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Mittag-Leffler Functions, Related Topics and Applications

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ISSN 1439-7382 ISSN 2196-9922 (electronic)
Springer Monographs in Mathematics
ISBN 978-3-662-43929-6 ISBN 978-3-662-43930-2 (eBook)
DOI 10.1007/978-3-662-43930-2
Springer Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014949196

Mathematics Subject Classification: 33E12, 26A33, 34A08, 45K05, 44Axx, 60G22

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Printed on acid-free paper

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*To the memory of our colleague and friend
Anatoly Kilbas*

Preface

The study of the Mittag-Leffler function and its various generalizations has become a very popular topic in mathematics and its applications. However, during the twentieth century, this function was practically unknown to the majority of scientists, since it was ignored in most common books on special functions. As a noteworthy exception the handbook *Higher Transcendental Functions*, vol. 3, by A. Erdelyi et al. deserves to be mentioned.

Now the Mittag-Leffler function is leaving its isolated role as *Cinderella* (using the term coined by F.G. Tricomi for the *incomplete gamma* function).

The recent growing interest in this function is mainly due to its close relation to the *Fractional Calculus* and especially to fractional problems which come from applications.

Our decision to write this book was motivated by the need to fill the gap in the literature concerning this function, to explain its role in modern pure and applied mathematics, and to give the reader an idea of how one can use such a function in the investigation of modern problems from different scientific disciplines.

This book is a fruit of collaboration between researchers in Berlin, Bologna and Minsk. It has highly profited from visits of SR to the Department of Physics at the University of Bologna and from several visits of RG to Bologna and FM to the Department of Mathematics and Computer Science at Berlin Free University under the European ERASMUS exchange. RG and SR appreciate the deep scientific atmosphere at the University of Bologna and the perfect conditions they met there for intensive research.

We are saddened that our esteemed and always enthusiastic co-author Anatoly A. Kilbas is no longer with us, having lost his life in a tragic accident on 28 June 2010 in the South of Russia. We will keep him, and our inspiring joint work with him, in living memory.

Berlin, Germany
Bologna, Italy
Minsk, Belarus
March 2014

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