Existence Result for Impulsive Third Order Periodic Boundary Value Problems

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Abstract. This paper is devoted to the study of periodic boundary value problems for nonlinear third order differential equations subjected to impulsive effects. We provide sufficient conditions on the nonlinearity and the impulse functions that guarantee the existence of at least one solution. Our approach is based on a priori estimates, the method of upper and lower solutions combined with an iterative technique, which need not be monotone.

Keywords: Third order nonlinear ordinary differential equation; impulsive effects; periodic solutions; a priori estimates; lower and upper solutions; Ascoli Arzela theorem.

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1 Introduction

The purpose of this paper is to establish the existence of periodic solutions for a class of third order ordinary differential equations subject to impulsive effects at fixed moments. More specifically,we consider the following problem:

$$-y'''(t) = f(t, y(t), y'(t), y''(t)), \text{ for a.e } t \in [0, 2\pi]\backslash\{t_1, ..., t_m\}, \tag{1}$$

$$y(0) = y(2\pi), y'(0) = y'(2\pi) \quad , y''(0) = y''(2\pi) \tag{2}$$

$$y(t_i^+) = g_i(y(t_i)), y'(t_i^+) = h_i(y'(t_i)) \quad , y''(t_i^+) = k_i(y''(t_i)) \;\; i = 1, ..., m \tag{3}$$

where $f : [0, 2\pi] \times \mathbb{R}^3 \to \mathbb{R}$ is an L^1-Carathéodory function, g_i, h_i and k_i are given real valued functions, and $t_i \in (0, 2\pi)$ for $i = 1, 2, ..., m$, such that $0 = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = 2\pi$.

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Impulsive differential equations arise naturally in the description of physical and biological phenomena that are subjected to instantaneous changes at some time instants called moments. For a good account on this theory, which has seen a significant development over the past decades we refer the interested reader to the monographs [1], [2] and the references therein. Nonlinear boundary value problems for second order impulsive differential equations have received a great deal of attention (see for instance [3], [4], [5], [6], [7], [8]). Several authors have studied nonlinear third order boundary value problems, which have applications in engineering and applied sciences. See [9], [10]. However, very few papers have been devoted to the study of higher order impulsive differential equations. We refer to [4] and [11]. In fact, very little is known in case of boundary value problems for third order impulsive differential equations. We can mention the paper [12]. It is our aim in this paper to present a self-contained contribution to this important area. We shall introduce some auxiliary functions that will play a fundamental role in our analysis. We provide sufficient conditions on the nonlinearity and the impulse functions that guarantee the existence of at least one solution. Our approach is based on a priori estimates, the method of upper and lower solutions combined with an iterative technique, which is not necessarily monotone (see [13]). This paper is organized as follows. In section 2, we introduce some notations, definitions and preliminary results that will be used in the remainder of the paper. The main result and its proof are given in section 3.

2 Preliminaries

In this section we present some notations, definitions and results that will be used in the remainder of the paper. Let I denote the closed interval $[0, 2\pi]$. For $m \in N$, let $0 = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = 2\pi$ be a subdivision of the interval $[0, 2\pi]$. Let $D = \{t_1, t_2, ..., t_m\}$. Denote by $PC(I)$ the set of functions $u : I \to \mathbb{R}$ continuous on $I\backslash D$, $u(t_k^+)$ and $u(t_k^-)$ exist, with $u(t_k^-) = u(t_k)$, for $k = 1, 2, ..., m$. For $u \in PC(I)$, we define its norm by $\|u\|_0 = \sup_{t\in I} |u(t)|$. Then $(PC(I), \|.\|_0)$ is a Banach space. Let $PC^{\ell}(I)$, $\ell = 1, 2$, denote the space of real-valued functions u, such that $u^{(\ell)} \in PC(I)$, $u^{(\ell)}(t_k^+)$ and $u^{(\ell)}(t_k^-)$ exist, with $u^{(\ell)}(t_k^-) = u^{(\ell)}(t_k)$, for $k = 1, 2, ..., m$.

Note that we can write $u \in PC^2(I)$ as

$$u(t) = \begin{cases} u_{(0)}(t) & if \quad t \in \quad [0, t_1] \\ u_{(1)}(t) & if \quad t \in \quad (t_1, t_2] \\ . & \\ . & \\ u_{(m)}(t) & if \quad t \in \quad (t_m, 2\pi] \end{cases}$$

where $u_{(i)}$ is twice continuously differentiable on (t_i, t_{i+1}) for $i = 1, ..., m$.

We shall denote by $PC_D^2(I)$ the set of functions $u \in PC^2(I)$ such that u'' is absolutely continuous on each interval (t_i, t_{i+1}) for $i = 1, ..., m$, and for each $u \in PC_D^2(I)$ we set

$$\|u\|_D := \|u\|_0 + \|u'\|_0 + \|u''\|_0 .$$

Then $PC_D^2(I)$ is a Banach space when equipped with the norm $\|u\|_D$.

Let $L^1(I) = \left\{ f : I \to \mathbb{R};\ f \text{ measurable and } \int_0^{2\pi} |f(t)|\, dt < +\infty \right\}$ equipped with its usual norm. Finally, we shall denote by $Car_{L^1}\left(I \times \mathbb{R}^3\right)$ the set of L^1-Carathéodory functions on $I \times \mathbb{R}^3$ and by $C(\mathbb{R})$ the set of continuous functions defined on $\mathbb{R}$.

