Nonlocal Problems for Parabolic Inclusions

Abdelkader Boucherif
Brown University Box F
Division of Applied Mathematics
Providence, RI 02912 USA
Abdelkader_Boucherif@brown.edu
and
King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics,
Box 5046 Dhahran, 31261, Saudi Arabia
aboucher@kfupm.edu.sa

Abstract. Let Ω be a an open bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$, and $T > 0$. Define $D = \Omega \times (0, T)$ and $\Gamma = \partial\Omega \times [0, T]$. In this paper we are concerned with the existence of solutions of the following parabolic inclusion $u_t + Lu \in F(x, t, u)$, $(x, t) \in D$, $u(x, t) = 0$, $(x, t) \in \Gamma$ subjected to the nonlocal condition $u(x, 0) = \int_0^T g(x, t, u(x, t))dt$ for $x \in \Omega$. We provide sufficient conditions on L, F, g that guarantee the existence of at least one solution. Our technique is based on the Green's function for linear parabolic partial differential equations, fixed point theorems for multivalued maps.
Keywords. parabolic problems; integral representation of solutions; multivalued maps; nonlocal conditions; fixed point theorems.
AMS (MOS) Subject Classification: 35K20; 35K55; 35K60; 35C15; 35A05; 35B50; 35R05; 35R70

1 Introduction

Let Ω be a an open bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$. We denote the norm (usually the Euclidean norm) of $x \in \Omega$ by $\|x\|$. Let T be a positive real number. Define $D = \Omega \times (0, T)$ and $\Gamma = \partial\Omega \times [0, T]$. Our objective is to investigate the existence of solutions of the following parabolic problem with a multivalued right-hand side and nonlocal initial condition

$$D_t u + Lu \in F(x,t,u) \qquad (x,t) \in D, \tag{1}$$

$$u(x,t) = 0 \qquad\qquad (x,t) \in \Gamma, \tag{2}$$

$$u(x,0) = \int_0^T g(x,t,u(x,t))dt \quad x \in \Omega, \tag{3}$$

where L is an elliptic operator given by

$$Lu = -\sum_{i,j=1}^{N} a_{ij}(x,t) D_i D_j u + c(x,t)u.$$

Parabolic problems with discontinuous nonlinearities arise as simplified models in the description of porous medium combustion (see for instance [16], [17]), chemical reactor theory (see [18]). Also, best response dynamics arising in game theory can be modeled by a parabolic equation with a discontinuous right hand side (see [12], [21] for details and references). Parabolic problems with discontinuous nonlinearities have been also investigated in the papers [7], [6], [35], [36], [38]. On the other hand parabolic problems with integral boundary conditions appear in the modeling of concrete problems, such as heat conduction [5], [23], [9], thermoelasticity [11]. Several papers have been devoted to the study of parabolic problems with integral conditions [10], [32], [41]. Many authors have dealt with parabolic problems with continuous nonlinearities and nonlocal conditions of the form $u(x,0) + \sum_{i=0}^{m} \beta_i(x) u(x,t_i) = \psi(x)$ for $x \in \Omega$. See for instance [3], [15], [24], [29], [30]. We refer to [8] for details and references concerning the linear problem with the above type of nonlocal conditions. A good account on numerical treatment of parabolic problems with integral conditions can be found in [13].

In this paper we consider a nonlocal problem for a class of nonlinear parabolic equations with a multivalued right hand side. We shall convert Problem (1), (2), (3), to an integral inclusion using the properties of the Green's function corresponding to the linear problem. We, then, provide sufficient conditions on the data that will guarantee that the problem under consideration has at least one solution. Our approach is based on fixed point theorems for suitable multivalued operators.

The outline of the paper is as follows. In section 2 we introduce notations and preliminary results which will be used in the paper. In section 3, we shall recall the main properties of upper semicontinuous multivalued maps. We state and prove our main results in section 4.

2 Preliminaries

In this section we introduce some notations and preliminary results which will be used in the paper. Let Ω be a an open bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$. Let T be a positive real number. Define $D = \Omega \times (0,T)$ and $\Gamma = \partial\Omega \times [0,T]$. Then Γ is smooth and any point on Γ satisfies the inside (and outside) strong sphere property , i.e. for any $(x_0,t_0) \in \Gamma$ there is a closed ball $B \subset \Omega$ (and a closed ball $\widetilde{B}$ outside Ω) such that $\Gamma \cap (B \times [0,T]) = \{(x_0,t_0)\}$, (and $\Gamma \cap \left(\widetilde{B} \times [0,T]\right) = \{(x_0,t_0)\}$) (see [19]). For $u : D \to \mathbb{R}$ we denote its partial derivatives (when they exists) by $D_t u = \partial u/\partial t$, $D_i u = \partial u/\partial x_i$, $D_i D_j u = \partial^2 u/\partial x_i \partial x_j$, $i, j = 1, ..., N$.

