The Role of Salvatore Pincherle in the Development of Fractional Calculus

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Abstract We revisit two contributions by Salvatore Pincherle (Professor of Mathematics at the University of Bologna from 1880 to 1928) published (in Italian) in 1888 and 1902 in order to point out his possible role in the development of Fractional Calculus. Fractional Calculus is that branch of mathematical analysis dealing with pseudo-differential operators interpreted as integrals and derivatives of non-integer order. Even if the former contribution (published in two notes on Accademia dei Lincei) on generalized hypergeomtric functions does not concern Fractional Calculus it contains the first example in the literature of the use of the so called Mellin–Barnes integrals. These integrals will be proved to be a fundamental task to deal with all higher transcendental functions including the Meijer and Fox functions introduced much later. In particular, the solutions of differential equations of fractional order are suited to be expressed in terms of these integrals. In the second paper (published on Accademia delle Scienze di Bologna), the author is interested to insert in the framework of his operational theory the notion of derivative of non integer order that appeared in those times not yet well established. Unfortunately, Pincherle's foundation of Fractional Calculus seems still ignored.

1 Some Biographical Notes of Salvatore Pincherle

Salvatore Pincherle was born in Trieste on 11 March 1853 and died in Bologna on 10 July 1936. He was Professor of Mathematics at the University of Bologna from 1880 to 1928. Furthermore, he was the founder and the first President of Unione Matematica Italiana (UMI) from 1922 to 1936.

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Pincherle retired from the University just after the International Congress of Mathematicians that he had organized in Bologna since 3–10 September 1928, following the invitation received at the previous Congress held in Toronto in 1924.[1]

Pincherle wrote several treatises and lecture notes on Algebra, Geometry, Real and Complex Analysis. His main book related to his scientific activity is entitled "Le Operazioni Distributive e loro Applicazioni all'Analisi"; it was written in collaboration with his assistant, Dr. Ugo Amaldi, and was published in 1901 by Zanichelli, Bologna, see [20]. Pincherle can be considered one of the most prominent founders of the Functional Analysis, as pointed out by J. Hadamard in his review lecture "Le développement et le rôle scientifique du Calcul fonctionnel", given at the Congress of Bologna (1928). A description of Pincherle's scientific works requested from him by Mittag-Leffler, who was the Editor of Acta Mathematica, appeared (in French) in 1925 on this prestigious journal [19]. A collection of selected papers (38 from 247 notes plus 24 treatises) was edited by Unione Matematica Italiana (UMI) on the occasion of the centenary of his birth, and published by Cremonese, Roma 1954.

2 Pincherle and the Mellin-Barnes Integrals

Here, we point out that the 1888 paper (in Italian) of S. Pincherle on the *Generalized Hypergeometric Functions* led him to introduce the afterwards named Mellin-Barnes integral to represent the solution of a generalized hypergeometric differential equation investigated by Goursat in 1883. Pincherle's priority was explicitly recognized by Mellin and Barnes themselves, as reported below.

In 1907 Barnes, see p. 63 in [1], wrote: "The idea of employing contour integrals involving gamma functions of the variable in the subject of integration appears to be due to Pincherle, whose suggestive paper was the starting point of the investigations of Mellin (1895) though the type of contour and its use can be traced back to Riemann." In 1910 Mellin, see p. 326ff in [15], devoted a section (Sect. 10: Proof of Theorems of Pincherle) to revisit the original work of Pincherle; in particular, he wrote "Before we are going to prove this theorem, which is a special case of a more general theorem of Mr. Pincherle, we want to describe more closely the lines L over which the integration preferably is to be carried out" [free translation from German].