Definition 1 .
We say that $f : I \times \mathbb{R}^3 \to \mathbb{R}$ *is an* L^1*-Carathéodory function on* $I \times \mathbb{R}^3$ *if :*
i) for all $(x, y, z) \in \mathbb{R}^3$, *the function* $f(., x, y, z)$ *is measurable on* I,
ii) for a.e $t \in I$, *the function* $f(t, ., ., .)$ *is continuous on* $\mathbb{R}^3$
iii) for each compact subset $\Omega \subset \mathbb{R}^3$ *,there exists a function* $f_\Omega(.) \in L^1(I)$ *such that:*

$$|f(t, x, y, z)| \leq f_\Omega(t) \text{ for a.e } t \in I \text{ and for all } (x, y, z) \in \Omega.$$

Definition 2 .
A function $u \in PC_D^2(I)$ *is a solution of problem* (1), (2), (3) *if it satisfies* (1) *almost everywhere on* $I \backslash D$, *the periodicity conditions* (2) *and the jump conditions* (3).

We shall assume that the impulse functions g_i, h_i and k_i, are continuous on $\mathbb{R}$ and satisfy for each $i = 1, ..., m$

(**H1**)(i) $m_g \leq g_i(u) \leq M_g \ \forall u \in \mathbb{R}$,
(ii) $h_i(u_1) - h_i(u_2) \geq (u_1 - u_2)$,
(iii) k_i is nondecreasing.

Let $\alpha, \beta \in PC_D^2(I)$ be such that $\alpha''(t) \leq \beta''(t)$ for each $t \in (t_i, t_{i+1})$, $i = 0, 1, 2, ..., m+1$. We introduce some auxiliary functions, which will play a fundamental role in our study.

Let

$$\mu = \max_{i=0,\ldots,m} \left(|\alpha'(t_i)|, \left|\alpha'\left(t_i^+\right)\right|\right)$$

and

$$\eta = \max_{i=0,\ldots,m} \left(\left|\beta'(t_i)\right|, \left|\beta'\left(t_i^+\right)\right|\right).$$

For $i = 0, 1, 2, ..., m$ and $t \in (t_i, t_{i+1})$ we define

$$\widetilde{\alpha}(t) = \alpha'(t) - \mu \text{ and } \tilde{\beta}(t) = \beta'(t) + \eta. \tag{4}$$

Next, for $i = 1, 2, ..., m$ and $t \in (t_i, t_{i+1})$ we let

$$\widehat{\alpha}(t) = \alpha(t) - \alpha\left(t_i^+\right) - \mu(t - t_i) + m_g, \tag{5}$$

and

$$\widehat{\beta}(t) = \beta(t) - \beta\left(t_i^+\right) + \eta(t - t_i) + M_g. \tag{6}$$

Finally, for $t \in (0, t_1)$ let

$$\hat{\alpha}(t) = \alpha(t) - \alpha\left(t_m^+\right) - 2\pi\mu + m_g, \tag{7}$$

and

$$\hat{\beta}(t) = \beta(t) - \beta\left(t_m^+\right) + 2\pi\eta + M_g. \tag{8}$$

Lemma 3 *Assume that* $\alpha, \beta \in PC_D^2(I)$ *satisfy*

(**H2**) (i) $\alpha''(t) \leq \beta''(t)$ on $I \backslash D$,
(ii) $\alpha(0^+) \leq \alpha(2\pi)$,
(iii) $\beta(2\pi) \leq \beta(0^+)$.

Then $\tilde{\alpha} \leq \tilde{\beta}$ and $\hat{\alpha} \leq \hat{\beta}$ on I.

Proof. If $\alpha'' < \beta''$ on $I \backslash D$, then for $i = 0, 1, ..., m$ and $t \in (t_i, t_{i+1})$, we have

$$\int_{t_i}^t \alpha''(s)\, ds \leq \int_{t_i}^t \beta''(s)\, ds,$$

so that

$$\alpha'(t) - \alpha'\left(t_i^+\right) \leq \beta'(t) - \beta'\left(t_i^+\right).$$

It follows from (4) that

$$\begin{aligned} \tilde{\alpha}(t) &= \alpha'(t) - \mu \leq \alpha'(t) - \alpha'\left(t_i^+\right) \\ &\leq \beta'(t) - \beta'\left(t_i^+\right) \leq \beta'(t) + \eta = \tilde{\beta}(t). \end{aligned}$$

Hence,

$$\tilde{\alpha}(t) \leq \tilde{\beta}(t) \quad \text{for all } t \in (t_i, t_{i+1}). \tag{9}$$

Now, integrating (9) on (t_i, t) for $t \in (t_i, t_{i+1})$ and $i = 1, ..., m$, we obtain, taking into account the definitions of $\tilde{\alpha}(t)$ and $\tilde{\beta}(t)$, μ, m_g and M_g, respectively,

$$\begin{aligned} & \alpha(t) - \alpha\left(t_i^+\right) - \mu(t - t_i) + m_g \\ \leq\ & \beta(t) - \beta\left(t_i^+\right) + \eta(t - t_i) + M_g, \end{aligned}$$

which, in view of (5) and (6), shows that

$$\hat{\alpha}(t) \leq \hat{\beta}(t), \text{ for all } t \in (t_i, t_{i+1}), \ i = 1, ..., m.$$

In particular, we have

$$\hat{\alpha}(2\pi) \leq \hat{\beta}(2\pi).$$

Next, for $t \in (0, t_1)$, we have

$$\int_0^t \tilde{\alpha}(s)\, ds \leq \int_0^t \tilde{\beta}(s)\, ds,$$

which implies

$$\alpha(t) - \alpha\left(0^+\right) - \mu t \leq \beta(t) - \beta\left(0^+\right) + \eta t.$$