$C(D)$ denotes the Banach space of continuous functions $u : D \to \mathbb{R}$, endowed with the norm

$$|u|_0 = \sup\{|u(x,t)|\,;\ (x,t) \in \overline{D}\}.$$

We say that $u \in C^{2,1}(D)$ if $u,\ D_i u,\ D_i D_j u$ and $D_t u$ exist and are continuous on D. In fact, we can write (see [4])

$$C^{2,1}(D) = \{u \in C(D);\ u(.,t) \in C^2(\Omega), t \in (0,T)\,,\ u(x,.) \in C^1\,(0,T)\,, x \in \Omega\}.$$

$u \in C(D)$ is called Hölder continuous of order $\alpha \in (0,1]$ if

$$H_\alpha\,(u) = \sup\{\frac{|u(x,t) - u(\xi,\tau)|}{\left(\|x-\xi\|^2 + |t-\tau|\right)^{\alpha/2}}; (x,t)\,, (\xi,\tau) \in D\} < +\infty.$$

In this case we write $u \in C^{\alpha}(D)$ and we define its norm by

$$|u|_{\alpha} = |u|_0 + H_{\alpha}(u).$$

If $\alpha = 1$, u is called Lipschitz continuous, and we write $u \in Lip(D)$. Note that the natural injection $i : C^{\alpha}(D) \to C(D)$ is continuous. We say that that $C^{\alpha}(D)$ is continuously embedded in $C(D)$, and we write $C^{\alpha}(D) \hookrightarrow C(D)$. Also, $u \in C^{2+\alpha,1+\alpha}(D)$ if $u(.,t) \in C^{2+\alpha}(\Omega)$ for all $t \in (0,T)$ and $u(x,.) \in C^{1+\alpha}(0,T)$ for all $x \in \Omega$. For $u \in C^{2+\alpha,1+\alpha}(D)$ we define its norm by

$$|u|_{2+\alpha,1+\alpha} = |u|_{\alpha} + \sum_{i=1}^{N} |D_i u|_{\alpha} + \sum_{i,j=1}^{N} |D_i D_j u|_{\alpha} + |D_t u|_{\alpha}.$$

We say that $\partial\Omega$ is in the class $C^{\ell+\alpha}, \ell \in \mathbb{N},\ \alpha \in [0,1)$ if in a neighborhood of each point of $\partial\Omega$ there is a local reprentation of $\partial\Omega$ having the form $x_i = \vartheta_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_N)$ with $\vartheta_i \in C^{\ell+\alpha}$.

Next, we introduce the Lebesgue spaces. For $1 \leq p < +\infty$, we say that $u : \Omega \to \mathbb{R}$ is in $L^p(\Omega)$ if u is measurable and $\int_{\Omega} |u(x)|^p \, dx < +\infty$, in which case we define its norm by

$$|u|_{L^p} = \left(\int_{\Omega} |u(x)|^p \, dx \right)^{1/p}.$$

For $p = +\infty$, we write

$$|u|_{\infty} = ess\sup\{|u(x)| \,; x \in \Omega\} = \inf_{N \subset \Omega, \mu(N)=0} \sup_{x \in \Omega \setminus N} |u(x)|\,,\ \mu = \text{Lebesgue measure}.$$

If $\mu(\Omega) < +\infty$ then we have the natural embeddings

$$L^q(\Omega) \subset L^p(\Omega) \text{ for } p < q.$$

In particular,

$$L^2(\Omega) \subset L^1(\Omega),$$

for, it is clear that

$$|u|_{L^1} \leq |u|_{L^2} \ (\mu(\Omega))^{1/2}.$$

Consider the linear nonhomogeneous problem

$$D_t u + Lu = f(x,t), \qquad (x,t) \in D, \tag{4}$$

$$u(x,t) = 0, \qquad (x,t) \in \Gamma, \tag{5}$$

with the following nonlocal boundary condition

$$u(x,0) = \int_0^T g(x,t,u(x,t))dt \quad x \in \Omega. \tag{6}$$

We shall assume throughout this paper that the functions $a_{ij}, c : D \to \mathbb{R}$ are Hölder continuous, $a_{ij} = a_{ji}$ and moreover, there exist positive numbers λ_0, λ_1 such that

$$\lambda_0 \left\|\xi\right\|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \leq \lambda_1 \ \left\|\xi\right\|^2, \ \forall \xi \in \mathbb{R}^N \text{ and } \forall (x,t) \in D.$$

Let $u_0 : \Omega \to \mathbb{R}$ be continuous. For the problem (4), (5) together with initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{7}$$

we have the classical result

Lemma 2.1 *(see [19], [27], [28], [34]). Assume that the functions f and u_0 are Hölder continuous. Then, Problem (4), (5), (7) has a unique solution $u \in C^{2,1}(D) \cap C(\overline{D})$, given by*

$$u(x,t) = \int_\Omega G(x,t;y,0)\ u_0\left(y\right)dy + \int_0^t \int_\Omega G(x,t;y,s)\ f\left(y,s\right)dyds, \quad (x,t) \in D, \tag{8}$$

where $G(x,t;y,s)$, is the Green's function corresponding to the linear homogeneous problem.

