15] and [9]. Equation (1.1) has a different structure in the nonlinear term from that of the dissipative quasi-geostrophic equations or the Navier-Stokes equation, since the velocity u in the nonlinear term is not divergence free. The divergence free property is heavily used and seems indispensable in [15], [9] and many other articles in this subject. We circumvent this technical difficulty by using bootstrap and backward bootstrap arguments with the aid of the localization property of Littlewood-Paley projections and Lemma A.5 proved in the appendix.

4.1 Basic a priori estimates

First we establish a few a priori bounds. Let θ be a solution of (1.1). Denote $\theta_j = \Delta_j \theta$. For each $j \in \mathbb{Z}$, we apply the operator Δ_j to the both sides of the first equation in (1.1) and get

$$\partial_t \theta_j + \Delta_j(u \theta_x) + \nu \Lambda^\gamma \theta_j = 0.$$

Thus,

$$\partial_t \theta_j + u(\theta_j)_x + \nu \Lambda^\gamma \theta_j = [u, \Delta_j] \theta_x. \quad (4.3)$$

After multiplying both sides of (4.3) by θ_j , taking integration in x and integrating by parts, we obtain by using Bernstein's inequality, and the localization property of Littlewood-Paley projections that for some constant $\lambda > 0$ independent of j ,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta_j\|_{L^2}^2 + 2^{\gamma j} \nu \lambda \|\theta_j\|_{L^2}^2 &\leq \int_{\mathbb{R}^2} \left([u, \Delta_j] \theta_x + u_x \theta_j / 2 \right) \theta_j \, dx \\ &= \int_{\mathbb{R}^2} \left(\tilde{\Delta}_j [u, \Delta_j] \theta_x + \tilde{\Delta}_j (u_x \theta_j / 2) \right) \theta_j \, dx \\ &\leq \left(\|\tilde{\Delta}_j [u, \Delta_j] \theta_x\|_{L^2} + \|\tilde{\Delta}_j (u_x \theta_j)\|_{L^2} \right) \|\theta_j\|_{L^2}. \end{aligned}$$

And therefore, after replacing λ by $\lambda/2$,

$$\frac{d}{dt} \|\theta_j\|_{L^2} + 2^{\gamma j} \nu \lambda \|\theta_j\|_{L^2} \leq 2 \|\tilde{\Delta}_j [u, \Delta_j] \theta_x\|_{L^2} + 2 \|\tilde{\Delta}_j (u_x \theta_j)\|_{L^2}. \quad (4.4)$$

Similarly,

$$\frac{d}{dt} \|\theta_j\|_{L^2} + 2^{\gamma j} \nu \bar{\lambda} \|\theta_j\|_{L^2} \geq -2 \|\tilde{\Delta}_j [u, \Delta_j] \theta_x\|_{L^2} - 2 \|\tilde{\Delta}_j (u_x \theta_j)\|_{L^2}. \quad (4.5)$$

for some constant $\bar{\lambda} > 0$ independent of j . Gronwall's inequality together with (4.4) yields

$$\begin{aligned} \|\theta_j(t, \cdot)\|_{L^2} &\leq e^{-2^{\gamma j} \nu \lambda t} \|\theta_j(0)\|_{L^2} \\ &+ 2 \int_0^t e^{-2^{\gamma j} \nu \lambda (t-s)} \left(\|\tilde{\Delta}_j [u, \Delta_j] \theta_x(s, \cdot)\|_{L^2} + \|\tilde{\Delta}_j (u_x \theta_j)(s, \cdot)\|_{L^2} \right) ds. \end{aligned} \quad (4.6)$$

Multiply both sides of (4.6) by $2^{(3/2-4\gamma/7)j}$ and get

$$2^{(3/2-4\gamma/7)j} \|\theta_j(t, \cdot)\|_{L^2} \leq I_0 + C(I_1 + I_2), \quad (4.7)$$

where

$$\begin{aligned} I_0 &= 2^{(3/2-4\gamma/7)j} e^{-2^{\gamma j} \nu \lambda t} \|\theta_j(0, \cdot)\|_{L^2} \\ I_1 &= \int_0^t 2^{(3/2-4\gamma/7)j} e^{-2^{\gamma j} \nu \lambda (t-s)} \|\tilde{\Delta}_j [u, \Delta_j] \theta_x(s, \cdot)\|_{L^2} \, ds, \\ I_2 &= \int_0^t 2^{(3/2-4\gamma/7)j} e^{-2^{\gamma j} \nu \lambda (t-s)} \|\tilde{\Delta}_j (u_x \theta_j)(s, \cdot)\|_{L^2} \, ds. \end{aligned}$$

For I_1 , we use Lemma A.4 with $m = 0$, $\alpha_1 = 3/2 - 4\gamma/7$ and $\alpha_2 = 1/2 - 4\gamma/7$ to obtain

$$\begin{aligned} I_1 &\leq Cc_j \int_0^t 2^{(4\gamma/7)j} e^{-2\gamma j \nu \lambda(t-s)} \|u(s, \cdot)\|_{\dot{H}^{3/2-4\gamma/7}} \|\theta(s, \cdot)\|_{\dot{H}^{3/2-4\gamma/7}} ds \\ &\leq Cc_j \int_0^t (t-s)^{-4/7} \|\theta(s, \cdot)\|_{\dot{H}^{3/2-4\gamma/7}}^2 ds, \end{aligned} \quad (4.8)$$

where $\{c_j\} \in l^2$ with $\|c_j\|_{l^2} \leq 1$, and in the second inequality we used the boundedness of the Hilbert transform in Sobolev spaces. Similarly, we apply Lemma A.5 with $m = 0$, $\alpha_1 = 1/2 - 4\gamma/7$ and $\alpha_2 = 3/2 - 4\gamma/7$ to estimate I_2

$$\begin{aligned} I_2 &\leq Cc_j \int_0^t 2^{(4\gamma/7)j} e^{-2\gamma j \nu \lambda(t-s)} \|u(s, \cdot)\|_{\dot{H}^{3/2-4\gamma/7}} \|\theta(s, \cdot)\|_{\dot{H}^{3/2-4\gamma/7}} ds \\ &\leq Cc_j \int_0^t (t-s)^{-4/7} \|\theta(s, \cdot)\|_{\dot{H}^{3/2-4\gamma/7}}^2 ds, \end{aligned} \quad (4.9)$$

Denote $\Theta(t) = G(t, \cdot) * \theta_0$. We take the l^2 norm of both sides of (4.7) with respect to j , and then take the $L^{7/3}$ norm in t . For some $T_1 > 0$ to be specified later, by using Lemma 3.1 and 3.2, it is easily seen

$$\left\| \|I_0\|_{l^2} \right\|_{L^{7/3}(0, T_1)} \leq C \|\Theta\|_{L_t^{7/3}(0, C_1 T_1) \dot{H}_x^{3/2-4\gamma/7}} \leq c(T_1) \|\theta_0\|_{\dot{H}^{\gamma_0}}, \quad (4.10)$$

where C_1 is a constant independent of t . Recall that $c(T_1) \rightarrow 0$ as $T_1 \rightarrow 0$. Due to the fractional integration, it holds that

$$\left\| \|I_1\|_{l^2} \right\|_{L^{7/3}(0, T_1)} + \left\| \|I_2\|_{l^2} \right\|_{L^{7/3}(0, T_1)} \leq C \|\theta\|_{L_t^{7/3}(0, T_1) \dot{H}_x^{3/2-4\gamma/7}}^2. \quad (4.11)$$

Combining (4.7), (4.10) and (4.11) together yields the following a priori estimate

$$\|\theta\|_{L_t^{7/3}(0, T_1) \dot{H}_x^{3/2-4\gamma/7}} \leq c(T_1) \|\theta_0\|_{\dot{H}^{\gamma_0}} + C \|\theta\|_{L_t^{7/3}(0, T_1) \dot{H}_x^{3/2-4\gamma/7}}^2. \quad (4.12)$$

Similarly, we multiply both sides of (4.6) by $2^{(3/2-\gamma/2)j}$, take l^2 norm and get

$$\|\theta\|_{L_t^2(0, T_1) \dot{H}_x^{3/2-\gamma/2}} \leq c(T_1) \|\theta_0\|_{\dot{H}^{\gamma_0}} + C \|\theta\|_{L_t^{7/3}(0, T_1) \dot{H}_x^{3/2-4\gamma/7}}^2, \quad (4.13)$$

where $c(T_1)$ is possibly a different constant from that in (4.12) but also goes to zero as $T_1 \rightarrow 0$. Moreover, we also have

$$\|\theta\|_{L_t^\infty(0,T_1)\dot{H}_x^{3/2-\gamma}} \leq \|\theta_0\|_{\dot{H}^{\gamma_0}} + C\|\theta\|_{L_t^2(0,T_1)\dot{H}_x^{3/2-\gamma/2}}^2, \quad (4.14)$$

We finish this subsection by deriving a lower bound of $\|\theta\|_{L^\infty(0,T_1)\dot{H}^{3/2-\gamma}}$. Gronwall's inequality together with (4.5) yields

$$\begin{aligned} \|\theta_j(t, \cdot)\|_{L^2} &\geq e^{-2^{\gamma j} \nu \bar{\lambda} t} \|\theta_j(0)\|_{L^2} \\ &\quad - 2 \int_0^t e^{-2^{\gamma j} \nu \bar{\lambda} (t-s)} \left(\|\tilde{\Delta}_j[u, \Delta_j]\theta_x(s, \cdot)\|_{L^2} + \|\tilde{\Delta}_j(u_x \theta_j)(s, \cdot)\|_{L^2} \right) ds. \end{aligned} \quad (4.15)$$

Arguing as before, it follows that

$$\begin{aligned} \|\theta(t, \cdot)\|_{\dot{H}^{3/2-\gamma}} &\geq \left(\sum_j e^{-2^{1+\gamma j} \nu \bar{\lambda} t} 2^{2(3/2-\gamma)j} \|\theta_j(0)\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\quad - C\|\theta\|_{L^2(0,T_1)\dot{H}^{3/2-\gamma/2}}^2, \end{aligned} \quad (4.16)$$

for any $t \in [0, T_1)$.