Thus,

$$\alpha(t) - \alpha\left(0^+\right) - \mu t + \hat{\alpha}(2\pi) \leq \beta(t) - \beta\left(0^+\right) + \eta t + \hat{\beta}(2\pi).$$

Since

$$\hat{\alpha}(2\pi) = \alpha(2\pi) - \alpha\left(t_m^+\right) - \mu(2\pi - t_m) + m_g,$$

$$\hat{\beta}(2\pi) = \beta(2\pi) - \beta\left(t_m^+\right) + \eta(2\pi - t_m) + M_g,$$

and

$$\alpha(2\pi) - \alpha\left(0^+\right) \quad \geq 0 \geq \beta\left(0^+\right) - \beta(2\pi)$$

It follows that

$$\begin{aligned} & \alpha(t) - \alpha\left(0^+\right) - \mu t + \alpha(2\pi) - \alpha\left(t_m^+\right) - \mu(2\pi - t_m) + m_g \\ \geq\ & \alpha(t) - \mu t - \alpha\left(t_m^+\right) - \mu(2\pi - t_m) + m_g \\ =\ & \alpha(t) - \alpha\left(t_m^+\right) - \mu 2\pi + \mu(t_m - t) + m_g \\ \geq\ & \alpha(t) - \alpha\left(t_m^+\right) - 2\pi\mu + m_g = \hat{\alpha}(t). \end{aligned}$$

Similarly,

$$
\begin{aligned}
& \beta(t)-\beta\left(0^{+}\right)+\eta t+\beta(2\pi)-\beta\left(t_m^{+}\right)+\eta(2\pi-t_m)+M_g \\
\leq\ & \beta(t)+\eta t-\beta\left(t_m^{+}\right)+\eta(2\pi-t_m)+M_g \\
=\ & \beta(t)-\beta\left(t_m^{+}\right)+2\pi\eta+\eta(t-t_m)+M_g \\
\leq\ & \beta(t)-\beta\left(t_m^{+}\right)+2\pi\eta+M_g=\hat{\beta}(t).
\end{aligned}
$$

Comparing with (7) and (8) we see that

$$\hat{\alpha}(t) \leq \hat{\beta}(t) \quad \text{for } t\in(0,t_1).$$

This completes the proof of the lemma. ■

Suppose that (**H2**) is satisfied. For $t\in I$, we let

$$p(u)(t) := \max\left(\hat{\alpha}(t),\ \min\left(u(t),\ \hat{\beta}(t)\right)\right), \tag{10}$$

and

$$q(v)(t) := \max\left(\tilde{\alpha}(t),\ \min\left(v(t),\ \tilde{\beta}(t)\right)\right). \tag{11}$$

Denote by $[\hat{\alpha},\ \hat{\beta}]$ the set of $u\in PC_D^2(I)$ such that $\hat{\alpha}(t)\leq y(t)\leq\hat{\beta}(t)$ for all $t\in I$. Similarly, $[\tilde{\alpha},\ \tilde{\beta}]$ is the set of $v\in PC^1(I)$ such that $\tilde{\alpha}(t)\leq v(t)\leq\tilde{\beta}(t)$ for all $t\in I$.

Lemma 4 *The operators $p: PC_D^2(I)\to[\hat{\alpha},\ \hat{\beta}]$ and $q: PC^1(I)\to[\tilde{\alpha},\ \tilde{\beta}]$ defined by (10) and (11), respectively, are continuous and bounded.*

Proof. Obvious. ■

We, now, introduce the notion of lower and upper solutions that are suitable for our setting.

Definition 5 *$\alpha\in PC_D^2(I)$ is called a lower solution of problem (1), (2), (3) if:*

$$
\begin{cases}
-\alpha'''(t)\geq f(t,\hat{\alpha}(t),\tilde{\alpha}(t),\alpha''(t)) \quad \text{for a.e } t\in(0,2\pi), \\
\alpha(0)\leq\alpha(2\pi),\quad \alpha'(0)\leq\alpha'(2\pi),\quad \alpha''(0)\leq\alpha''(2\pi), \\
\alpha(t_i^{+})\leq g_i(\alpha(t_i)),\ \alpha'(t_i^{+})\leq h_i(\alpha'(t_i)),\ \alpha''(t_i^{+})\leq k_i(\alpha''(t_i)) \quad i=1,...,m.
\end{cases} \tag{12}
$$

Similarly, $\beta\in PC_D^2(I)$ is called an upper solution of problem (1), (2), (3) if the reverse of the inequalities (12) are satisfied when we substitute β for α.

Lemma 6 *Let $\varphi: I\times\mathbb{R}\to\mathbb{R}$ be a bounded L^1-Carathéodory function. Assume that there exists a positive function $\psi\in L^1(I)$ such that*

$$\varphi(t,w_2)-\varphi(t,w_1)\leq\psi(t)(w_2-w_1) \quad for\ w_1\geq w_2 \quad w_1,w_2\in\mathbb{R}$$

Then for each $a_i,b_i,c_i\in\mathbb{R}$, $i=1,2,...,m$, the following initial value problem

$$-u'''(t)=\varphi(t,u''(t)) \quad a.e \quad t\in(0,2\pi),$$

$$u\left(t_i^{+}\right)=a_i,\ \ u'\left(t_i^{+}\right)=b_i\ ,\ u''\left(t_i^{+}\right)=c_i, \qquad i=0,1,2,...,m,$$

has a unique solution $u\in PC_D^2(I)$.