3 Multivalued Functions

We, now, introduce some useful definitions and properties from set-valued analysis. For complete details on multivalued maps we refer the interested reader to the books [1], [2], [14], and [22].

Let $(X, |.|_X)$ and $(Y, |\cdot|_Y)$ be Banach spaces. We shall denote the set of all subsets of X having property ℓ by $P_\ell(X)$. For instance, $U \in P_{cl}(X)$ means U closed in X; when $\ell = b$ we have the bounded subsets of X, $\ell = cv$ for convex subsets, $\ell = cp$ for compact subsets and $\ell = cp, cv$ for compact and convex subsets. The domain of a multivalued map $F : X \to 2^Y$ is the set $domF = \{z \in Z;\ R(z) \neq \emptyset\}$. F is convex (closed) valued if $F(z)$ is convex (closed) for each $z \in X$. F is bounded on bounded sets if $F(A) = \cup_{z\in A} F(z)$ is bounded in Y for all $A \in P_b(X)$ (i.e. $\sup_{z\in A}\{\sup\{|y|\,; y \in F(z)\}\} < \infty$). F is called upper semicontinuous (usc) on X if for each $z \in X$ the set $F(z) \in P_{cl}(Y)$ is nonempty, and for each open subset Y_0 of Y containing $F(z)$, there exists an open neighborhood Π of z such that $F(\Pi) \subset Y_0$. In terms of sequences, F is usc if for each sequence $(z_n) \subset X$, $z_n \to z_0$, and B a closed subset of Y such that $F(z_n) \cap B \neq \emptyset$ then $F(z_0) \cap B \neq \emptyset$.

The set-valued map F is called completely continuous if $F(A)$ is relatively compact in Y for every $A \in P_b(X)$. If F is completely continuous with nonempty compact values, then F is usc if and only if F has a closed graph (i.e. $z_n \to z$, $w_n \to w$, $w_n \in F(z_n) \Rightarrow w \in F(z)$). When $X \subset Y$ then F has a fixed point if there exists $z \in X$ such $z \in F(z)$. A multivalued map $F : \overline{D} \to P_{cl}(\mathbb{R})$ is called measurable if for every $\theta \in \mathbb{R}$, the function $v \longmapsto dist(\theta, F(v)) = \inf\{|\theta - z|\,; z \in F(v)\}$ is measurable. Finally, we let $|F(x,t,u)| := \sup\{|v|\,; v \in F(x,t,u)\}$.

Definition 3.1 *A multivalued map $F : D \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \emptyset$ is called an L^2- Carathéodory multifunction if*

(i) $(x,t) \mapsto F(x,t,u)$ is measurable for each $u \in \mathbb{R}$,

(ii) $u \mapsto F(x,t,u)$ is upper semicontinuous for almost all $(x,t) \in D$,

(iii) for each $r > 0$ there exists $\omega_r \in L^2(D)$ such that $|F(x,t,u)| \leq \omega_r(x,t)$ a.e. on D whenever $|u| \leq r$.

Definition 3.2 *For $u \in C(D)$, the set of L^2-selections of the multivalued map F is defined by*

$$S_{F,u} = \{v \in L^2(D);\ v(x,t) \in F(x,t,u(x,t)),\ a.e.\, (x,t) \in D\}.$$

This set may be empty. However, it can be shown that if $\inf\{|v|\,;\ v \in F(.,.,u\}$ is in $L^2(D)$ then $S_{F,u} \neq \emptyset$. See [40, Theorem 5.10].

Definition 3.3 *Let $F : D \times \mathbb{R} \to 2^{\mathbb{R}}$ be an L^2- Carathéodory multifunction with nonempty compact values. The Nemitsky operator of F is the set-valued operator $\mathcal{F} : C(D) \to L^2(D)$, defined by*

$$\mathcal{F}(u) := \{v \in L^2(D);\ v(x,t) \in F(x,t,u(x,t)),\ a.e.\, (x,t) \in D\}.$$

Notice that $\mathcal{F}(u) = S_{F(.,.,u(.,.))}$.

Using the properties of the Green's function we get the following results (see [33], [37]).

Lemma 3.1 *Assume the single valued map $h \in C(D)$. Let $\gamma : L^2(D) \to C(D)$ be defined by*

$$\gamma(f)(x,t) = h(x,t) + \int_0^t \int_\Omega G(x,t;y,s) f(y,s)\,dyds$$

is continuous.