4.2 *A priori bounds for Higher regularity*

We shall derive some a priori estimates in this subsection in order to prove (4.1) and (4.2) for any $\beta > 0$. Firstly, let us consider the case when $\beta = 3\gamma/7$. Proceeding as in the previous subsection, we multiply both sides of (4.6) by $2^{(3/2-\gamma+\beta)j}$, use Bernstein's inequality and get

$$\|\theta_j(t, \cdot)\|_{\dot{H}^{3/2-\gamma+\beta}} \leq C(I_3 + I_4 + I_5), \quad (4.17)$$

where

$$\begin{aligned}
I_3 &= 2^{(3/2-\gamma+\beta)j} e^{-2^{\gamma j} \nu \lambda t} \|\theta_j(0, \cdot)\|_{L^2} \\
I_4 &= \int_0^t 2^{(3/2-\gamma+\beta)j} e^{-2^{\gamma j} \nu \lambda (t-s)} \left\| \tilde{\Delta}_j[u, \Delta_j] \theta_x(s, \cdot) \right\|_{L^2} ds, \\
I_5 &= \int_0^t 2^{(3/2-\gamma+\beta)j} e^{-2^{\gamma j} \nu \lambda (t-s)} \left\| \tilde{\Delta}_j(u_x \theta_j)(s, \cdot) \right\|_{L^2} ds.
\end{aligned}$$

To estimate I_4 , we apply Lemma A.4 with $m = 0$, $\alpha_1 = 3/2 - \gamma + \beta$, $\alpha_2 = 1/2 - \gamma + \beta$. To estimate I_5 , we apply Lemma A.5 with $m = 0$, $\alpha_1 = 1/2 - \gamma + \beta$, $\alpha_2 = 3/2 - \gamma + \beta$. Taking the l^2 norm of both sides of (4.17) and using Lemma 3.1 and 3.2, we obtain

$$\begin{aligned}
&\|\theta(t, \cdot)\|_{\dot{H}^{\frac{3}{2}-\gamma+\beta}} \\
&\leq c(T_1) t^{-\frac{\beta}{\gamma}} \|\theta_0\|_{\dot{H}^{\gamma_0}} + C \int_0^t (t-s)^{-1+\frac{\beta}{\gamma}} \|\theta(s, \cdot)\|_{\dot{H}^{\frac{3}{2}-\gamma+\beta}}^2 ds. \quad (4.18)
\end{aligned}$$

Here we also used the simple inequality

$$2^{(-\beta+\gamma)j} e^{-2^{\gamma j} \nu \lambda (t-s)} \leq C(t-s)^{-1+\frac{\beta}{\gamma}}.$$

Multiply both sides of (4.18) by $t^{\beta/\gamma}$ and get

$$\begin{aligned}
&\sup_{t \in (0, T_1)} t^{\frac{\beta}{\gamma}} \|\theta(t, \cdot)\|_{\dot{H}^{\frac{3}{2}-\gamma+\beta}} \leq c(T_1) \|\theta_0\|_{\dot{H}^{\gamma_0}} \\
&+ C \left(\sup_{t \in (0, T_1)} t^{\frac{\beta}{\gamma}} \|\theta(t, \cdot)\|_{\dot{H}^{\frac{3}{2}-\gamma+\beta}} \right)^2 \int_0^t t^{\frac{\beta}{\gamma}} (t-s)^{-1+\frac{\beta}{\gamma}} s^{-2\frac{\beta}{\gamma}} ds. \quad (4.19)
\end{aligned}$$

Notice that the last integral is finite when $\beta = 3\gamma/7$. So to get an a priori estimate we can choose an even smaller T_1 so that the left-hand side of (4.19) is finite and goes to zero as $T_1 \rightarrow 0$. Simple interpolation gives the estimate for $\beta \in (0, 3\gamma/7)$.

Next let us consider the case $\beta > 3\gamma/7$. We shall use the idea in [9] to make use of the smoothing effect of the kernel. Write

$$I_4 = I'_4 + I''_4, \quad I_5 = I'_5 + I''_5,$$

where

$$I'_4 = \int_0^{t/2} \cdots ds, \quad I''_4 = \int_{t/2}^t \cdots ds,$$

$$I'_5 = \int_0^{t/2} \cdots ds, \quad I''_5 = \int_{t/2}^t \cdots ds.$$

To estimate I'_4 , we use Lemma A.4 with $m = 0$, $\alpha_1 = 3/2 - 3\gamma/5$ and $\alpha_2 = 1/2 - 3\gamma/5$. To estimate I''_4 , we use Lemma A.4 with $m = \beta - 3\gamma/7$, $\alpha_1 = 3/2 - 3\gamma/5$ and $\alpha_2 = 1/2 - 3\gamma/5$. Similarly, to estimate I'_5 , we use Lemma A.5 with $m = 0$, $\alpha_1 = 1/2 - 3\gamma/5$ and $\alpha_2 = 3/2 - 3\gamma/5$. To estimate I''_5 , we use Lemma A.5 with $m = \beta - 3\gamma/7$, $\alpha_1 = 1/2 - 3\gamma/5$ and $\alpha_2 = 3/2 - 3\gamma/5$. Taking the l^2 norm of both sides of (4.17) and using Lemma 3.1 and 3.2, we reach

$$\begin{aligned} & \|\theta(t, \cdot)\|_{\dot{H}^{\frac{3}{2}-\gamma+\beta}} \\ & \leq c(T_1)t^{-\frac{\beta}{\gamma}}\|\theta_0\|_{\dot{H}^{\gamma_0}} + C \int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{5}-\frac{\beta}{\gamma}} \|\theta(s, \cdot)\|_{\dot{H}^{\frac{3}{2}-\frac{3\gamma}{5}}}^2 ds \\ & + C \int_{\frac{t}{2}}^t (t-s)^{-\frac{22}{35}} \|\theta(s, \cdot)\|_{\dot{H}^{\frac{3}{2}-\frac{36\gamma}{35}+\beta}} \|\theta(s, \cdot)\|_{\dot{H}^{\frac{3}{2}-\frac{3\gamma}{5}}} ds. \end{aligned} \quad (4.20)$$

Multiplying both sides of (4.20) by $t^{\beta/\gamma}$ yields

$$\begin{aligned} & \sup_{t \in (0, T_1)} t^{\frac{\beta}{\gamma}} \|\theta(t, \cdot)\|_{\dot{H}^{\frac{3}{2}-\gamma+\beta}} \leq c(T_1) \|\theta_0\|_{\dot{H}^{\gamma_0}} \\ & + C \left(\sup_{t \in (0, T_1)} t^{\frac{2}{5}} \|\theta(t, \cdot)\|_{\dot{H}^{\frac{3}{2}-\frac{3\gamma}{5}}} \right)^2 \int_0^{\frac{t}{2}} t^{\frac{\beta}{\gamma}} (t-s)^{-\frac{1}{5}-\frac{\beta}{\gamma}} s^{-\frac{4}{5}} ds \\ & + C \left(\sup_{t \in (0, T_1)} t^{\frac{2}{5}} \|\theta(t, \cdot)\|_{\dot{H}^{\frac{3}{2}-\frac{3\gamma}{5}}} \right) \left(\sup_{t \in (0, T_1)} t^{\frac{\beta}{\gamma}-\frac{1}{35}} \|\theta(t, \cdot)\|_{\dot{H}^{\frac{3}{2}-\frac{36\gamma}{35}+\beta}} \right) \\ & \cdot \int_{\frac{t}{2}}^t t^{\frac{\beta}{\gamma}} (t-s)^{-\frac{22}{35}} s^{-\frac{\beta}{\gamma}-\frac{13}{35}} ds. \end{aligned} \quad (4.21)$$

Note that the last two integrals are finite and

$$\frac{3}{2} - \frac{3\gamma}{5} < \frac{3}{2} - \gamma + \beta, \quad \frac{3}{2} - \frac{36\gamma}{35} + \beta < \frac{3}{2} - \gamma + \beta.$$

Therefore, we can use an induction on β and bootstrap from $\beta \in (0, 3\gamma/7]$ to $(0, \infty)$. This gives a priori estimates of (4.1) and (4.2).

4.3 A priori L^2 bound

The usual contraction argument doesn't give an L^2 bound directly, since the velocity is not divergence free. Here we use a *backward* bootstrap. Again we first derive an a priori estimate. By (4.6), it holds that

$$\|\theta_j(t, \cdot)\|_{L^2} \leq e^{-2\gamma^j \nu \lambda t} \|\theta_j(0)\|_{L^2} + C(I_6 + I_7), \quad (4.22)$$

where

$$\begin{aligned} I_6 &= \int_0^t e^{-2\gamma^j \nu \lambda (t-s)} \left\| \tilde{\Delta}_j [u, \Delta_j] \theta_x(s, \cdot) \right\|_{L^2} ds \\ I_7 &= \int_0^t e^{-2\gamma^j \nu \lambda (t-s)} \left\| \tilde{\Delta}_j (u_x \theta_j)(s, \cdot) \right\|_{L^2} ds. \end{aligned}$$

To estimate I_6 , we use Lemma A.4 with $m = 0$, $\alpha_1 = 3/2 - \gamma/4$ and $\alpha_2 = -1$.