Proof. **Existence**. For $\lambda \in [0,1]$ and $i = 1, 2, ..., m$, we consider the one-parameter family of problems

$$\begin{cases} -u'''(t) = \lambda\varphi\left(t\ , u''\left(t\right)\right) \quad a.e\ \ t \in (t_i, t_{i+1}) \\ u\left(t_i^+\right) = a_i \quad , \qquad u'\left(t_i^+\right) = b_i \quad , \qquad u''\left(t_i^+\right) = c_i \quad . \end{cases} \tag{13}$$

We show that there exists $\delta > 0$, not depending on λ, such that all possible solutions u of (13) satisfy

$$\|u\|_D \leq \delta. \tag{14}$$

It is easily seen that the solutions of (13) are given by

$$u\left(t\right) = \lambda[a_i + b_i\left(t - t_i^+\right) + c_i\frac{\left(t - t_i^+\right)^2}{2} - \int_{t_i^+}^{t}\int_{t_i^+}^{s}\int_{t_i^+}^{\tau}\varphi\left(\sigma\ , u''\left(\sigma\right)\right)d\sigma d\tau ds]. \tag{15}$$

This implies that

$$u'(t) = \lambda[b_i + c_i\left(t - t_i^+\right) - \int_{t_i^+}^{t}\int_{t_i^+}^{s}\varphi\left(\tau\ , u''\left(\tau\right)\right)d\tau ds],$$

and

$$u''(t) = \lambda[c_i - \int_{t_i^+}^{t}\varphi\left(s\ , u''\left(s\right)\right)ds].$$

Since φ is bounded, $N = \sup\left\{\left|\varphi\left(t\ , x\right)\right| ; \left(t\ , x\right) \in I \times \mathbb{R}\right\}$ is well defined, and for all $t \in (t_i, t_{i+1})$ we have the following estimates

$$|u(t)| \leq |a_i| + 2\pi\,|b_i| + 2\pi^2\,|c_i| + N\,(2\pi)^3\,,$$

$$|u'(t)| \leq |b_i| + 2\pi\,|c_i| + N\,(2\pi)^2\,,$$

$$|u''(t)| \leq |c_i| + N\,(2\pi)\,.$$

Letting

$$\delta := |a_i| + |b_i| + |c_i| + 2\pi\left(N + |b_i| + |c_i|\right) + (2\pi)^2\left(N + |c_i|\right) + (2\pi)^3 N,$$

we see that

$$\|u\|_D = \|u\|_0 + \|u'\|_0 + \|u''\|_0\ \leq \delta.$$

For each $z \in PC_D^2\left(I\right)$, let

$$Tz(t) := a_i + b_i\left(t - t_i^+\right) + c_i\frac{\left(t - t_i^+\right)^2}{2} - \int_{t_i^+}^{t}\int_{t_i^+}^{s}\int_{t_i^+}^{\tau}\varphi\left(\sigma\ , z''\left(\sigma\right)\right)d\sigma d\tau ds. \tag{16}$$

Eq.(16) defines an operator

$$T : PC_D^2\left(I\right) \to PC_D^2\left(I\right).$$

(a) T is continuous, for let $z_n \to z$ in $PC_D^2\left(I\right)$. Then $z_n'' \to z''$ in $PC(I)$ and

$$|Tz_n(t) - Tz(t)| \leq \int_{t_i^+}^{t}\int_{t_i^+}^{s}\int_{t_i^+}^{\tau}\left|\varphi\left(\sigma\ , z_n''\left(\sigma\right)\right) - \varphi\left(\sigma\ , z''\left(\sigma\right)\right)\right|d\sigma d\tau ds.$$

Since φ is an L^1-Carathéodory function $|\varphi(\sigma, z_n''(\sigma)) - \varphi(\sigma, z''(\sigma))| \to 0$ as $n \to +\infty$. Also, $|\varphi(\sigma, z_n''(\sigma))| \leq N$ for all $n \in \mathbb{N}$. By the Lebesgue dominated convergence theorem we have

$$\int_{t_i^+}^{t}\int_{t_i^+}^{s}\int_{t_i^+}^{\tau} |\varphi(\sigma, z_n''(\sigma)) - \varphi(\sigma, z''(\sigma))|\, d\sigma d\tau ds \to 0 \text{ as } n \to +\infty.$$

This implies that

$$\|Tz_n - Tz\|_0 \to 0 \text{ as } n \to +\infty.$$

Similarly, we can show that

$$\left\|(Tz_n)' - (Tz)'\right\|_0 \to 0 \text{ as } n \to +\infty,\ \left\|(Tz_n)'' - (Tz)''\right\|_0 \to 0 \text{ as } n \to +\infty.$$

Hence

$$\|Tz_n - Tz\|_D \to 0 \text{ as } n \to +\infty.$$

(b) Let B be a bounded subset of $PC_D^2(I)$. Then, there exists $\rho > 0$ such that

$$\|z\|_D \leq \rho \text{ for all } z \in B.$$

We show that $T(B)$ relatively compact, i.e uniformly bounded and equicontinuous. In fact, let $z \in B$. Then (14) implies that

$$\|Tz\|_D \leq \delta.$$

Also, it follows from the boundedness of φ that, for $\xi \leq \zeta$,

$$|Tz(\zeta) - Tz(\xi)| \leq \int_{\xi}^{\zeta}\int_{t_i^+}^{s}\int_{t_i^+}^{\tau} |\varphi(\sigma, z(\sigma))|\, d\sigma d\tau ds \leq N(2\pi)^2 |\zeta - \xi|.$$

By Ascoli-Arzela's theorem T is a compact operator. Moreover, the set of solutions of the equation

$$z = \lambda Tz, \text{ for } 0 < \lambda < 1,$$

is bounded. Schaefer's theorem (see [14]) implies that T has a fixed point $z \in PC_D^2(I)$. This fixed point satisfies

$$z(t) = a_i + b_i\left(t - t_i^+\right) + c_i\frac{\left(t - t_i^+\right)^2}{2} - \int_{t_i^+}^{t}\int_{t_i^+}^{s}\int_{t_i^+}^{\tau} \varphi(\sigma, z''(\sigma))\, d\sigma d\tau ds], \tag{17}$$

and $z(t_i^+) = a_i$ and $z'(t_i^+) = b_i$. Comparing Eq.(15), with $\lambda = 1$, and Eq.(17) we see that z is a solution of (13) for $\lambda = 1$.