Lemma 3.2 *Assume $F : D \times \mathbb{R} \to \mathbb{R}$ is an L^2- Carathéodory multifunction with nonempty, compact, convex values and $h \in C(D)$. Then the operator $\gamma \circ F$ is of usc type; i.e. is usc, completely continuous and has nonempty, compact, convex values.*

Theorem 3.1 *(Nonlinear alternative for multivalued maps [20], [33]). Let K be a convex subset of a Banach space E, $U \subseteq K$ be relatively open, and $p \in U$. Suppose $F : \overline{U} \to K$ is an usc compact multivalued operator with nonempty, compact, convex values. Then either*

(i) there is $u \in \overline{U}$ such that $u \in Fu$; or

(ii) there is $u \in \partial U$ and a $\lambda \in (0,1)$ such that $u \in \lambda Fu + (1-\lambda)\, p$.

Definition 3.4 *Let (Z,d) be a metric space and let A, B be two nonempty substes of Z. The Hausdorf distance between A and B is defined by*

$$d_H(A,B) := \max\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\},$$

where $d(a,B) = \inf\{d(a,b); b \in B\}$ and $d(A,b) = \inf\{d(a,b); a \in A\}$. Then one can show that $(P_{cl}(Z),\ d_H)$ is a metric space.

Definition 3.5 *A multivalued operator $F : Z \to P_{cl}(Z)$ is called*

(i) $\delta-$Lipschitz if and only if there exists $\delta > 0$ such that $d_H\left(F(u), F(v)\right) \leq \delta\ d\left(u,v\right)$ for all $u, v \in Z$

(ii) a contraction if and only if it is $\delta-$Lipschitz with $\delta < 1$.

4 Existence Results

Before stating and proving our main results we introduce the notion of strong solutions of Problem (1), (2), (3).

Definition 4.1 $u \in C^{2,1}(D) \cap C\left(\overline{D}\right)$ *is called a strong solution of (1), (2), (3) if there exists a single-valued function $f \in Lip(D)$, i.e. $|f(x,t) - f(y,s)| \leq \ell_f\left(\|x-y\| + |t-s|\right)$ such that $f(x,t) \in F(x,t,u(x,t))$ and (4), (5), (6) hold.*

We shall assume throughout the rest of the paper that the multivalued map F has a Lipschitz selection. This is not a restrictive condition, as shown by the following result.

Theorem 4.1 *[26, Theorem 2]. Let $F : \mathbb{R} \to 2^{\mathbb{R}} \backslash \emptyset$ be a Lipschitz multivalued map with closed values. Then F has a Lipschitz selection.*

Also, we shall suppose that $g : D \times \mathbb{R} \to \mathbb{R}$ is continuous and $g(x,t,0) = 0$ for all $(x,t) \in D$.

The integral representation (8) shows that u is a solution of problem (4), (5), (6) if and only if u satisfies

$$u(x,t) = \int_{\Omega} G(x,t;y,0) \int_0^T g(y,s,u(y,s))dsdy + \int_0^t \int_{\Omega} G(x,t;y,s)\, f\,(y,s)\, dyds,\ (x,t) \in D. \tag{9}$$

Theorem 4.2 *Suppose that $F : D \times \mathbb{R} \to 2^{\mathbb{R}}$ is an L^2-Carathéodory multifunction with closed values and the following conditions are satisfied.*

(H1) There exists $\ell_0 \in L^2(D, \mathbb{R}^+)$ such that

$d_H\left(F(x,t,u), F(x,t,z)\right) \leq \ell_0\,(x,t)\,|u - z|\,,\ $ *a.e.* $(x,t) \in D,\ u, z \in \mathbb{R},$

(H2) there exist $\eta \in Lip(\Omega)$, $\omega \in L^1(0,T)$, $\sigma_0 \in C(\mathbb{R}_+)$ nondecreasing with $\sigma_0(r) < r$ such that $|g(x,t,u) - g(x,t,v)| \leq \eta\,(x)\,\omega(t)\sigma_0\,(|u - v|)\,,$

(H3) $|G|_{L^2}\,|\ell_0|_{L^2} + (max_{(x,t)\in D} \int_{\Omega} G(x,t;y,0)\eta\,(y)\,dy)\,|\omega|_{L^1} < 1.$

Then Problem (1), (2), (3) has a strong solution.