This is possible because

$$m + \alpha_1 + \alpha_2 = 1/2 - \gamma/4 > 0.$$

To estimate I_7 , we use Lemma A.5 with $m = 0$, $\alpha_1 = 1/2 - \gamma/4$ and $\alpha_2 = 0$.

Taking the l^2 norm of both sides of (4.22) yields

$$\|\theta(t, \cdot)\|_{L^2} \leq \|\theta_0\|_{L^2} + C \int_0^t (t-s)^{-\frac{1}{4}} \|\theta(s, \cdot)\|_{L^2} \|\theta(s, \cdot)\|_{\dot{H}^{\frac{3}{2}-\frac{\gamma}{4}}} ds. \quad (4.23)$$

Thus,

$$\begin{aligned} &\sup_{t \in [0, T_1]} \|\theta(t, \cdot)\|_{L^2} \leq \|\theta_0\|_{L^2} \\ &\quad + C \sup_{t \in [0, T_1]} \|\theta(t, \cdot)\|_{L^2} \sup_{t \in (0, T_1)} t^{\frac{3}{4}} \|\theta(t, \cdot)\|_{\dot{H}^{\frac{3}{2}-\frac{\gamma}{4}}} \int_0^t (t-s)^{-\frac{1}{4}} s^{-\frac{3}{4}} ds \\ &\leq \|\theta_0\|_{L^2} + C \sup_{t \in [0, T_1]} \|\theta(t, \cdot)\|_{L^2} \sup_{t \in (0, T_1)} t^{\frac{3}{4}} \|\theta(t, \cdot)\|_{\dot{H}^{\frac{3}{2}-\frac{\gamma}{4}}}. \end{aligned} \quad (4.24)$$

Starting from (4.5), similarly we obtain a lower bound

$$\begin{aligned} \|\theta(t, \cdot)\|_{L^2} &\geq \left(\sum_j e^{-2^{1+\gamma j} \nu \bar{\lambda} t} \|\theta_j(0)\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &- C \sup_{t \in [0, T_1]} \|\theta(t, \cdot)\|_{L^2} \sup_{t \in (0, T_1)} t^{\frac{3}{4}} \|\theta(t, \cdot)\|_{\dot{H}^{\frac{3}{2}-\frac{\gamma}{4}}}. \end{aligned} \quad (4.25)$$

4.4 Strong contraction

With those a priori estimates derived in the previous subsections, we are now able to show that there exists a unique solution

$$\theta \in C([0, T_1], H^{3/2-\gamma}) \cap L^2((0, T_1), H^{3/2-\gamma/2}),$$

for some $T_1 > 0$. As usual, our proof is based on a contraction argument.

For this purpose, consider the following successive approximation:

$$\begin{cases} \theta_t^0 + \nu(-\Delta)^{\gamma/2} \theta^0 = 0 & \text{on } \mathbb{R} \times (0, \infty), \\ \theta^0(0, x) = \theta_0(x) & x \in \mathbb{R}, \end{cases} \quad (4.26)$$

and for $k = 0, 1, 2, \dots$,

$$\begin{cases} \theta_t^{k+1} + (H\theta^k)\theta_x^{k+1} + \nu(-\Delta)^{\gamma/2} \theta^{k+1} = 0 & \text{on } \mathbb{R} \times (0, \infty), \\ \theta^{k+1}(0, x) = \theta_0(x) & x \in \mathbb{R}, \end{cases} \quad (4.27)$$

As in (4.12), it is easily seen that

$$\|\theta^0\|_{L_t^{7/3}(0, T_1) \dot{H}_x^{3/2-4\gamma/7}} \leq c(T_1) \|\theta_0\|_{\dot{H}^{\gamma_0}}. \quad (4.28)$$

and for any $k \geq 0$,

$$\begin{aligned} \|\theta^{k+1}\|_{L_t^{7/3}(0,T_1)\dot{H}_x^{3/2-4\gamma/7}} &\leq c(T_1)\|\theta_0\|_{\dot{H}^{\gamma_0}} \\ &+ C\|\theta^k\|_{L_t^{7/3}(0,T_1)\dot{H}_x^{3/2-4\gamma/7}}\|\theta^{k+1}\|_{L_t^{7/3}(0,T_1)\dot{H}_x^{3/2-4\gamma/7}}. \end{aligned} \quad (4.29)$$

We choose T_1 sufficiently small and obtain a uniform bound

$$\|\theta^k\|_{L_t^{7/3}(0,T_1)\dot{H}_x^{3/2-4\gamma/7}} \leq 2c(T_1)\|\theta_0\|_{\dot{H}^{\gamma_0}}. \quad (4.30)$$

Taking (4.13) and (4.14) into account, we choose T_1 even smaller to get another two uniform bounds

$$\|\theta^k\|_{L_t^2(0,T_1)\dot{H}_x^{3/2-\gamma/2}} \leq 2c(T_1)\|\theta_0\|_{\dot{H}^{\gamma_0}} \quad (4.31)$$

$$\|\theta^k\|_{L_t^\infty(0,T_1)\dot{H}_x^{3/2-\gamma}} \leq (1 + c(T_1))\|\theta_0\|_{\dot{H}^{\gamma_0}}. \quad (4.32)$$

Moreover, by virtue of (4.16) and the dominated convergence theorem, there is also a uniform lower bound

$$\inf_{t \in [0, T_1]} \|\theta^k(t, \cdot)\|_{\dot{H}^{3/2-\gamma}} \geq (1 - c(T_1))\|\theta_0\|_{\dot{H}^{\gamma_0}}. \quad (4.33)$$

By using the estimate in Subsection 4.2, we then get uniform bounds for higher order homogeneous Sobolev norms of θ^k :

$$\|t^{\beta/\gamma}\theta^k\|_{L_t^\infty(0,T_1)\dot{H}_x^{3/2-\gamma+\beta}} \leq C_\beta c(T_1)\|\theta_0\|_{\dot{H}^{\gamma_0}}, \quad (4.34)$$

for any $\beta > 0$. From the estimates (4.24) and (4.25), the following uniform upper and lower bounds of L^2 norm hold

$$\sup_{t \in [0, T_1]} \|\theta^k\|_{L^2} \leq (1 + c(T_1))\|\theta_0\|_{L^2}, \quad (4.35)$$

$$\inf_{t \in [0, T_1]} \|\theta^k\|_{L^2} \geq (1 - c(T_1))\|\theta_0\|_{L^2}, \quad (4.36)$$

for an even smaller T_1 . Notice that in the previous argument we only make four times choices of T_1 . Therefore, T_1 is small but bounded away from zero.

To get a contraction, let us consider the equations satisfied by the differences $\delta\theta^{k+1} = \theta^{k+1} - \theta^k$ for $k = 1, 2, 3, \dots$:

$$\begin{cases} \delta\theta_t^{k+1} + (\delta u^k)\theta_x^k + u^k(\delta\theta^{k+1})_x + \nu\Lambda^{\gamma/2}\theta^{k+1} = 0 & \text{on } \mathbb{R} \times (0, \infty), \\ \delta\theta^{k+1}(0, x) = 0 & x \in \mathbb{R}, \end{cases} \quad (4.37)$$

where $u^k = H\theta^k$ and $\delta u^k = u^k - u^{k-1}$. Denote $\delta\theta_j^{k+1} = \Delta_j(\theta^{k+1} - \theta^k)$. Similar to (4.6), by using Bernstein's inequality, the localization property of Littlewood-Paley projection and Gronwall's inequality, we reach

$$\|\delta\theta_j^{k+1}(t, \cdot)\|_{L^2} \leq C(I_8 + I_9 + I_{10}), \quad (4.38)$$

where

$$\begin{aligned} I_8 &= \int_0^t e^{-2\gamma j \nu \lambda(t-s)} \left\| \tilde{\Delta}_j[u^k, \Delta_j] \delta\theta_x^{k+1}(s, \cdot) \right\|_{L^2} ds, \\ I_9 &= \int_0^t e^{-2\gamma j \nu \lambda(t-s)} \left\| \tilde{\Delta}_j(u_x^k \delta\theta_j^{k+1})(s, \cdot) \right\|_{L^2} ds, \\ I_{10} &= \int_0^t e^{-2\gamma j \nu \lambda(t-s)} \left\| \Delta_j((\delta u^k)\theta_x^k) \right\|_{L^2} ds. \end{aligned}$$

We apply Lemma A.4 with $(m, \alpha_1, \alpha_2) = (0, 3/2 - \gamma/5, -1)$ to estimate I_8 , Lemma A.5 with $(m, \alpha_1, \alpha_2) = (0, 0, 1/2 - \gamma/5)$ to estimate I_9 , and Lemma A.1 iii) with $(\alpha_1, \alpha_2) = (0, 1/2 - \gamma/5)$ to estimate I_{10} . Then after taking the l^2 norm of (4.38) with respect to j and using the boundedness of the Hilbert transform in Sobolev spaces, it follows that

$$\begin{aligned} &\|\delta\theta^{k+1}\|_{L_t^\infty[0, T_1]L_x^2} \\ &\leq C_1 \|t^{\frac{4}{5}}\theta^k\|_{L_t^\infty[0, T_1]\dot{H}_x^{\frac{3}{2}-\frac{\gamma}{5}}} \left(\|\delta\theta^{k+1}\|_{L_t^\infty[0, T_1]L_x^2} + \|\delta\theta^k\|_{L_t^\infty[0, T_1]L_x^2} \right). \end{aligned} \quad (4.39)$$