[1]More precisely, as we know from the recent biography of the Swedish mathematician Mittag-Leffler by Arild Stubhaug [22]: *The final decision was to be made as to where the next international mathematics congress (in 1928) would be held; the options were Bologna and Stockholm. One strike against Stockholm was the strength of the Swedish currency; it was said that it would simply be too expensive in Stockholm. Mittag-Leffler was also in favor of Bologna, and in that context he had contacted both the Canadian J.C. Fields and the Italian Salvatore Pincherle. The latter even asked Mittag-Leffler whether he would preside at the opening meeting of what in reality would be the first international congress for mathematicians since 1912. This was because mathematicians from Germany and the other Central Powers would be invited to Bologna.*

The purpose of this section is, following our 2003 paper [8], to let know the community of scientists interested in special functions the pioneering 1888 work by Pincherle on Mellin–Barnes integrals, that, in the author's intention, was devoted to compare two different generalizations of the Gauss hypergeometric function due to Pochhammer and to Goursat. In fact, dropping the details on which the interested reader can be informed from our paper [8], Pincherle arrived at the following expression of the Goursat hypergeometric function

$$\psi(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(x-\rho_1)\,\Gamma(x-\rho_2)\ldots\Gamma(x-\rho_m)}{\Gamma(x-\sigma_1)\,\Gamma(x-\sigma_2)\ldots\Gamma(x-\sigma_{m-1})}\, \mathrm{e}^{xt}\,\mathrm{d}x, \tag{1}$$

where $a > \Re\{\rho_1, \rho_2, \ldots, \rho_m\}$, and ρ_k and σ_k are the roots of certain algebraic equations of order m and $m-1$, respectively. We recognize in (1) the first example in the literature of the (afterwards named) Mellin–Barnes integral. The convergence of the integral was proved by Pincherle by using his asymptotic formula[2] for $\Gamma(a+i\eta)$ as $\eta \to \pm\infty$. So, for a solution of a particular case of the Goursat equation, Pincherle provided an integral representation that later was adopted by Mellin and Barnes for their treatment of the generalized hypergeometric functions ${}_pF_q(z)$. Hince then, the merits of Mellin and Barnes were so well recognized that their names were attached to the integrals of this type; on the other hand, after the 1888 paper (written in Italian), Pincherle did not pursue on this topic, so his name was no longer related to these integrals and, in this respect, his 1888 paper was practically ignored.

In more recent times other families of higher transcendental functions have been introduced to generalize the hypergeometric function based on their representation by Mellin–Barnes type integrals. We especially refer to the so-called G and H functions introduced by Meijer [13] in 1946, and by Fox [4] in 1961, respectively, so Pincherle can be considered their precursor. For an exhaustive treatment of the Mellin–Barnes integrals we refer to the recent monograph by Paris and Kaminski [16].

In the original part of our 2003 paper, we have shown that, by extending the original arguments by Pincherle, we have been able to provide the Mellin-Barnes integral representation of the transcendental functions introduced by Meijer (the so-called G functions). In fact, we have shown how to formally derive the ordinary differential equation and the Mellin–Barnes integral representation of the G functions introduced by Meijer in 1936–1946. So, in principle, Pincherle could

[2] We also note the priority of Pincherle in obtaining this asymptotic formula, as outlined by Mellin, see e.g. [14], pp. 330–331, and [15], p. 309. In his 1925 "Notices sur les travaux" [19] (p. 56, Sect. 16) Pincherle wrote "Une expression asymptotique de $\Gamma(x)$ pour $x \to \infty$ dans le sens imaginaire qui se trouve dans [17] a été attribuée à d'autres auteurs, mais M. Mellin m'en a récemment révendiqué la priorité." This formula is fundamental to investigate the convergence of the Mellin–Barnes integrals, as one can recognize from the detailed analysis by Dixon and Ferrar [3], see also [16].

have introduced the G functions much before Meijer if he had intended to pursue his original arguments in this direction.

Finally, we like to point out that the so-called Mellin–Barnes integrals are an efficient tool to deal with the higher transcendental functions. In fact, for a pure mathematics view point they facilitate the representation of these functions (as formerly indicated by Pincherle), and for an applied mathematics view point they can be successfully adopted to compute the same functions. In particular we like to refer to our papers [9, 10] where we have derived the solutions of diffusion-wave equations of fractional order and their subordination properties by using the Mellin-Barnes integrals.

3 Pincherle's Foundation of Fractional Derivatives

The interest of S. Pincherle about Fractional Calculus was mainly motivated by the fact that literature definitions of derivation of not integer order, now called fractional derivation, were arbitrary introduced as generalization of some aspects of the ordinary integer order derivation. This lack of a rigorous foundation attracted him.