■

Uniqueness.

Suppose that problem (13), with $\lambda = 1$, has two solutions v_1 and v_2.
Let $z(t) = v_1''(t) - v_2''(t)$, for $t \in [t_i, t_{i+1})$. Then $z\left(t_i^+\right) = v_1''\left(t_i^+\right) - v_2''\left(t_i^+\right) = 0$, and $z(t_{i+1}) = v_1''(t_{i+1}) - v_2''(t_{i+1}) = 0$. Suppose that $z(t) \geq 0$ for all $t \in (t_i, t_{i+1})$. Then for $t \in (t_i, t_{i+1})$,

$$\begin{aligned} z'(t) &= v_1'''(t) - v_2'''(t) = \varphi(t, v_2''(t)) - \varphi(t, v_1''(t)) \\ &\leq \psi(t)(v_2''(t) - v_1''(t)) \\ &\leq -\psi(t)(v_1''(t) - v_2''(t)) = -\psi(t)z(t) \leq 0. \end{aligned}$$

We see that for $t \in (t_i, t_{i+1})$,

$$z'(t) \leq 0.$$

Hence

$$0 \geq z(t)z'(t) = \frac{1}{2}\frac{d}{dt}(z(t))^2.$$

This implies that

$$0 \leq (z(t))^2 \leq \left(z(t_i^+)\right)^2 + \int_{t_i}^{t} \frac{d}{dt}(z(t))^2\, dt \leq 0,$$

which gives

$$z(t) = 0,\ t \in [t_i, t_{i+1}).$$

Similarly, if we suppose $z(t) \leq 0$ for all $t \in (t_i, t_{i+1})$ we will arrive at the same conclusion

$$z(t) = 0,\ t \in [t_i, t_{i+1}).$$

Hence

$$(v_1 - v_2)''(t) = v_1''(t) - v_2''(t) = 0 \text{ for } t \in [t_i, t_{i+1}).$$

Since

$$(v_1 - v_2)'\left(t_i^+\right) = 0 \text{ and } (v_1 - v_2)\left(t_i^+\right) = 0$$

then

$$(v_1 - v_2)'(t) = 0 \text{ and } (v_1 - v_2)(t) = 0 \text{ for } t \in [t_i, t_{i+1})$$

Consequently,

$$\text{for each } t \in [t_i, t_{i+1}) \qquad v_1(t) = v_2(t).$$

We conclude that:

$$v_1 = v_2 \quad \text{in } I,$$

which completes the proof of the lemma.

3 Main Result

In this section we state and prove our main result. For this purpose we shall assume that the nonlinearity $f \in Car_{L^1}\left(I \times \mathbb{R}^3\right)$ satisfies

(**H3**) there exist positive functions θ_1 and θ_2 such that for

$$f(t, u_1, v_1, w) - f(t, u_2, v_2, w) > -\theta_1(t)(u_1 - u_2) - \theta_2(t)(v_1 - v_2),$$

for $t \in I$, $\hat{\alpha} \leq u_1 \leq u_2 \leq \hat{\beta}$, $\tilde{\alpha} \leq v_1 \leq v_2 \leq \tilde{\beta}$, and $w \in \mathbb{R}$,

(**H4**) there exists a positive function $\gamma \in L^1(I)$ such that

$$f(t, u, v, w_2) - f(t, u, v, w_1) \leq \gamma(t)(w_2 - w_1)$$

for $t \in I$, $u \in [\hat{\alpha},\ \hat{\beta}]$, $v \in [\tilde{\alpha},\ \tilde{\beta}]$, $w_1, w_2 \in \mathbb{R}$ and $w_1 \geq w_2$.

Theorem 7 .
*Let α, β be, respectively, a lower and an upper solution of problem (1), (2), (3) such that $\alpha'' \leq \beta''$ on I . Assume that the conditions (**H1**), (**H2**), (**H3**) and (**H4**) are satisfied. Then problem (1), (2), (3) has at least one solution y such that :*

$$\tilde{\alpha} \leq y' \leq \tilde{\beta} \qquad \text{and} \qquad \hat{\alpha} \leq y \leq \hat{\beta} \qquad \text{in } I.$$

Proof. .
It follows from (H2) that $\tilde{\alpha} \leq \tilde{\beta}$ and $\hat{\alpha} \leq \hat{\beta}$; so that the operators p and q, given, respectively, by (10) and (11) are well defined. We modify the problem (1), (2), (3) as follows. For $u \in PC_D^2$ let $F: \ I \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ be defined by

$$F(t,u,v,w) = \begin{cases} f(t,p(u),q(v),\beta'') & \text{if} \quad w \geq \beta'' \\ f(t,p(u),q(v),w) & \text{if} \quad \alpha'' \leq w \leq \beta'' \\ f(t,p(u),q(v),\alpha'') & \text{if} \quad w \leq \alpha'' \end{cases}$$

Remark 8 *Since $p(u)$ and $q(v)$ are continuous and bounded, and $f \in Car_{L^1}\left(I \times \mathbb{R}^3\right)$ it follows that $F \in Car_{L^1}\left(I \times \mathbb{R}^3\right)$ and is bounded.*