For T_1 sufficiently small such that

$$\|t^{\frac{4}{5}}\theta^k\|_{L_t^\infty[0, T_1]\dot{H}_x^{\frac{3}{2}-\frac{\gamma}{5}}} \leq 1/(3C_1),$$

we get

$$\|\delta\theta^{k+1}\|_{L_t^\infty[0, T_1]L_x^2} \leq \frac{1}{2}\|\delta\theta^k\|_{L_t^\infty[0, T_1]L_x^2} \leq \frac{C}{2^k}. \quad (4.40)$$

This implies that $\{\theta^{k+1}\}$ is a Cauchy sequence in the Banach space $L_t^\infty[0, T_1]L_x^2$, and they converge in the same space to a function θ . Due to the uniform bound (4.32) and (4.34),

$$t^{\beta/\gamma}\theta \in L_t^\infty(0, T_1)\dot{H}_x^{\frac{3}{2}-\gamma+\beta}, \quad (4.41)$$

for any $\beta \geq 0$, and

$$\lim_{t \rightarrow 0} t^{\beta/\gamma}\|\theta(t, \cdot)\|_{\dot{H}^{\gamma_0+\beta}} = 0, \quad (4.42)$$

for any $\beta > 0$. Moreover, by interpolating the uniform bound (4.34) and (4.40), it is clear that

$$\|t^{\beta\tau/\gamma}\delta\theta^{k+1}\|_{L_t^\infty(0, T_1)\dot{H}_x^{(3/2-\gamma+\beta)\tau}} \leq C_\beta 2^{-k(1-\tau)}$$

for any $\tau \in [0, 1)$ and $\beta \geq 0$. Consequently, $\{t^{\bar{\beta}/\gamma'}\theta^{k+1}\}$ also is a Cauchy sequence in $L_t^\infty(0, T_1)\dot{H}_x^{\bar{\beta}}$ for any $\bar{\beta} \geq 0$ and $\gamma' > \gamma$, and converge to θ in those spaces. By the Sobolev embedding theorem, $\theta^k, k = 0, 1, 2, \dots$ are smooth and converge, along with all the derivatives, to θ for any $t \in (0, T_1)$ uniformly in x . It follows from (4.31) and (4.32) that

$$\theta \in L^\infty([0, T_1]; H^{\gamma_0}(\mathbb{R})) \cap L^2(0, T_1; H^{\gamma_0+\gamma/2}(\mathbb{R}))$$

is a classical solution of the first equation in (1.1).

We still need to show that

$$\theta \in C([0, T_1]; H^{\gamma_0}(\mathbb{R})). \quad (4.43)$$

The right continuity at $t = 0$ follows immediately from the uniform upper and lower bounds (4.32), (4.33), (4.35) and (4.36). For any $\delta \in (0, T_1)$, from (4.41) and the first equation in (1.1) we see that

$$\theta_t \in L^1([\delta, T_1]; H^{3/2-\gamma}).$$

This gives

$$\theta \in C([\delta, T_1]; H^{3/2-\gamma}(\mathbb{R})). \quad (4.44)$$

Since δ is arbitrary and the right continuity at $t = 0$ is already known, (4.43) follows. Finally, the uniqueness follows from the local uniqueness, which in turn is implied by the previous contraction argument (see (4.39)). This finishes the proof of Theorem 4.1.

5 Local well-posedness and smoothness when $\gamma \in (3/2, 2]$

We shall prove an analogue local well-posedness of (1.1) when the range of γ is $(3/2, 2]$. Our main result of this section is the following.

Theorem 5.1. For $\gamma \in (3/2, 2]$ and $\theta_0 \in L^2(\mathbb{R})$. Then there exists $T_1 > 0$ such that the initial value problem of (1.1) has a unique solution

$$\theta(t, x) \in C([0, T_1]; L^2(\mathbb{R})) \cap L^2(0, T_1; H^{\gamma/2}(\mathbb{R})).$$

Moreover the solution θ satisfies

$$\sup_{0 < t < T_1} t^{\beta/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^\beta} < \infty, \quad (5.1)$$

for any $\beta \geq 0$ and

$$\lim_{t \rightarrow 0} t^{\beta/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^\beta} = 0, \quad (5.2)$$

for any $\beta > 0$.

The proof basically follows the line of the previous section with a few modifications. Actually, in some sense this case is easier than the previous one, since the L^2 space is a subcritical space of (1.1) when $\gamma \in (3/2, 2]$. For brevity, we will only point out the differences from the corresponding proofs in Section 4.

5.1 Basic a priori estimates

Multiply both sides of (4.6) by 2^j , use Bernstein's inequality and get

$$\|\theta_j(t, \cdot)\|_{\dot{H}^{\frac{3j}{4}}} \leq C(I_0 + I_1 + I_2), \quad (5.3)$$

where

$$\begin{aligned} I_0 &= 2^{\frac{3j}{4}} e^{-2^{\gamma j} \nu \lambda t} \|\theta_j(0, \cdot)\|_{L^2} \\ I_1 &= \int_0^t 2^{\frac{3j}{4}} e^{-2^{\gamma j} \nu \lambda (t-s)} \left\| \tilde{\Delta}_j [u, \Delta_j] \theta_x(s, \cdot) \right\|_{L^2} ds, \\ I_2 &= \int_0^t 2^{\frac{3j}{4}} e^{-2^{\gamma j} \nu \lambda (t-s)} \left\| \tilde{\Delta}_j (u_x \theta_j)(s, \cdot) \right\|_{L^2} ds. \end{aligned}$$

For I_1 , we use Lemma A.4 with $(m, \alpha_1, \alpha_2) = (0, 3/4, -1/4)$. To estimate I_2 , we apply Lemma A.5 with $(m, \alpha_1, \alpha_2) = (0, -1/4, 3/4)$. It follows that

$$I_1 + I_2 \leq C c_j \int_0^t (t-s)^{-\frac{3}{4\gamma}} \|\theta(s, \cdot)\|_{\dot{H}^{\frac{3}{4}}}^2 ds, \quad (5.4)$$

for some $\{c_j\} \in l^2$ satisfying $\|c_j\|_{l^2} \leq 1$. Taking the l^2 norm on both sides of (5.3), using Lemma 3.1 and 3.2, we obtain

$$\left\| \|I_0\|_{l^2} \right\|_{L^{\frac{4\gamma}{3}}(0, T_1)} \leq C \|\Theta\|_{L_t^{\frac{4\gamma}{3}}(0, C_1 T_1) \dot{H}_x^{\frac{3}{4}}} \leq c(T_1) \|\theta_0\|_{L^2}. \quad (5.5)$$

where $C_1 > 0$ is independent of t . Due to the fractional integration, (5.4) and (5.5), it holds that

$$\begin{aligned} & \|\theta\|_{L_t^{\frac{4\gamma}{3}}(0, T_1) \dot{H}_x^{\frac{3}{4}}} \\ & \leq c(T_1) \|\theta_0\|_{L^2} + C t^{1-\frac{3}{2\gamma}} \left\| \int_0^t (t-s)^{-1+\frac{3}{4\gamma}} \|\theta(s, \cdot)\|_{\dot{H}^{\frac{3}{4}}}^2 ds \right\|_{L_t^{\frac{4\gamma}{3}}(0, T_1)} \\ & \leq c(T_1) \|\theta_0\|_{L^2} + C t^{1-\frac{3}{2\gamma}} \|\theta\|_{L_t^{\frac{4\gamma}{3}}(0, T_1) \dot{H}_x^{\frac{3}{4}}}^2. \end{aligned} \quad (5.6)$$

Multiply both sides of (4.6) by $2^{\gamma j/2}$, take l^2 norm and get

$$\|\theta\|_{L_t^2(0,T_1)\dot{H}_x^{\frac{\gamma}{2}}} \leq c(T_1)\|\theta_0\|_{L^2} + Ct^{1-\frac{3}{2\gamma}}\|\theta\|_{L_t^{\frac{4\gamma}{3}}(0,T_1)\dot{H}_x^{\frac{3}{4}}}^2, \quad (5.7)$$

Similarly, we also have

$$\|\theta\|_{L_t^\infty(0,T_1)L_x^2} \leq \|\theta_0\|_{L^2} + Ct^{1-\frac{3}{2\gamma}}\|\theta\|_{L_t^{\frac{4\gamma}{3}}(0,T_1)\dot{H}_x^{\frac{3}{4}}}^2, \quad (5.8)$$

and

$$\|\theta(t, \cdot)\|_{L_x^2} \geq \left(\sum_j e^{-2^{1+\gamma j} \nu \bar{\lambda} t} \|\theta_j(0)\|_{L^2}^2 \right)^{\frac{1}{2}} - Ct^{1-\frac{3}{2\gamma}}\|\theta\|_{L_t^{\frac{4\gamma}{3}}(0,T_1)\dot{H}_x^{\frac{3}{4}}}^2. \quad (5.9)$$

5.2 A priori bounds for Higher regularity

One can argue as in Subsection 4.2. Firstly we derive a priori estimates of (5.1) and (5.2) for $\beta = 3\gamma/7$. Then one uses interpolation to get them for $\beta \in (0, 3\gamma/7)$. Notice that in the nonlinear terms, e.g. I_4 and I_5 , there will be an extra factor $t^{1-3\gamma/2}$ reflecting the subcriticality as in (5.7) and (5.8), which is not harmful if we assume $T_1 \leq 1$. After that, we can adapt the bootstrap argument as in Section 4.2 by appealing Lemma A.4 and A.5 with suitable combinations of m , α_1 and α_2 . The details are left to interested readers.