Proof. It follows from the representation (9) that u is a solution of problem (1), (2), (3) if and only if u is a fixed point of the multivalued operator

$$\Lambda : X \to 2^X,\quad X = C(\overline{D}),$$

defined by

$$\Lambda u = h_g + G\mathcal{F}(u) \tag{10}$$

where

$$h_g(x,t) = \int_\Omega G(x,t;y,0) \int_0^T g(y,s,u(y,s))dsdy, \quad (x,t) \in D, \tag{10.a}$$

and

$$G\mathcal{F}(u)(x,t) = \int_0^t \int_\Omega G(x,t;y,s)\ \mathcal{F}(u)\,(y,s))dyds, \quad (x,t) \in D. \tag{10.b}$$

Notice that Λ is the sum of the single-valued operator $h_g \in C(D)$ and a multivalued operator $G\mathcal{F}$, where $\mathcal{F}$ is the Nemitsky operator associated with the multifunction F. We show that $\Lambda u \in P_{cl}(X)$ for any $u \in X$. For, let $(z_n)_{n\in\mathbb{N}} \subset X$, $z_n \in \Lambda u$, $z_n \to z$ in X. Then, $z \in X$ and there exists $f_n \in S_{F,u}$, i.e. $f_n \in L^2(D)$ and $f_n(x,t) \in F(x,t,u(x,t))$ such that

$$z_n(x,t) = h_g(x,t) + \int_0^t \int_\Omega G(x,t;y,s)\ f_n(y,s)dyds, \quad (x,t) \in D.$$

Since F is an L^2-Carathéodory it follows that $(\ f_n)_{n\in\mathbb{N}}$ is bounded, and passing to subsequences if necessary, converges to some $f \in L^2(D)$. By the Lebesgue dominated convergence we get

$$z(x,t) = h_g(x,t) + \int_0^t \int_\Omega G(x,t;y,s)\ f(y,s)dyds, \quad (x,t) \in D,$$

which shows that $z \in \Lambda u$. Hence Λu is nonempty and closed.

Next, we show that Λ is a contraction. For, let $u_1, u_2 \in X$ and consider $z_i \in \Lambda u_i$, $i = 1,2$. Then, there exist $h_i \in S_{F,u_i}$, $i = 1,2$ such that for every $(x,t) \in D$ and $i = 1,2$

$$z_i(x,t) = \int_\Omega G(x,t;y,0) \int_0^T g(y,s,u_i(y,s))dsdy + \int_0^t \int_\Omega G(x,t;y,s)\ h_i(y,s)dyds, \quad .$$

Then

$$z_1\,(x,t) - z_2\,(x,t) =$$

$$\int_\Omega G(x,t;y,0)\ \int_0^T [g(y,s,u_1(y,s)) - g(y,s,u_1(y,s))]dsdy$$

$$+ \int_0^t \int_\Omega G(x,t;y,s)\ [h_1(y,s) - h_2(y,s)]dyds, \quad (x,t) \in D.$$

(H1) and (H2) yield

$$|z_1\,(x,t) - z_2\,(x,t)| \leq$$

$$\int_\Omega G(x,t;y,0)\eta\,(y) \int_0^T \omega(s)\sigma_0\,(|u_1(y,s) - u_2(y,s)|)\,dsdy$$

$$+ \int_0^t \int_\Omega G(x,t;y,s)\ell(y,s)\ \ |u_1(y,s) - u_2(x,t)|\,dyds \leq$$

$$\int_\Omega G(x,t;y,0)\eta\,(y)\,dy\,|\omega|_{L^1}\,\sigma_0\,(|u_1 - u_2|_0)$$

$$+ \int_0^T \int_\Omega G(x,t;y,s)\ \ \ell_0(y,s)dyds\,|u_1 - u_2|_0$$

Hence

$$|z_1\,(x,t) - z_2\,(x,t)| \leq$$

$$\int_\Omega G(x,t;y,0)\eta\,(y)\,dy\,|\omega|_{L^1}\,\sigma_0\,(|u_1 - u_2|_0) + |G|_{L^2}\,|\ell_0|_{L^2}\,|u_1 - u_2|_0 \leq$$

$$(\int_\Omega G(x,t;y,0)\eta\,(y)\,dy\,|\omega|_{L^1} + |G|_{L^2}\,|\ell_0|_{L^2})\,|u_1 - u_2|_0\,.$$

Interchanging the role of z_1 and z_2 we see that

$$d_H(\Lambda u_1, \Lambda u_2) \leq \delta\,\|u_1 - u_2\|_0\,,$$

where

$$\delta := \max_{(x,t)\in D}(\int_\Omega G(x,t;y,0)\eta\,(y)\,dy)\,|\omega|_{L^1} + |G|_{L^2}\,|\ell_0|_{L^2}\,.$$

It follows from (H3) that Λ is a contraction. Nadler's theorem ([31], see also [14, Theorem 11.1]) implies that Λ has a fixed point u_0, which is a solution of problem (1), (2), (3).

Remark. It what follows we shall denote by $C(\eta)$ the generic constant

$\max_{(x,t)\in D}(\int_\Omega G(x,t;y,0)\eta\,(y)\,dy)$, depending on a function $\eta \in Lip(\Omega)$.