Remembering that one of the research field of S. Pincherle was the operational calculus, it seems straightforward to think that for him it was natural to apply his knowledge in this field to derive the most rigorous definition of derivation of not integer order. In fact, in the book [20] entitled *Le operazioni distributive* published in 1901 in collaboration with U. Amaldi, S. Pincherle analyzed the general properties of operators, and in particular of differential operators. This background inducted him to search for a rigorous foundation of Fractional Calculus, which overcomes the arbitrariness of literature definitions, deriving a generalized derivative operator which meets all the properties of differential operators in the most general sense. This problem was addressed by S. Pincherle in the memoir *Sulle derivate ad indice qualunque* [18].

In 1902, Fractional Calculus had put its basis with the works by Liouville [7], Riemann [21] Tardy [23], Holmgrem [6]. S. Pincherle was acquainted about these works that however he considered to have a paramount flaw because these fractional derivation was arbitrary defined.

In particular, with respect to Liouville definition of not integer derivation of order s

$$D^s \mathrm{e}^{zx} = z^s \mathrm{e}^{zx} \,, \tag{2}$$

he observed that, from this arbitrary definition as an ingenuos extension for not integer s of the derivation of the exponential function, serious objections arise about the application of the distributive property of D^s for a sum of infinite terms. Arbitrariness has been highlighted also for Riemann definition of D^s, which was related to the coefficient of the term h^s when a function $f(x+h)$ is developed by a power series of terms $h^{\xi+n}$ with $n \in N$. Finally, also Holmgrem arbitrary assumed the derivation of not integer order s as the integral

$$D^s f(x) = \frac{1}{\Gamma(m-s)} D^m \int^x (x-z)^{m-s-1} f(z)\,\mathrm{d}z, \tag{3}$$

which was presented by Riemann himself. The same integral was used by Hadamard [5] in his research on Taylor series.

The general properties of ordinary differential operator of integer order are

1. It is uniquely defined for any analytical fuction,
2. It is distributive,
3. It satisfies an index law,
4. It meets a composition law for $D^m(\phi\psi)$ by the application of D^n (with $n \leq m$) to ϕ and ψ, as for example $D^m(x\phi) = xD^m\phi + mD^{m-1}\phi$.

Then, the generalization of the derivative operator for not integer order s is obtained by the construction of an operator $\mathcal{A}_s$ with the same properties of derivation of integer order. Hence, for the whole space of analytical functions or a part of it named $\mathcal{Q}$, the derivation of not integer order s meets the following constraints:

1. It is defined for any value of s, both real and complex, and for any function of $\mathcal{Q}$ generating at least one function belonging to the same space $\mathcal{Q}$,
2. It is distributive
$$\mathcal{A}_s(\phi + \psi) = \mathcal{A}_s(\phi) + \mathcal{A}_s(\psi)\,,$$
3. It solves the equation
$$D\mathcal{A}_{s-1} = \mathcal{A}_s\,,$$
4. It solves the equation
$$\mathcal{A}_s(x\phi) = x\mathcal{A}_s(\phi) + s\mathcal{A}_{s-1}(\phi)\,,$$
5. It reduces for integer value of $s = m$ to the operator D^m.

In this case, in the space $\mathcal{Q}$ it is defined the derivation of not integer order s and then $\mathcal{A}_s$ corresponds to D^s.

S. Pincherle derives such generalized operator D^s which emerges to be

$$x^s D^s \phi = \sum_{n=0}^{\infty} \binom{s}{n} \frac{x^n}{\Gamma(1-s+n)} D^n \phi\,, \tag{4}$$

where

$$\binom{s}{n} = \frac{s(s-1)\ldots(s-n+1)}{n!} = \frac{\Gamma(1+s)}{\Gamma(1+n)\Gamma(1+s-n)}\,, \tag{5}$$

is the binomial coefficient and $\Gamma(z)$, with $z \in C$, is the Euler gamma function that for integer argoument is related to the factorial by $\Gamma(1+n) = n!$. Furthermore, by adopting the Gamma function property $\Gamma(1+z) = z\Gamma(z)$ it results that

$$\begin{aligned}\Gamma(1-s+n) &= \Gamma(1-s)(1-s)\dots(n-s)\\ &= (-1)^n\Gamma(1-s)(s-1)\dots(s-n)\\ &= (-1)^n\frac{\Gamma(1-s)\Gamma(1+s)}{\Gamma(1+s-n)}\frac{(s-n)}{s}\,,\end{aligned}$$

and by the definition of binomial coefficient (5) formula (4) can be rewritten as

$$x^s D^s \phi = \frac{s}{\Gamma(1-s)}\sum_{n=0}^{\infty}\frac{(-x)^n}{(s-n)n!}D^n\phi\,. \tag{6}$$