5.3 Strong contraction

As in Subsection 4.4, we construct a sequence θ^k of successive approximation by (4.26) and (4.27). Similarly, by virtue of (5.6)-(5.9), we obtain the following uni-

form bounds for T_1 sufficiently small:

$$\|\theta^k\|_{L_t^{\frac{4\gamma}{3}}(0, T_1) \dot{H}_x^{\frac{3}{4}}} \leq 2c(T_1) \|\theta_0\|_{L^2}, \quad (5.10)$$

$$\|\theta^k\|_{L_t^2(0, T_1) \dot{H}_x^{\frac{7}{2}}} \leq 2c(T_1) \|\theta_0\|_{L^2}, \quad (5.11)$$

$$\|\theta^k\|_{L_t^\infty(0, T_1) L_x^2} \leq (1 + c(T_1)) \|\theta_0\|_{L^2}, \quad (5.12)$$

$$\inf_{t \in (0, T_1)} \|\theta^k(t, \cdot)\|_{L_x^2} \geq (1 - c(T_1)) \|\theta_0\|_{L^2}. \quad (5.13)$$

Also we have uniform bounds for higher order Sobolev norms:

$$\|t^{\frac{\beta}{\gamma}} \theta^k\|_{L_t^\infty(0, T_1) \dot{H}^\beta} \leq c(T_1) \|\theta_0\|_{L^2}. \quad (5.14)$$

As in (4.40), we see that $\{\theta^k\}$ is a Cauchy sequence in the Banach space $L_t^\infty[0, T_1] L_x^2$, and they converge in the same space to a function θ . Moreover,

$$t^{\beta/\gamma} \theta \in L_t^\infty(0, T_1) \dot{H}_x^\beta, \quad (5.15)$$

for any $\beta \geq 0$, and $\{t^{\beta/\gamma} \theta^k\}$ is a Cauchy sequence in $L_t^\infty(0, T_1) \dot{H}_x^\beta$ for any $\beta \geq 0$, and converge to θ in those spaces. By the Sobolev embedding theorem, $\theta^k, k = 0, 1, 2, \dots$ are smooth and converge, along with all the derivatives, to θ for any $t \in (0, T_1)$ uniformly in x . As a conclusion,

$$\theta \in L^\infty([0, T_1]; L^2(\mathbb{R})) \cap L^2(0, T_1; H^{\gamma/2}(\mathbb{R}))$$

is a classical solution of the first equation in (1.1). Similar argument as in the last part to Subsection 4.4 shows that

$$\theta \in C([0, T_1]; L^2(\mathbb{R})),$$

and such solution is actually unique. The proof of Theorem 5.1 is completed. \square

Remark 5.2. Because L^2 is a subcritical space for (1.1) with $\gamma \in (3/2, 1]$ and there is an extra factor $t^{1-3/(2\gamma)}$ in the nonlinear term, the time of existence T_1 can be chosen to only depend on γ and $\|\theta_0\|_{L^2}$.

6 Global well-posedness

6.1 A blow-up criterion

Due to Theorems 1.1 and 1.2, we may assume $\theta_0 \in H^s$ for integers $s \geq 2$ by shifting the time variable if necessary. Since $H^s, s \geq 2$ is a subcritical space for (1.1), if T^* is a finite blow-up time, it must hold that

$$\limsup_{t \rightarrow T^*} \|\theta(t)\|_{H^s} = \infty.$$

For the purpose of establishing a blow-up criterion for (1.1), we shall have a good estimate of $H\theta$. Although the Hilbert transform is not bounded in L^∞ , we still have the following logarithmic bound.

Lemma 6.1. For any $f \in H^1(\mathbb{R})$, it holds that

$$\|Hf\|_{L^\infty} \leq C \left[1 + \|f\|_{L^\infty} \log(e + \|f\|_{H^1}) + \|f\|_{L^2} \right]. \quad (6.1)$$

Proof. This estimate is classical. Here we give a proof for the sake of completeness. It suffices to prove $\text{PV} \int f(x)/x \, dx$ is bounded by the right-hand side of (6.1). Let $\rho \in (0, 1]$ be a number to be specified later. Write

$$\begin{aligned} \text{PV} \int_{\mathbb{R}} \frac{f(x)}{x} \, dx &= \text{PV} \int_{-\rho}^{\rho} \frac{f(x)}{x} \, dx + \int_{|x| \in (\rho, 1]} \frac{f(x)}{x} \, dx + \int_{|x| > 1} \frac{f(x)}{x} \, dx \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

It is clear that

$$|I_2| \leq \|f\|_{L^\infty} |\log \rho|, \quad |I_3| \leq \|f\|_{L^2}. \quad (6.2)$$

To estimate I_1 , we compute

$$\begin{aligned}
|I_1| &\leq \int_0^\rho \frac{|f(x) - f(-x)|}{x} dx \leq \int_0^\rho \int_{-1}^1 |f'(rx)| dr dx \\
&= \int_{-1}^1 \int_0^\rho |f'(rx)| dx dr \leq \rho^{1/2} \int_{-1}^1 \left(\int_0^\rho |f'(rx)|^2 dx \right)^{1/2} dr \\
&\leq \rho^{1/2} \|f'\|_{L^2} \int_{-1}^1 |r|^{-1/2} dr \leq C \rho^{1/2} \|f'\|_{L^2}.
\end{aligned} \tag{6.3}$$

Combining (6.2) and (6.3) together and optimizing in ρ give (6.1). The lemma is proved. \square

To get global regularity, we need the following Beale-Kato-Majda type blow-up criterion for (1.1) (see [2] and [8]).

Proposition 6.2. Let θ be the solution of (1.1) described above, and suppose T^* is the first finite blow-up time such that the solution cannot be continued in $C([0, T^*]; H^s)$.

Then we have

$$\int_0^{T^*} \|\theta_x(t, \cdot)\|_{L^\infty} dt = \infty. \tag{6.4}$$

Proof. We follow the idea of [2] and also take care of the fact that u is not divergence free. In the first equation of (1.1), we take one derivative with respect to x , multiply both sides by θ_x , integrate in x . Noting that the contribution of the dissipation term is non-negative, we have

$$\frac{1}{2} \frac{d}{dt} \int |\theta_x(t)|^2 dx + \int \left(u_x (\theta_x)^2 + u \theta_{xx} \theta_x \right) dx \leq 0. \tag{6.5}$$

Integrating by parts and using $u_x = \Lambda \theta$, we get from (6.5)

$$\frac{1}{2} \frac{d}{dt} \int |\theta_x(t)|^2 dx + \int \left(u_x (\theta_x)^2 - \frac{1}{2} \Lambda \theta (\theta_x)^2 \right) dx \leq 0.$$

By using Hölder's inequality and the boundedness of the Hilbert transform in L^2 ,

it holds that

$$\frac{d}{dt} \int |\theta_x(t, \cdot)|^2 dx \leq C \|\theta_x(t, \cdot)\|_{L^\infty} \int |\theta_x(t, \cdot)|^2 dx.$$

This together with Gronwall's inequality yields

$$\|\theta_x(t, \cdot)\|_{L^2} \leq e^{Cm(t)} \|\theta_x(0, \cdot)\|_{L^2}, \quad (6.6)$$

where $m(t) = \int_0^t \|\theta_x(s, \cdot)\|_{L^\infty} ds$. For any integer $\alpha \in [0, s]$, we take D_x^α on both sides of the first equation in (1.1), and get

$$(D_x^\alpha \theta)_t + [D_x^\alpha, u] \theta_x + u D_x^\alpha \theta_x + \nu \Lambda^\gamma D_x^\alpha = 0. \quad (6.7)$$

Now multiply both sides of the equation above by $D_x^\alpha \theta$ and integrate in x . Noting that the contribution of the dissipation term is non-negative, we have

$$\frac{1}{2} \frac{d}{dt} \int |D_x^\alpha \theta(t)|^2 dx + \int \left([D_x^\alpha, u] \theta_x D_x^\alpha \theta + u D_x^{\alpha+1} \theta (D_x^\alpha \theta) \right) dx \leq 0 \quad (6.8)$$

By using integrating by parts in the third term above, Hölder's inequality, and Lemma A.3 with $f = u$ and $g = \theta_x$, we obtain, after adding up in α ,

$$\frac{d}{dt} \|\theta(t, \cdot)\|_{H^s} \leq C \left(\|u_x(t, \cdot)\|_{L^\infty} + \|\theta_x(t, \cdot)\|_{L^\infty} \right) \|\theta(t, \cdot)\|_{H^s} \quad (6.9)$$

Now we put $s = 2$. Due to Lemma 6.1, we can estimate $\|u_x(t, \cdot)\|_{L^\infty}$ through $\|\theta_x(t, \cdot)\|_{L^\infty}$, $\|\theta(t, \cdot)\|_{H^2}$ and $\|\theta_x(t, \cdot)\|_{L^2}$. Denote

$$y(t) = \log(e + \|\theta(t, \cdot)\|_{H^2}).$$

Then (6.9) and (6.1) give

$$y'(t) \leq C \left[1 + \|\theta_x(t, \cdot)\|_{L^\infty} y(t) + \|\theta_x(t, \cdot)\|_{L^2} \right].$$

This together with (6.6) completes the proof of the proposition by using Gronwall's inequality. \square

6.2 Proof of Theorem 1.4

For the solution θ , we have the following maximum principle, which is a necessary ingredient in the derivation of Lemma 6.6 and 6.7 (see [9]).

Lemma 6.3. The sup norm $\|\theta(t, \cdot)\|_{L^\infty}$ is non-increasing in $t \in (0, T)$.