■
Consider the following modified problem :

$$\begin{cases} -y''' = F(t,y,y',y'') \ \text{ for } t \in (0,2\pi) \ \text{ and } t \neq t_i \\ y(0) = y(2\pi), \, y'(0) = y'(2\pi) \ \ , y''(0) = y''(2\pi) \\ y(t_i^+) = g_i\left(y(t_i)\right), y'(t_i^+) = h_i\left(y'(t_i)\right), y''(t_i^+) = k_i\left(y''(t_i)\right), i = 1, ..., m. \end{cases} \tag{18}$$

To prove the existence of at least one solution of problem (18) we rely on an iterative technique,which is not monotone. Let $(y_j)_{j\in\mathbb{N}}$ be the sequence of functions in $PC_D^2\left(I\right)$ defined as follows, $y_0 = \hat{\alpha}$ and for $j = 1, 2, ...$

$$\begin{cases} -y_j'''\left(t\right) = F(t, y_{j-1}, y_{j-1}', y_j'') \ \text{ for } \ a.e \ \ t \in (0,2\pi) \\ y_j(0) = y_{j-1}(2\pi) \ , y_j'(0) = y_{j-1}'(2\pi) \ , \ y_j''(0) = y_{j-1}''(2\pi) \\ y_j(t_i^+) = g_i\left(y_{j-1}(t_i)\right), y_j'(t_i^+) = h_i\left(y_{j-1}'(t_i)\right), y_j''(t_i^+) = k_i\left(y_{j-1}''(t_i)\right), i = 1, ..., m. \end{cases} \tag{19}$$

Claim 9 *The sequence $(y_j)_{j=1}^{\infty}$ is well defined.*

Proof. For each $j = 0, 1, 2, ...$, the function $\varphi : I \times \mathbb{R} \to \mathbb{R}$, defined by $\varphi\left(t, z\right) = F(t, y_{j-1}, y_{j-1}', z)$ for all $t \in I$ and $z \in \mathbb{R}$, is an L^1-Carathéodory and bounded. Condition (H4) implies that

$$\varphi t, z_2) - \varphi\left(t, z_1\right) \leq \gamma\left(t\right)\left(z_2 - z_1\right).$$

The assumptions of Lemma 6 are satisfied. This shows that the problem (19) admits a unique solution $y_j \in PC_D^2\left(I\right)$.Thus,the sequence $(y_j)_{j\in N}$ is well defined . ■

Claim 10 *For all $j \in \mathbb{N}$ we have for every $t \in I$*

(i) $\alpha''(t) \leq y_j''(t) \leq \beta''(t)$
(ii) $\tilde{\alpha}\left(t\right) \leq y_j'(t) \leq \tilde{\beta}\left(t\ \right)$
(iii) $\hat{\alpha}\left(t\right) \leq y_j\left(t\right) \leq \hat{\beta}\left(t\right)$.

Proof. Since $y_0 = \hat{\alpha}$ we have $y_0 \in [\hat{\alpha}, \hat{\beta}]$. Also, the definition of $\widehat{\alpha}$ implies that

$$y_0'(t) = \hat{\alpha}'(t) = \alpha'(t) \text{ and } y_0''(t) = \hat{\alpha}''(t) = \alpha''(t).$$

Since $\alpha'(t) = \tilde{\alpha}(t) + \mu \geq \tilde{\alpha}(t)$, we see that $y_0'(t) \geq \tilde{\alpha}(t)$. Similarly, we can show that $y_0'(t) \leq \widetilde{\beta}(t)$. Also, $y_0''(t) = \alpha''(t)$ implies that $\alpha''(t) \leq y_0''(t)$. In a similar way, we show that $\beta''(t) \geq y_0''(t)$. ■
Suppose that for $n = 1, ..., j-1$, we have

$$\alpha'' \leq y_n'' \leq \beta'', \ \widetilde{\alpha} \leq y_n' \leq \widetilde{\beta}, \ \widehat{\alpha} \leq y_n \leq \widehat{\beta}.$$

(i) We show that $\alpha'' \leq y_j''$. Let

$$w(t) = y_j''(t) - \alpha''(t).$$

First, $w\left(t_i^+\right) \geq 0$. This follows from the above induction assumption and the fact that k_i is nondecreasing.

$$\alpha''\left(t_i^+\right) \leq k_i\left(\alpha''(t_i)\right) \leq k_i\left(y_{j-1}''(t)\right) = y_j''\left(t_i^+\right).$$

Suppose, next, that there exists $\tau \in (t_i, t_{i+1})$, $i = 0, 1, ..., m$, such that $w(\tau) < 0$. Then, there exists $t^* \in (t_i, t_{i+1})$ such that

$$w(t^*) = \min\{w(t); t_i \leq t \leq t_{i+1}\} < 0, \text{ and } w'(t^*) = 0.$$

Hence

$$y_j''(t^*) < \alpha''(t^*),$$

and

$$\begin{aligned} 0 &= y_j'''(t^*) - \alpha'''(t^*) = \\ &\quad -F(t^*, y_{j-1}(t^*), y_{j-1}'(t^*), y_j''(t^*)) - \alpha'''(t^*) \\ &\geq f(t^*, \hat{\alpha}(t^*), \tilde{\alpha}(t^*), \alpha''(t^*)) - f(t^*, p(y_{j-1}(t^*)), q(y_{j-1}'(t^*)), \alpha''(t^*)) \\ &\geq f(t^*, \hat{\alpha}(t^*), \tilde{\alpha}(t^*), \alpha''(t^*)) - f(t^*, y_{j-1}(t^*), y_{j-1}'(t^*), \alpha''(t^*)) \end{aligned}$$