S. Pincherle recognized that formula (6) was originally obtained by Bourlet [2]. However, S. Pincherle highlighted also that Bourlet obtained (6) assuming as definition of derivation of not integer order the same one given by Riemann, while S. Pincherle derived (6) as the necessery consequence of those properties that an operator must satisfy to be intended as derivation of not integer order.

Before concluding the memoir, S. Pincherle also shows that when s is a negative integer, i.e. $s = -m$, formula (4) becomes

$$D^{-m}\phi = \frac{x^m}{\Gamma(m)}\sum_{n=0}^{\infty}\frac{(-x)^n}{n!(m+n)}D^n\phi\,, \tag{7}$$

which he recognized to be the generalized Bernoulli formula given in *Le operazioni distributive* [20].

Then the fractional derivation in the Pincherle sense is defined by (4) and it is emerged to be a series of weighted integer derivations up to the infinite order. This fractional derivation is not related to usual definitions in literature. But we remark that, differently from generally accepted definitions of fractional derivation, S. Pincherle derived an operator which meets all the general constraints that a derivation operator must satisfies. Unfortunately, in spite of the rigorous foundation, the fractional derivation in the Pincherle sense is not considered by the community of fractional analysts.

Moreover, we would like conclude this section stressing that, since Pincherle's fractional derivation has been obtained by strong analogy with ordinary differentiation satisfying all differential operator constraints, it could have a straightforward physical and geometrical interpretation, at variance with the actual literature of Fractional Calculus. This topic would be the argument of future analysis.

In fact, considering the interpretation of Pincherle fractional derivative as a weighted sum of infinity integer derivative of order n, $n = 0, 1, \dots, \infty$, then

$$D^s\phi = \sum_{n=0}^{\infty} w(n; x, s) D^n\phi\,, \tag{8}$$

where $w(n; x, s)$ is the weight of derivative of order n given the point x and the fractional order s and from normalization

$$\sum_{n=0}^{\infty} w(n; x, s) = 1\,. \tag{9}$$

From (4) and (6) it follows that

$$w(n; x, s) = \binom{s}{n} \frac{x^{n-s}}{\Gamma(1-s+n)} = \frac{s\, x^{-s}}{\Gamma(1-s)} \frac{(-x)^n}{(s-n)n!}\,, \tag{10}$$

and the series (9) becomes

$$\begin{aligned} \sum_{n=0}^{\infty} w(n; x, s) &= \frac{\Gamma(1+s)}{x^s} \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(1+n)\Gamma(1+s-n)\Gamma(1-s+n)} \\ &= \frac{s\, x^{-s}}{\Gamma(1-s)} \sum_{n=0}^{\infty} \frac{(-x)^n}{(s-n)n!}\,. \end{aligned} \tag{11}$$

Finally, computation of series (11) will give also a solution to the problem of the interpretation of fractional derivation.

4 Conclusions

We have revisited two contributions (in Italian) by Pincherle on generalized hypergeometric functions, dated 1888, and on derivatives of any order, dated 1902, in order to point out a possible role that he could have played in the development of the Fractional Calculus in Italy and abroad. As a matter of fact, unfortunately, these contributions remained practically unknown to the specialists of Fractional Calculus. However, we have recognized, since our 2003 paper [8], that the 1888 contribution if suitably continued could have led to the introduction of G functions before Meijer and used to deal with differential equations of fractional order. Up to nowadays the 1902 contribution has been ignored but in our opinion the approach by Pincherle is worth to be pursued in the framework of the modern theory of Fractional Calculus. It is interesting to note that even a former pupil of Pincherle at the University of Bologna, Antonio Mambriani, ignored the approach of his mentor in his papers on differential equations of fractional order, preferring the approach by Holmgren, see e.g. [11, 12].

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