Proof. For $\gamma = 2$, the lemma follows from the maximum principle for second order parabolic equations. For $\gamma \in (0, 2)$, the proof is also standard. Choose $t_1, t_2 \in (0, T_1)$ such that $t_1 < t_2$. For a fixed $\varepsilon > 0$, we consider $\bar{\theta} = \theta - \varepsilon t$. Since $\bar{\theta}$ is smooth in $E := [t_1, t_2] \times \mathbb{R}$ with bounded derivatives and $\theta \in L^2(E)$, we can find $(t', x') \in E$ such that $\bar{\theta}(t', x') = \sup_E \bar{\theta}$. Due to the integral representation

$$\Lambda^\gamma \theta(t, x) = \int_{\mathbb{R}} \frac{\theta(t, x) - \theta(t, y)}{|x - y|^{1+\gamma}} dy, \quad \gamma \in (0, 2),$$

we get

$$\bar{\theta}_t(t', x') = \theta_t(t', x') - \varepsilon = -u(t', x')\theta_x(t', x') - \Lambda^\gamma \theta(t', x') - \varepsilon < 0.$$

Therefore, $t' = t_1$. Since $\varepsilon > 0$ is arbitrary, it holds that

$$\sup_x \theta(t_1, x) = \sup_E \theta.$$

Similarly,

$$\inf_x \theta(t_1, x) = \inf_E \theta.$$

Because t_1 and t_2 are also arbitrary, the lemma is proved. \square

Remark 6.4. Since the velocity u is not divergence free and the nonlinear term is not in divergence form, unlike solutions to the quasi-geostrophic equations or the Burger's equation, the L^p norm of θ for general $p \in [1, \infty)$ is not non-increasing. Another difficulty is that there seems no easy control of the sup norm of the velocity

u . Because of these reasons, it seems that the global well-posedness of (1.1) doesn't follow easily from the local well-posedness in a traditional way even in the subcritical case.

In the sequel, we shall use the idea of the non-local maximum principle for a suitably chosen modulus of continuity. This method was first used by Kiselev, Nazarov and Volberg in [13], where they proved the global regularity for the 2D critical dissipative quasi-geostrophic equations with periodic C^∞ data. With suitable modifications, this idea was later adapted by the author and D. Du in [9], [10] to establish the global regularity for the same equation with either periodic or non-periodic H^1 data.

Definition 6.5. We say a function f has modulus of continuity ω if $|f(x) - f(y)| \leq \omega(|x - y|)$, where ω is an increasing continuous concave function $\omega : [0, +\infty) \rightarrow [0, +\infty)$. We say f has strict modulus of continuity ω if the inequality is strict for $x \neq y$.

In what follows, we will choose ω satisfying

$$\omega(0) = 0, \quad \omega' > 0, \quad \omega'(0) < +\infty, \quad \lim_{\xi \rightarrow 0^+} \omega''(\xi) = -\infty, \quad (6.10)$$

As before, we may assume $\theta_0 \in H^{20} \cup C^\infty$. Due to the scaling property of (1.1), $\theta_c(t, x) = C^{\gamma-1}\theta(C^\gamma t, Cx)$ is also a solution of (1.1) with initial data $C^{\gamma-1}\theta_0(C^\gamma x)$. Thus if we can show that θ_c is a global solution, the same remains true for θ . Note that for any ω satisfying (6.10) we can always find a constant $C > 0$ such that $\omega(\xi)$ is a *strict* modulus of continuity of $C^{\gamma-1}\theta_0(C^\gamma x)$ if $\gamma > 1$. While in the critical case, i.e. $\gamma = 1$, this still holds for any unbounded ω satisfying (6.10).

We shall show that for suitably chosen ω , the modulus of continuity is preserved

for all the time. This together with Lemma B.2 i) and the blow-up criterion (6.4) implies Theorem 1.4. The next lemma is proved in [10], which says that the strict modulus of continuity is preserved at least for a short time.

Lemma 6.6. Assume $\theta(t, \cdot)$ has strict modulus of continuity ω for all $t \in [0, T_2]$. Then there exists $\delta > 0$ such that $\theta(t, \cdot)$ has strict modulus of continuity ω for all $t \in [0, T_2 + \delta)$.

Clearly, if $\theta(t, \cdot)$ has strict modulus of continuity ω for all $t \in [0, T_2)$, then θ is smooth up to T_2 and $\theta(T_2, \cdot)$ has modulus of continuity ω by continuity. Therefore, to show that the modulus of continuity is preserved for all the time, it suffices to rule out the case that

$$\sup_{x \neq y} \frac{\theta(T_2, x) - \theta(T_2, y)}{\omega(|x - y|)} = 1.$$

Also we recall the following lemma (see [10]).

Lemma 6.7. Under the conditions above, there exist two different points $x, y \in \mathbb{R}$ satisfying

$$\theta(T_2, x) - \theta(T_2, y) = \omega(|x - y|). \quad (6.11)$$

This possibility can be eventually ruled out if we are able to chose suitable ω such that under the conditions above we have

$$\frac{\partial}{\partial t} (\theta(T_2, x) - \theta(T_2, y)) < 0. \quad (6.12)$$

Due to Lemma B.2 ii) and iii), the left-hand side of (6.12) is less than or equal to

$I_4 + I_5 + I_6$, where

$$\begin{aligned} I_4 &= \Omega(r)\omega'(r), \quad \Omega(\xi) = C \left(\int_0^\xi \omega(s)/s \, dr + \xi \int_\xi^\infty \omega(s)/s^2 \, dr \right), \\ I_5 &= C(\gamma) \int_0^{\frac{r}{2}} \frac{\omega(r+2s) + \omega(r-2s) - 2\omega(r)}{r^{1+\gamma}} \, ds, \\ I_6 &= C(\gamma) \int_{\frac{r}{2}}^\infty \frac{\omega(2s+r) - \omega(2s-r) - 2\omega(r)}{r^{1+\gamma}} \, ds, \quad r = |x-y|. \end{aligned}$$

By the concavity of ω , both I_5 and I_6 are strictly negative.

We are now ready to prove Theorem 1.4.

Subcritical case, $\gamma \in (1, 2]$: Set for $r > 0$

$$\omega''(r) = -\frac{\delta_1}{r^{1-\gamma/3} + r^4}, \quad \omega'(r) = -\int_r^\infty \omega''(s) \, ds, \quad \omega(0) = 0, \quad (6.13)$$

where $\delta_1 > 0$ is a small number to be specified later. Clearly, such ω satisfies (6.10).

It is easily seen that as $r \rightarrow 0$,

$$\begin{aligned} \omega'(r) &\sim \delta_1, \quad \omega(r) \sim \delta_1 r, \quad \Omega(r) \sim -\delta_1 r \log r, \\ I_4 = \Omega\omega' &\sim -\delta_1^2 r \log r, \quad I_5 \leq C\delta_1 r^{2-\gamma}\omega''(r) \sim -\delta_1 r^{1-2\gamma/3}. \end{aligned} \quad (6.14)$$

The last inequality is because of the mean value theorem and the monotonicity of ω'' . Similarly, as $r \rightarrow \infty$,

$$\begin{aligned} \omega'(r) &\sim \delta_1 r^{-3}, \quad \omega(r) \sim \delta_1, \quad \Omega(r) \sim \delta_1 \log r, \\ I_4 = \Omega\omega' &\sim \delta_1^2 r^{-3} \log r, \quad I_6 \leq -C\delta_1 r^{-\gamma}. \end{aligned} \quad (6.15)$$

From (6.14), (6.15) and the continuity, we can choose δ_1 sufficiently small such that $I_4 + I_5 + I_6$ is strictly negative on $(0, +\infty)$. Thus, we obtain (6.12).

Critical case, $\gamma = 1$: Set for $r > 0$

$$\omega''(r) = -\frac{\delta_2}{r^{1/2} + r^2 \log r}, \quad \omega'(r) = -\int_r^\infty \omega''(s) \, ds, \quad \omega(0) = 0,$$

which clearly satisfies (6.10). It is easily seen that as $r \rightarrow 0$, (6.14) still holds. As $r \rightarrow \infty$, it is not difficult to check that

$$\begin{aligned} \omega'(r) &\sim \delta_2 (r \log r)^{-1}, \quad \omega(r) \sim \delta_2 \log \log r, \quad \Omega(r) \sim \delta_2 \omega(r) \log r, \\ I_4 = \Omega \omega' &\sim \delta_2^2 r^{-1} \log \log r, \quad I_6 \leq -C \delta_2 r^{-1} \log \log r. \end{aligned} \quad (6.16)$$

From (6.14), (6.16) and the continuity, we can choose $\delta_2 > 0$ sufficiently small such that $I_4 + I_5 + I_6$ is strictly negative on $(0, +\infty)$. Thus, for $\gamma = 1$ we obtain (6.12) too.

The proof of Theorem 1.4 is finished. \square

7 Further results

7.1 Remark of the periodic case

It is worth noting that all the well-posedness results in Section 1 also remains true in the periodic setting, i.e. θ_0 is periodic in x and (1.1) is equipped with a periodic boundary condition. The proofs are not much different from the non-periodic setting and in some places even simpler. For further details, we refer the readers to [9] and [13].

7.2 Small data global well-posedness in the supercritical case

Naturally, we also have global well-posedness for small data in the supercritical case.

Theorem 7.1. Let $\gamma \in (0, 1)$ and $\theta_0 \in H^{\gamma_0}(\mathbb{R})$, where $\gamma_0 = 3/2 - \gamma$. There

exists a positive constant ε depending on γ , such that if the initial data satisfies $\|\theta_0\|_{H^{\gamma_0}(\mathbb{R})} < \varepsilon$, then (1.1) has a unique solution

$$\theta(t, x) \in C([0, \infty); H^{\gamma_0}(\mathbb{R})) \cap L^2(0, \infty; H^{\gamma_0+\gamma/2}(\mathbb{R})).$$

Moreover, (4.1) and (4.2) hold true with $T_1 = \infty$.