Theorem 4.3 *Let $F : D \times \mathbb{R} \to \mathbb{R}$ be an L^2- Carathéodory multifunction with nonempty, compact, convex values. Assume the following conditions are satisfied*

(H4) there exist $M_g > 0$ and $\eta \in Lip(\Omega)$ such that $\int_0^T |g(y,s,u(y,s))|\, ds \le M_g \eta(y)$ for all $y \in \Omega$,

(H5) there exist $q \in L^2(D)$ and $\Psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that $|F(x,t,u| \le q(x,t)\Psi(|u|)$ for almost all $(x,t) \in D$ and $u \in \mathbb{R}$,

(H6) $sup_{\rho\in[0,\infty)} \dfrac{\rho}{M_g C(\eta) + |G|_{L^2} |q|_{L^2} \Psi(\rho)} > 1.$

Then problem (1), (2), (3) has a strong solution.

Proof. Recall that u is a solution of (1), (2), (3) if and only if u is a fixed point of the multivalued operator Λ given by (10). Lemma 3.1 implies that Λ is of upper semi-continuous type. By (H6) there exists $M_0 > 0$ such that

$$\frac{M_0}{M_g C(\eta) + |G|_{L^2} |q|_{L^2} \Psi(M_0)} > 1.$$

Consider $U := \{u \in X;\ |u|_0 < M_0\}$. Then U is relatively open in $K = X = C(\overline{D})$. We shall apply Theorem 3.2 to the operator Λ, and show that the second alternative does not hold. Let $u \in X$ be a solution of

$$u(x,t) \in \lambda(h_g(x,t) + \int_0^t \int_\Omega G(x,t;y,s)\ F(y,s,u(y,s))dyds), \quad (x,t) \in D. \qquad (11)$$

with $\lambda \in (0,1)$. From (11) and (H5) we obtain for each $(x,t) \in D$

$$\begin{aligned} |u(x,t)| &\le |h_g(x,t)| + \int_0^T \int_\Omega G(x,t;y,s)\ q(y,s)\Psi(|u(y,s)|)\, dyds) \\ &\le |h_g|_0 + \int_0^T \int_\Omega G(x,t;y,s) q(y,s) dyds) \Psi(|u|_0) \end{aligned}$$

Since $h_g(x,t) = \int_\Omega G(x,t;y,0)\ \int_0^T g(y,s,u(y,s))dsdy, \quad (x,t) \in D$, (H4) implies

$$|h_g|_0 \le M_g C(\eta) \qquad (12)$$

Hence

$$|u|_0 \leq M_g C(\eta) + |G|_{L^2} |q|_{L^2} \Psi(|u|_0). \quad (13)$$

Suppose now that there exist $u \in \partial U$ and $\lambda \in (0,1)$ such that $u \in \lambda \Lambda u$. Then u satisfies (11) and $|u|_0 = M_0$. It follows from the condition on Ψ and (13) that

$$M_0 \leq M_g C(\eta) + |G|_{L^2} |q|_{L^2} \Psi(M_0).$$

This, obviously, contradicts the definition of M_0. Consequently, the first alternative in Theorem 3.2 holds; i.e. the multivalued operator Λ has a fixed point u. Therefore u is a solution of (1), (2), (3).

Theorem 4.4 *Let $F : D \times \mathbb{R} \to \mathbb{R}$ be an L^2- Carathéodory multifunction with nonempty, compact, convex values. Assume that, in addition to (H4), the following conditions are satisfied*

(H7) there exists $\Phi : D \times \mathbb{R} \to \mathbb{R}_+$ an L^2- Carathéodory function, nondecreasing with respect to its third argument such that

(i) $limsup_{\rho \to \infty} \frac{1}{\rho} \int_0^T \int_\Omega G(x,t;y,s) \Phi(y,s,\rho) dyds < 1$

(ii) $|F(x,t,u| \leq \Phi(x,t,|u|)$ for a.e. $(x,t) \in D$, $u \in \mathbb{R}$.

Then problem (1), (2), (3) has at least one solution.

Proof. First, we show that all possible solutions of our problem are a priori bounded, i.e. there exists $\rho^* > 0$ such that any solution u of the problem satisfies $|u|_0 \leq \rho^*$. We have $u(x,t) \in h_g(x,t) + \int_0^t \int_\Omega G(x,t;y,s)\ F(y,s,u(y,s)) dyds, \quad (x,t) \in D$, where h_g is given by (10.a).