Condition (H3) implies that

$$\begin{aligned} &f(t^*, \hat{\alpha}(t^*), \tilde{\alpha}(t^*), \alpha''(t^*)) - f(t^*, y_{j-1}(t^*), y_{j-1}'(t^*), \alpha''(t^*)) \\ > \ &-\theta_1(t^*)(\hat{\alpha}(t^*) - y_{j-1}(t^*)) - \theta_2(t^*)\left(\tilde{\alpha}(t^*) - y_{j-1}'(t^*)\right) > 0. \end{aligned}$$

Hence, we arrive at a contradiction. Proceeding as above we show that $y_j'' \leq \beta''$.
(b) We show that

$$\widetilde{\alpha}(t) \leq y_j'(t) \leq \widetilde{\beta}(t) \text{ for all } t \in I.$$

We prove the first inequality only, the second one can be handled in a similar way.
(ii.1) Let $t \in (0, t_1)$. Since $y_j''(t) - \alpha''(t) \geq 0$, a simple integration on $[0, t]$ gives

$$\begin{aligned} \alpha'(t) - \alpha'\left(0^+\right) &\leq y_j'(t) - y_j'\left(0^+\right) \\ &= y_j'(t) - y_{j-1}'(2\pi) \\ &\leq y_j'(t) - \widetilde{\alpha}(2\pi). \end{aligned}$$

This implies that

$$\alpha'(t) - \alpha'\left(0^+\right) + \widetilde{\alpha}(2\pi) \leq y_j'(t).$$

Since $\alpha'(0^+) = \widetilde{\alpha}(0^+) + \mu$ it follows that

$$\alpha'(t) + \widetilde{\alpha}(2\pi) - \widetilde{\alpha}\left(0^+\right) - \mu \leq y_j'(t).$$

But,

$$\widetilde{\alpha}(2\pi) - \widetilde{\alpha}\left(0^+\right) = \alpha'(2\pi) - \alpha'\left(0^+\right) \geq 0.$$

Hence,

$$\widetilde{\alpha}(t) = \alpha'(t) - \mu \leq y_j'(t) \text{ for all } t \in (0, t_1).$$

(ii.2) Next, let $t \in (t_i, t_{i+1})$, $i = 1, 2, ..., m$. We have

$$y_j'(t) - y_j'\left(t_i^+\right) = \int_{t_i}^{t} y_j''(s)ds \geq \int_{t_i}^{t} \alpha''(s)ds.$$

Thus,

$$\begin{aligned} y_j'(t) &\geq y_j'\left(t_i^+\right) + \alpha'(t) - \alpha'(t_i^+) \\ &\geq \alpha'(t) - \alpha'(t_i^+) + h_i\left(y_{j-1}'(t_i)\right) \\ &\geq \alpha'(t) - h_i\left(\alpha'(t_i)\right) + h_i\left(y_{j-1}'(t_i)\right) \\ &\geq \alpha'(t) + h_i\left(y_{j-1}'(t_i)\right) - h_i\left(\alpha'(t_i)\right). \end{aligned}$$

From the condition on h_i, (see (H1)), we infer that

$$\begin{aligned} \left(h_i\left(y_{j-1}'(t_i)\right) - h_i\left(\alpha'(t_i)\right)\right) &\geq \left(y_{j-1}'(t_i) - \alpha'(t_i)\right) \\ &\geq \widetilde{\alpha}(t_i) - \alpha'(t_i) = -\mu \end{aligned}$$

Therefore,

$$y_j'(t) \geq \alpha'(t) - \mu = \widetilde{\alpha}(t) \text{ for all } t \in (t_i, t_{i+1}),\ i = 1, 2, ..., m.$$

(iii) We show that $\hat{\alpha}(t) \leq y_j(t) \leq \hat{\beta}(t)$. Here again, we prove only the first inequality, since the second one can be proved in a similar way.
(iii.1) Let $t \in (0, t_1)$. We have

$$\begin{aligned} y_j(t) - y(0^+) &= \int_0^t y_j'(s)ds \\ &\geq \int_0^t \widetilde{\alpha}(s)\,ds = \int_0^t \left(\alpha'(s) - \mu\right)ds \\ &= \alpha(t) - \alpha\left(0^+\right) - \mu t \end{aligned}$$

Since $y(0^+) = y_{j-1}(2\pi)$, it follows that

$$\begin{aligned} y_j(t) &\geq y_{j-1}(2\pi) + \alpha(t) - \alpha\left(0^+\right) - \mu t \\ &\geq \alpha(t) - \alpha\left(0^+\right) - \mu t + \widehat{\alpha}(2\pi) \end{aligned}$$

Now, we have $\alpha(2\pi) \geq \alpha(0^+)$, so that

$$y_j(t) \geq \alpha(t) - \alpha(2\pi) - \mu t + \widehat{\alpha}(2\pi).$$

Recall that $\widehat{\alpha}(2\pi) = \alpha(2\pi) - \alpha(t_m^+) - \mu(2\pi - t_m^+) + m_g$. The above inequality becomes

$$\begin{aligned} y_j(t) &\geq \alpha(t) - \mu t - \alpha\left(t_m^+\right) - \mu\left(2\pi - t_m^+\right) + m_g \\ &\geq \alpha(t) - 2\pi\mu - \alpha\left(t_m^+\right) + \mu\left(t_m^+ - t\right) + m_g \\ &\geq \alpha(t) - 2\pi\mu - \alpha\left(t_m^+\right) + m_g = \widehat{\alpha}(t), \ t \in (0, t_1). \end{aligned}$$