Indeed, from the proof of Theorem 1.1, we can easily see that if the \dot{H}^{γ_0} norm of θ_0 is sufficiently small, then we can choose $T_1 = \infty$. Consequently, because of (4.1), higher order homogeneous Sobolev norms of the solution θ decay polynomially in time.

7.3 Some decay estimates

Next we shall discuss some interesting decay in time estimates of Sobolev norms and the C^k norms of θ in the critical or subcritical cases, without any smallness assumption. We have the following result.

Theorem 7.2. Let $\gamma \in [1, 2]$ and $\theta_0 \in H^{\gamma_0}(\mathbb{R})$, where $\gamma_0 = \max(3/2 - \gamma, 0)$. Suppose $\theta_0 \leq 0$. Then for any $\beta > 0$ and integer $k \geq 0$, the \dot{H}^β norm and the C^k norm of $\theta(t, \cdot)$ decay polynomially as t goes to infinity.

Proof. We outline a sketch of the proof. Firstly, with non-positiveness assumption, it is proved in [4] that for any $t \geq 0$,

$$\|\theta(t, \cdot)\|_{L_x^2} \leq \|\theta_0\|_{L^2}, \quad \int_0^t \|\Lambda^{\gamma/2}\theta\|_{L_x^2}^2 dt \leq \frac{\|\theta_0\|_{L^2}}{2\nu}. \quad (7.1)$$

Remember that $\theta(t, \cdot)$ is in H^k for any $k \geq 0$ and $t > 0$. We consider the subcritical case and the critical case separately.

Case 1: $\gamma > 1$. Theorem 3.2 of [4] tells us

$$\sup_{t \in [1, \infty)} \|\Lambda^{1/2}\theta(t, \cdot)\|_{L^2} \leq C. \quad (7.2)$$

Note that $H^{1/2}$ is a subcritical space of (1.1) when $\gamma > 1$. Thanks to (7.2) and the first inequality of (7.1), by using the bootstrap argument in Section 4 we get a polynomial decay of $\|\theta(t, \cdot)\|_{\dot{H}^\beta}$ for any $\beta > 0$. Due to the Sobolev imbedding theorem, for any $k \geq 0$, $\|\theta_x(t, \cdot)\|_{C^k}$ has a polynomial decay in t . In particular, there exist an $\alpha > 0$ and $C_0 > 0$ such that

$$\|\theta_x(t, \cdot)\|_{L^\infty} \leq C_0 t^{-\alpha} \quad (7.3)$$

for any $t \geq 1$. Therefore, to prove the decay of the $\|\theta(t, \cdot)\|_{C^k}$, it suffices to show a decay of $\|\theta(t, \cdot)\|_{L^\infty}$. Fix a $t \geq 0$ and $x \in \mathbb{R}$, we take a positive constant R to be specified later. Because of the first inequality of (7.1), there exists a point $y \in [x - R, x]$ such that $|\theta(t, y)| \leq \|\theta_0\|_{L^2} R^{-1/2}$. This together with (7.3) gives

$$|\theta(t, x)| \leq |\theta(t, y)| + R\|\theta_x(t, \cdot)\|_{L^\infty} \leq \|\theta_0\|_{L^2} R^{-1/2} + C_0 t^{-\alpha} R.$$

Optimizing in R gives the result in the case $\gamma > 1$.

Case 2: $\gamma = 1$. As in the proof of Theorem 3.2 [4], we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} |\Lambda\theta(t, x)|^2 dx &\leq C \int_{\mathbb{R}} |\Lambda\theta|^3 dx - \nu \|\Lambda^{3/2}\theta(t, \cdot)\|_{L^2}^2 \\ &\leq C \|\theta(t, \cdot)\|_{L^\infty} \|\Lambda^{3/2}\theta(t, \cdot)\|_{L^2}^2 - \nu \|\Lambda^{3/2}\theta(t, \cdot)\|_{L^2}^2, \end{aligned} \quad (7.4)$$

where we used the Gagliardo-Nirenberg inequality in the second inequality. Arguing as in [3] and using the first inequality of (7.1), we get a polynomial decay of $\|\theta(t, \cdot)\|_{L^\infty}$. Thus there exists $T_3 > 0$ such that $\|\theta(t, \cdot)\|_{L^\infty} < \nu/(2C)$ for any $t \geq T_3$. Therefore, $\|\Lambda\theta(t, \cdot)\|_{L^2}$ is non-increasing in $t \in [T_3, \infty)$. This and (7.1)

imply that

$$\sup_{t \in [1, \infty)} \|\theta(t, \cdot)\|_{H^1} \leq C. \quad (7.5)$$

Now since H^1 is a subcritical space of (1.1) when $\gamma = 1$, (7.5) together with the bootstrap argument in Section 4 yields a polynomial decay of homogeneous Sobolev norms, from which we obtain the result. \square

7.4 Further applications of the method

Finally, we remark that our method can also be applied to many other models, for instance, the Burger's equation

$$\begin{cases} u_t + uu_x + \nu(-\Delta)^{\gamma/2}u = 0 & \text{on } \mathbb{R} \times (0, \infty), \\ u(0, x) = u_0(x) & x \in \mathbb{R}, \end{cases} \quad (7.6)$$

the Burger's system and the (modified) quasi-geostrophic equations without the divergence free property of the velocity (cf. [12]), e.g.

$$\begin{cases} \theta_t + u \cdot \nabla \theta + (-\Delta)^{\gamma/2}\theta = 0 & \text{on } \mathbb{R}^2 \times (0, \infty), \\ \theta(0, x) = \theta_0(x) & x \in \mathbb{R}^2, \end{cases} \quad (7.7)$$

where the velocity

$$u = (\partial_{x_2}(-\Delta)^{1/2}\theta, \partial_{x_1}(-\Delta)^{1/2}\theta).$$

Equation (7.6) has a simpler structure in the nonlinear term and there are some nice properties of solutions to it, which are not possessed by solutions to (1.1). We refer interested readers to [14] and [11].

A Some multiplicative inequalities

Define paraproduct operators by

$$T_f g := \sum_{j \in \mathbb{Z}} S_j f \Delta_j g, \quad R(f, g) := \sum_{|i-j| \leq 2} \Delta_i f \Delta_j g,$$

where $S_j f = \sum_{k \leq j-3} \Delta_k f$. Then we have the decomposition

$$fg = T_f g + T_g f + R(f, g).$$

We need the following lemma (see, e.g. [17]).

Lemma A.1. i) Let $d \geq 1$ be an integer. If $\alpha_1 < d/2, \alpha_2 \in \mathbb{R}$, there exists a positive constant C depending only on d, α_1 and α_2 such that for any $f \in \dot{H}^{\alpha_1}(\mathbb{R}^d)$ and $g \in \dot{H}^{\alpha_2}(\mathbb{R}^d)$ we have

$$\|T_f g\|_{\dot{H}^{\alpha_1 + \alpha_2 - d/2}} \leq C \|f\|_{\dot{H}^{\alpha_1}} \|g\|_{\dot{H}^{\alpha_2}}. \quad (\text{A.1})$$

ii) If $\alpha_1 + \alpha_2 > 0$, there exists a positive constant C depending only on d, α_1 and α_2 such that for any $f \in \dot{H}^{\alpha_1}(\mathbb{R}^d)$ and $g \in \dot{H}^{\alpha_2}(\mathbb{R}^d)$ we have

$$\|R(f, g)\|_{\dot{H}^{\alpha_1 + \alpha_2 - d/2}} \leq C \|f\|_{\dot{H}^{\alpha_1}} \|g\|_{\dot{H}^{\alpha_2}}. \quad (\text{A.2})$$

iii) If $\alpha_1, \alpha_2 < d/2$ and $\alpha_1 + \alpha_2 > 0$, there exists a positive constant C depending only on d, α_1 and α_2 such that for any $f \in \dot{H}^{\alpha_1}(\mathbb{R}^d)$ and $g \in \dot{H}^{\alpha_2}(\mathbb{R}^d)$ we have

$$\|fg\|_{\dot{H}^{\alpha_1 + \alpha_2 - d/2}} \leq C \|f\|_{\dot{H}^{\alpha_1}} \|g\|_{\dot{H}^{\alpha_2}}. \quad (\text{A.3})$$

The following fractional Leibniz's rule and commutator estimate are well-known.

Lemma A.2. Assume $\alpha_1 \geq 0$ and $p \in (1, \infty)$. Then we have

$$\|fg\|_{\dot{W}^{\alpha_1, p}} \leq C\|f\|_{\dot{W}^{\alpha_1, p_1}}\|g\|_{L^{p_2}} + C\|f\|_{L^{p'_1}}\|g\|_{\dot{W}^{\alpha_1, p'_2}}$$

if the right-hand side is finite. Here $p_1, p_2, p'_1, p'_2 \in (1, +\infty)$ satisfy

$$1/p = 1/p_1 + 1/p_2 = 1/p'_1 + 1/p'_2.$$

Lemma A.3. Assume $\alpha_1 > 0$ and $|\alpha| = \alpha_1$. Then

$$\|D^\alpha(fg) - fD^\alpha g\|_{L^2} \leq C\|f\|_{H^{\alpha_1}}\|g\|_{L^\infty} + C\|\nabla f\|_{L^\infty}\|g\|_{H^{\alpha_1-1}} \quad (\text{A.4})$$

holds if the right-hand side is finite.

Denote $f_j = \Delta_j f$ and $g_j = \Delta_j g$. The next two lemmas play together a key role in our bootstrap arguments.