Taking into account (H4) and (12) we have by (H7) (ii)

$$|u(x,t)| \leq M_g C(\eta) + \int_0^t \int_\Omega G(x,t;y,s)\ \Phi(y,s,|u(y,s)|) dyds.$$

Hence

$$|u|_0 \leq M_g C(\eta) + \int_0^T \int_\Omega G(x,t;y,s)\ \Phi(y,s,|u|_0) dyds. \quad (14)$$

Let $\rho_0 := |u|_0$. Inequality (14) yields

$$1 \leq \frac{1}{\rho_0}\{M_g C(\eta) + \int_0^T \int_\Omega G(x,t;y,s)\ \Phi(y,s,\rho_0)dyds\}. \tag{15}$$

It follows from (H7) (i) that there exists $\rho^* > 0$, independent of u, such that for all $\rho > \rho^*$

$$\frac{1}{\rho}[M_g C(\eta) + \int_0^T \int_\Omega G(x,t;y,s)\ \Phi(y,s,\rho)dyds] > 1. \tag{16}$$

Comparing inequalities (15) and (16) we see that $\rho_0 \leq \rho^*$.

Next, define a truncation function $\zeta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\zeta(\rho^*, \xi) = \begin{cases} 1 & 0 \leq \xi \leq \rho^*, \\ 2 - \dfrac{\xi}{\rho^*} & \rho^* \leq \xi \leq 2\rho^*, \\ 0 & \xi \geq 2\rho^*. \end{cases}$$

It is clear that ζ is continuous and $0 \leq \zeta(\rho^*, \xi) \leq 1$ for all $\xi \in \mathbb{R}_+$.

Let

$$H(x,t,u) := \zeta(\rho^*, |u|)\, F(x,t,u).$$

Then H is an L^2-Carathéodory multifunction and there exists $\omega_H \in L^2(D)$ such that $|\varphi(x,t)| \leq \omega_H(x,t)$ for a.e. $(x,t) \in D$ and all $\varphi \in S_{H,u}$. In fact, we have $\omega_H(x,t) := \sup\{|H(x,t,u)|\, ; |u| \leq 2\rho^*\}$. Hence H is a bounded multifunction.

Consider the modified problem

$$\begin{cases} D_t u + Lu \in H(x,t,u), & (x,t) \in D, \\ u(x,t) = 0, & (x,t) \in \Gamma, \\ u(x,0) = \int_0^T g(x,s,u(x,s))ds, & x \in \Omega. \end{cases} \tag{17}$$

We show that (17) has at least one solution u satisfying the estimate $|u|_0 \leq \rho^*$.

The set $Y := \{u \in X; |u|_0 \leq M_g C(\eta) + |G|_{L^2} |\omega_H|_{L^2}\}$ is nonempty, bounded, closed and convex. From the above results, we know that the solutions of (17) are fixed points of the multivalued operator $\Upsilon := h_g + G\widehat{H}$, where $\widehat{H}$ is the Nemitsky operator of H. Moreover Υ is of usc type and maps Y into itself. It follows from Bohnenblust-Karlin Theorem (see [14, Cor.11.3(e)]) that Υ has a fixed point v, which is a solution of (17).

It remains to show that $|v|_0 \leq \rho^*$. Indeed, we have that $|H(x,t,u)| \leq \Phi_0(x,t,|u|) := \zeta(\rho^*, |u|)\,\Phi(x,t,|u|)$ for a.e. $(x,t) \in D$, $u \in \mathbb{R}$. It is easily seen that Φ_0 satisfies condition (H7). Hence, the first part of the proof shows that $|v|_0 \leq \rho^*$. But, for all u such that $|u|_0 \leq \rho^*$, the multivalued functions F and H coincide, and problem (17) reduces to our original problem. Therefore (1), (2), (3) has at least one solution.

Acknowledgements: The author is visiting the Division of Applied Mathematics at Brown University through a Grant from The Arab Fund For Economic And Social Development, Kuwait. He is grateful to both Institutions and Professor John Mallet-Paret for making the visit possible.

References

[1] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer Verlag, Berlin, New york, 1984.

[2] J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhauser, Boston, 1990.

[3] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl. 162 (1991), 494-505

[4] F. Calabro and P. Zunino, *Analysis of parabolic problems on partitioned domains with nonlinear conditions at the interface. Application to mass transfer through semi-permeable membranes*, Math. Models Methods Appl. Sci. 16 (2006), 479-501.

[5] J. R. Cannon, *The solution of the heat equation subject to the specification of energy*, Quart. Appl. Math. 21 (1963), 155-160.

[6] S. Carl and S. Heikkilä, *On a parabolic boundary value problem with discontinuous nonlinearity*, Nonl.Anal. 15 (1990), 1091-1095.

[7] S. Carl, C. Grossmann and C. V. Pao, *Existence and monotone iterations for parabolic differential inclusions*, Comm. Appl. Nonl. Anal. 3 (1996), 1-24.

[8] J. Chabrowski, *On nonlocal problems for parabolic equations*, Nagoya Math. J. 93 (1984), 109-131.

[9] R. Yu Chegis, *Numerical solution of a heat conduction problem with an integral boundary condition*, Litovsk. Mat. Sb. 24 (1984), 209-215.