(iii.2) Now, let $t \in (t_i, t_{i+1})$, $i = 1, 2, ..., m$. We have

$$y_j(t) - y(t_i^+) = \int_{t_i^+}^{t} y_j'(s)ds \geq \int_{t_i^+}^{t} (\alpha'(s) - \mu)\, ds$$

Hence,

$$\begin{aligned} y_j(t) &\geq y(t_i^+) + \alpha(t) - \alpha\left(t_i^+\right) - \mu\left(t - t_i^+\right) \\ &= g_i(y_{j-1}(t_i)) + \alpha(t) - \alpha\left(t_i^+\right) - \mu\left(t - t_i^+\right) \\ &\geq m_g + \alpha(t) - \alpha\left(t_i^+\right) - \mu\left(t - t_i^+\right) = \widehat{\alpha}(t). \end{aligned}$$

This completes the proof of the claim.
Let $C = \max\{\|\alpha''\|_0, \|\beta''\|_0\}$. It follows from the above claim that the following estimates hold.

$$\|y_j\|_0 \leq \delta_0 := \max\{\|\widehat{\alpha}\|_0, \left\|\widehat{\beta}\right\|_0\}, \tag{20}$$

$$\left\|y_j'\right\|_0 \leq \delta_1 := \max\{\|\widetilde{\alpha}\|_0, \left\|\widetilde{\beta}\right\|_0\}, \tag{21}$$

$$\left\|y_j''\right\|_0 \leq C. \tag{22}$$

Hence, there exists $M > 0$, $M = \delta_0 + \delta_1 + C$, such that, for all $j = 0, 1, 2, ...$

$$\|y_j\|_D \leq M. \tag{23}$$

The set $\Delta := \{(u, v, w) \in \mathbb{R}^3; |u| \leq \delta_0, |v| \leq \delta_1, |w| \leq C\}$ is a compact subset of $\mathbb{R}^3$. Since $F \in Car_{L^1}\left(I \times \mathbb{R}^3\right)$ there exists $\chi_\Delta \in L^1$ such that

$$|F(t, u, v, w)| \leq F_\Delta(t) \text{ for all } (u, v, w) \in \Delta. \tag{24}$$

The definition of y_j, the estimates (20), (21), and (22) imply that for all $j = 1, 2, ...$

$$\left(y_{j-1}, y_{j-1}', y_j''\right) \in \Delta. \tag{25}$$

The differential equation in (19), (24) and (25) yield

$$\left|y_j'''(t)\right| \leq F_\Delta(t) \text{ for a.e.} t \in I. \tag{26}$$

Thus,

$$y_j''' \in L^1(I).$$

Since $y_j''(t) = y_j''(0) + \int_0^t y_j'''(s)\, ds$, it follows that y_j'' is absolutely continuous.
Therefore

$$y_j \in PC_D^2(I).$$

Moreover, the sequence $(y_j)_{j\in\mathbb{N}}$ is uniformly bounded (see (23)).

By Ascoli-Arzela theorem $(y_j)_{j\in\mathbb{N}}$ has a subsequence, which we label the same, and which converges, in $PC_D^2(I)$, to some function y, which satisfies the estimates (20), (21), (22). Moreover $(y, y', y'') \in \Delta$.
Lebesgue dominated convergence theorem implies that, for all $t \in (t_i, t_{i+1})$

$$\int_{t_i}^{t} F(s, y_{j-1}(s), y'_{j-1}(s), y''_j(s))ds \rightarrow \int_{t_i}^{t} F(s, y(s), y'(s), y''(s))ds, \text{ as } j \rightarrow \infty.$$

Thus, as $j \rightarrow \infty$

$$y''_j(t) = y''_j(t_i^+) - \int_{t_i}^{t} F(s, y_{j-1}(s), y'_{j-1}(s), y''_j(s))ds \rightarrow y''(t) = y''(t_i^+) - \int_{t_i}^{t} F(s, y(s), y'(s), y''(s))ds.$$

Recall that $F \in Car_{L^1}\left(I \times \mathbb{R}^3\right)$. It follows that y'' is absolutely continuous; so that $y \in PC_D^2$ and

$$-y'''(t) = F(t, y(t), y'(t), y''(t)) \text{ for all } t \neq t_i. \tag{27}$$

Taking limit as $j \rightarrow \infty$ in the periodicity conditions in (19) we obtain

$$y(0) = y(2\pi),\ y'(0) = y'(2\pi),\ y''(0) = y''(2\pi). \tag{28}$$

Since the impulse functions are continuous we will have

$$y(t_i^+) = g_i(y(t_i)),\ y'(t_i^+) = h_i(y'(t_i)),\ y''(t_i^+) = k_i(y''(t_i)),\ i = 1, ..., m. \tag{29}$$

From (27), (28) and (29) we see that y is a solution of problem (18).
The definition of F and (27) imply

$$-y'''(t) = f(t, p(y(t)), q(y'(t)), y''(t)) \text{ for all } t \neq t_i. \tag{30}$$

Claim 10 implies that

$$y \in [\hat{\alpha},\ \hat{\beta}],\ y' \in [\tilde{\alpha}, \tilde{\beta}],\ y'' \in [\alpha'', \beta'']. \tag{31}$$

From (10) and (11) we infer that

$$p(y) = y \text{ and } q(y') = y'.$$

Hence (30) becomes

$$-y'''(t) = f(t, y(t), y'(t), y''(t)) \text{ for all } t \neq t_i. \tag{32}$$

Combining (32), (28) and (29) we conclude that y is a solution of (1), (2), (3). This completes the proof of the theorem.

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