Lemma A.4. i) Assume $m \geq 0$, $d/2 \leq \alpha_1 < 1 + d/2$, $\alpha_2 < d/2$ satisfying $m + \alpha_2 + \alpha_1 > 0$. Then there exists positive constant $C = C(d, m, \alpha_1, \alpha_2)$ such that

$$\|[f, \Delta_j]g\|_{\dot{H}^m} \leq C2^{-(\alpha_1 + \alpha_2 - d/2)j} c_j (\|f\|_{\dot{H}^{m+\alpha_1}}\|g\|_{\dot{H}^{\alpha_2}} + \|f\|_{\dot{H}^{\alpha_1}}\|g\|_{\dot{H}^{m+\alpha_2}})$$

for any $j \in \mathbb{Z}$, $f \in \dot{H}^{\alpha_1} \cap \dot{H}^{m+\alpha_1}$, $g \in \dot{H}^{\alpha_2} \cap \dot{H}^{m+\alpha_1}$ and some $\{c_j\} \in l^2$ with $\|c_j\|_{l^2} \leq 1$. Here,

$$[f, \Delta_j]g = f\Delta_j g - \Delta_j(fg).$$

ii) Moreover, a similar estimate

$$\|\tilde{\Delta}_j[f, \Delta_j]g\|_{L^2} \leq C2^{-(m+\alpha_1+\alpha_2-d/2)j} c_j (\|f\|_{\dot{H}^{m+\alpha_1}}\|g\|_{\dot{H}^{\alpha_2}} + \|f\|_{\dot{H}^{\alpha_1}}\|g\|_{\dot{H}^{m+\alpha_2}})$$

holds if $m \geq 0$, $\alpha_1 < 1 + d/2$, $\alpha_2 < d/2$ and $m + \alpha_2 + \alpha_1 > 0$.

Proof. When $d = 2$, the lemma is proved in [9]. The general case follows from the proof there with minor modifications. \square

Lemma A.5. Let $k, j \in \mathbb{Z}$ and $k \leq j + 10$. Assume $m \geq 0$, $\alpha_1 < d/2$ and $\alpha_2 \in \mathbb{R}$. Then there exists a constant $C = C(d, m, \alpha_1, \alpha_2) > 0$ such that

$$\|S_k f g_j\|_{\dot{H}^m} \leq C 2^{-(\alpha_1 + \alpha_2 - d/2)j} c_j (\|f\|_{\dot{H}^{m+\alpha_1}} \|g\|_{\dot{H}^{\alpha_2}} + \|f\|_{\dot{H}^{\alpha_1}} \|g\|_{\dot{H}^{m+\alpha_2}})$$

for any $f \in \dot{H}^{\alpha_1} \cap \dot{H}^{m+\alpha_1}$, $g \in \dot{H}^{\alpha_2} \cap \dot{H}^{m+\alpha_2}$ and some $\{c_j\} \in l^2$ satisfying $\|c_j\|_{l^2} \leq 1$. Consequently, under the same assumptions we have

$$\|\tilde{\Delta}_j(fg_j)\|_{\dot{H}^m} \leq C 2^{-(\alpha_1 + \alpha_2 - d/2)j} c_j (\|f\|_{\dot{H}^{m+\alpha_1}} \|g\|_{\dot{H}^{\alpha_2}} + \|f\|_{\dot{H}^{\alpha_1}} \|g\|_{\dot{H}^{m+\alpha_2}}).$$

Proof. Choose $p_1 \in (2, \infty)$ sufficiently large so that $\alpha_1 + d/p_1 < d/2$. This is possible because $\alpha_1 < d/2$. Let $p_2 \in (2, \infty)$ be a number satisfying $1/p_1 + 1/p_2 = 1/2$. Due to Lemma A.2, we get

$$\|S_k f g_j\|_{\dot{H}^m} \leq C \left(\|S_k f\|_{\dot{W}^{m,p_1}} \|g_j\|_{L^{p_2}} + \|S_k f\|_{L^{p_1}} \|g_j\|_{\dot{W}^{m,p_2}} \right). \quad (\text{A.5})$$

Then by Hölder's inequality,

$$|S_k f| \leq C 2^{(d/2 - \alpha_1 - d/p_1)k} \|2^{(\alpha_1 + d/p_1 - d/2)j} f_j\|_{l^2}.$$

Therefore,

$$\begin{aligned} \|S_k f\|_{\dot{W}^{m,p_1}} &\leq C 2^{(d/2 - \alpha_1 - d/p_1)k} \left\| \left\| 2^{(\alpha_1 + d/p_1 - d/2)j} f_j \right\|_{l^2} \right\|_{\dot{W}^{m,p_1}} \\ &= C 2^{(d/2 - \alpha_1 - d/p_1)k} \|f\|_{\dot{W}^{m+\alpha_1+d/p_1-d/2,p_1}} \\ &\leq C 2^{(d/2 - \alpha_1 - d/p_1)k} \|f\|_{\dot{H}^{m+\alpha_1}}, \end{aligned}$$

where in the last inequality we used Sobolev embedding theorem. Similarly,

$$\|S_k f\|_{L^{p_1}} \leq C 2^{(d/2 - \alpha_1 - d/p_1)k} \|f\|_{\dot{H}^{\alpha_1}}.$$

These estimates together with (A.5) and Bernstein's inequality yield

$$\begin{aligned}
& \|S_k f g_j\|_{\dot{H}^m} \\
& \leq C 2^{(d/2 - \alpha_1 - d/p_1)j} \left(\|f\|_{\dot{H}^{m+\alpha_1}} \|g_j\|_{L^{p_2}} + \|f\|_{\dot{H}^{\alpha_1}} \|g_j\|_{\dot{W}^{m,p_2}} \right) \\
& \leq C 2^{(d/2 - \alpha_1 - \alpha_2)j} \left(\|f\|_{\dot{H}^{m+\alpha_1}} 2^{(\alpha_2 - d/p_1)j} \|g_j\|_{L^{p_2}} + \|f\|_{\dot{H}^{\alpha_1}} 2^{(\alpha_2 - d/p_1)j} \|g_j\|_{\dot{W}^{m,p_2}} \right) \\
& \leq C 2^{(d/2 - \alpha_1 - \alpha_2)j} \left(\|f\|_{\dot{H}^{m+\alpha_1}} 2^{tj} \|g_j\|_{L^2} + \|f\|_{\dot{H}^{\alpha_1}} 2^{tj} \|g_j\|_{\dot{H}^m} \right) \\
& \leq C 2^{(d/2 - \alpha_1 - \alpha_2)j} c_j (\|f\|_{\dot{H}^{m+\alpha_1}} \|g\|_{\dot{H}^{\alpha_2}} + \|f\|_{\dot{H}^{\alpha_1}} \|g\|_{\dot{H}^{m+\alpha_2}}),
\end{aligned}$$

where

$$c_j = \frac{2^{tj} \|g_j\|_{L^2}}{10 \|g\|_{\dot{H}^{\alpha_2}}} + \frac{2^{tj} \|g_j\|_{\dot{H}^m}}{10 \|g\|_{\dot{H}^{m+\alpha_2}}}.$$

It is clear that $\|c_j\|_{l^2} \leq 1$, and the lemma is proved. \square

B Estimates for moduli of continuity

Assume R is a singular integral operator with kernel $K(r, \zeta) = r^{-d} K_1(\zeta)$, where (r, ζ) are the polar coordinates and K_1 is a smooth function on S^{d-1} satisfying the cancelation property $\int_{S^{d-1}} K_1 = 0$.

Lemma B.1. Let θ be a function in \mathbb{R}^d having modulus of continuity ω . Then $R\theta$ has modulus of continuity

$$\Omega(\xi) = C \left(\int_0^\xi \omega(s)/s \, dr + \xi \int_\xi^\infty \omega(s)/s^2 \, dr \right). \quad (\text{B.1})$$

Proof. In [13], this lemma is stated and proved for Riesz transforms in \mathbb{R}^2 . The proof there can be easily modified to deal with the general case. So the details are omitted. \square

Lemma B.2. i) If $\theta \in H^4(\mathbb{R})$ has modulus of continuity ω satisfying (6.10), we

have $\|\theta_x\|_{L^\infty} < \omega'(0)$.

ii) Assume θ and u have moduli of continuity ω and Ω respectively, and $\theta(x) - \theta(y) = \omega(|x - y|)$ for some $x \neq y$. Then we have

$$|u(x)\nabla\theta(x) - u(y)\nabla\theta(y)| \leq \Omega(|x - y|)\omega'(|x - y|),$$

where Ω is defined in (B.1).

iii) Moreover, for any $\gamma \in (0, 2)$, it holds that

$$\begin{aligned} -\Lambda^\gamma\theta(x) + \Lambda^\gamma\theta(y) &\leq C(\gamma) \int_0^{\frac{r}{2}} \frac{\omega(r + 2s) + \omega(r - 2s) - 2\omega(r)}{r^{1+\gamma}} ds \\ &\quad + C(\gamma) \int_{\frac{r}{2}}^\infty \frac{\omega(2s + r) - \omega(2s - r) - 2\omega(r)}{r^{1+\gamma}} ds, \end{aligned}$$

where $r = |x - y|$.

Proof. Part i) is an easy consequence of the Sobolev embedding theorem, Taylor's formula and a spatial decay of u_x (see [13] and [9]). The proof of ii) can be found in [13], in which part iii) for $\gamma = 1$ is also proved. For $\gamma \in (0, 2)$, the proof can be found in [21]. The argument in [13] or [21] covers the case $\gamma = 2$ as well, by simply noticing that

$$\frac{1}{h\sqrt{2\pi h}} e^{-\frac{r^2}{h}} \leq C \frac{1}{r^3}.$$

□

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