[10] D-Q. Dai and Yu Huang, *Remarks on a semilinear heat equation with integral boundary conditions*, Nonl. Anal. 67 (2007), 468-475.

[11] W. A. Day, *A decreasing property of solutions of parabolic equations with applicatons to thermoelasticity*, Quart. Appl. Math. 41 (1983), 468-475.

[12] H. Deguchi, *On weak solutions of parabolic initial value problems with discontinuous nonlinearities*, Nonl. Anal. 63 (2005), e1107-e1117.

[13] M. Dehghan, *Efficient techniques for the second order parabolic equation subject to nonlocal specifications*, Appl. Numer. Math. 52 (2005), 39-62.

[14] K. Deimling, *Multivalued Differential Equations*, W. de Gruyter, Berlin, 1992.

[15] K. Deng, *Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions*, J. Math. Anal. Appl. 179 (1993), 630-637

[16] E. Feireisl, *A note on uniqueness for parabolic problems with discontinuous nonlinearities*, Nonl. Anal. 16 (1991), 1053-1056.

[17] E. Feireisl and J. Norbury, *Some existence, uniqueness and nonuniqueness theorems for solutions of parabolic equations with discontinuous nonlinearities*, Proc. Royal Soc. Edinburgh 119 A (1991), 1-17.

[18] B. A. Fleishman and T. J. Mahar, *A step function model in chemical reactor theory: Multiplicity and stability of solutions,* Nonl. Anal. 5 (1981), 645-654

[19] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[20] M. Frigon, *Application de la théorie de la transversalité topologique à des problèmes non linéaires pour des équations differentielles ordinaires*, Dissertationes Math. 296 (1990), 1-79.

[21] J. Hofbauer and P. L. Simon, *An existence theorem for parabolic equations on* $\mathbb{R}^N$ *with discontinuous nonlinearity*, Electron. J. Qual. Theory Diff. Eqns. 2001, No. 8, pp.1-9

[22] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis, Vol. I: Theory*, Kluwer, Dordrecht, The Netherlands, 2000.

[23] N. I. Ionkin, *Solution of a boundary value problem in heat conduction theory with nonlocal boundary conditions*, Differential Equations 13 (1977), 204-211.

[24] D. Jackson, *Existence and uniqueness of solutions to semilinear nonlocal parabolic equations*, J. Math. Anal. Appl. 172 (1993), 256-265.

[25] A. A. Kerefov, *Nonlocal boundary value problems for parabolic equations*, Differential Equations 15 (1979), 52-54.

[26] I. Kupka, *Continuous selections for Lipschitz multifunctions*, Acta Math. Univ. Comeniane Vol. LXXIV (2005), 133-144

[27] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, Nauka, Moscow, 1967 [English translation: Amer. Math. Soc., Providence, R.I. 1968.]

[28] G. M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, River Edge, New York, 1996.

[29] G. M. Lieberman, *Nonlocal problems for quasilinear parabolic equations*, in "Nonlinear Problems in Mathematical Physics and related Topics I", Kluwer 2002, 247-270.

[30] Y. Lin, *Analytical and numerical solutions for a class of nonlocal nonlinear parabolic differential equations*, SIAM J. Math. Anal. 25 (1994), 1577-1594.

[31] S. B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math. 30 (1969), 475-488

[32] W. E. Olmstead and C. A. Roberts, *The one dimensional heat equation with a nonlocal initial condition*, Appl. Math. Lett. 10 (1997), 89-94.

[33] D. O'Regan, *Integral inclusions of upper semi-continuous or lower semi-continuous type*, Proc. Amer. Math. Soc. 124 (1996), 2391-2399

[34] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.

[35] N. S. Papageorgiou, *On the existence of solutions for nonlinear parabolic problems with nonmonotone discontinuities*, J. Math. Anal. Appl. 205 (1997), 434-453.

[36] R. Pisani, *Problemi al contorno per operatori parabolici con non linearita discontinua*, Rend. Ist. Mat. Univ. Trieste 14 (1982), 85-98.

[37] T. Pruszko, *Some applications of the topological degree theory to multivalued boundary value problems*, Dissertationes Math. 299 (1984).

[38] J. Rauch, *Discontinuous semilinear differential equations and multiple valued maps*, Proc. Amer. Math. Soc. 64 (1977), 277-282.

[39] P. N. Vabishchevich, *Nonlocal parabolic problems and the inverse heat-conduction problem*, Differential Equations 17 (1981), 761-765.

[40] D. Wagner, *Survey of measurable selection theorems*, SIAM J. Control Optim. 15 (1977), 857-903

[41] H-M. Yin, *On a class of parabolic equations with nonlocal boundary conditions*, J. Math. Anal. Appl. 294 (2004), 712-728.