CISM LECTURE NOTES
International Centre for Mechanical Sciences
Palazzo del Torso, Piazza Garibaldi, Udine, Italy

FRACTIONAL CALCULUS :

Integral and Differential Equations of Fractional Order

Rudolf GORENFLO and **Francesco MAINARDI**

Department of Mathematics and Informatics
Free University of Berlin
Arnimallee 3
D-14195 Berlin, Germany
gorenflo@mi.fu-berlin.de

Department of Physics
University of Bologna
Via Irnerio 46
I-40126 Bologna, Italy
francesco.mainardi@unibo.it

URL: www.fracalmo.org

FRACALMO PRE-PRINT 54 pages : pp. 223-276

The paper is based on the lectures delivered by the authors at the CISM Course *Scaling Laws and Fractality in Continuum Mechanics: A Survey of the Methods based on Renormalization Group and Fractional Calculus*, held at the seat of CISM, Udine, from 23 to 27 September 1996, under the direction of Professors A. Carpinteri and F. Mainardi.

This TEX pre-print is a revised version (December 2000) of the chapter published in

A. Carpinteri and F. Mainardi (Editors): Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Wien and New York 1997, pp. 223-276.

Such book is the volume No. 378 of the series CISM COURSES AND LECTURES [ISBN 3-211-82913-X]

FRACTIONAL CALCULUS :
Integral and Differential Equations of Fractional Order

Rudolf GORENFLO and Francesco MAINARDI

Department of Mathematics and Informatics
Free University of Berlin
Arnimallee 3
D-14195 Berlin, Germany
`gorenflo@math.fu-berlin.de`

Department of Physics
University of Bologna
Via Irnerio 46
I-40126 Bologna, Italy
`francesco.mainardi@unibo.it`

URL: www.fracalmo.org

ABSTRACT

In these lectures we introduce the linear operators of fractional integration and fractional differentiation in the framework of the Riemann-Liouville fractional calculus. Particular attention is devoted to the technique of Laplace transforms for treating these operators in a way accessible to applied scientists, avoiding unproductive generalities and excessive mathematical rigor. By applying this technique we shall derive the analytical solutions of the most simple linear integral and differential equations of fractional order. We shall show the fundamental role of the Mittag-Leffler function, whose properties are reported in an *ad hoc* Appendix. The topics discussed here will be: (a) essentials of Riemann-Liouville fractional calculus with basic formulas of Laplace transforms, (b) Abel type integral equations of first and second kind, (c) relaxation and oscillation type differential equations of fractional order.

2000 Math. Subj. Class.: 26A33, 33E12, 33E20, 44A20, 45E10, 45J05.

This research was partially supported by Research Grants of the Free University of Berlin and the University of Bologna. The authors also appreciate the support given by the National Research Councils of Italy (CNR-GNFM) and by the International Centre of Mechanical Sciences (CISM).

1. INTRODUCTION TO FRACTIONAL CALCULUS

1.1 Historical Foreword

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. The term *fractional* is a misnomer, but it is retained following the prevailing use.

The fractional calculus may be considered an *old* and yet *novel* topic. It is an *old* topic since, starting from some speculations of G.W. Leibniz (1695, 1697) and L. Euler (1730), it has been developed up to nowadays. A list of mathematicians, who have provided important contributions up to the middle of our century, includes P.S. Laplace (1812), J.B.J. Fourier (1822), N.H. Abel (1823-1826), J. Liouville (1832-1873), B. Riemann (1847), H. Holmgren (1865-67), A.K. Grünwald (1867-1872), A.V. Letnikov (1868-1872), H. Laurent (1884), P.A. Nekrassov (1888), A. Krug (1890), J. Hadamard (1892), O. Heaviside (1892-1912), S. Pincherle (1902), G.H. Hardy and J.E. Littlewood (1917-1928), H. Weyl (1917), P. Lévy (1923), A. Marchaud (1927), H.T. Davis (1924-1936), A. Zygmund (1935-1945), E.R. Love (1938-1996), A. Erdélyi (1939-1965), H. Kober (1940), D.V. Widder (1941), M. Riesz (1949).

However, it may be considered a *novel* topic as well, since only from a little more than twenty years it has been object of specialized conferences and treatises. For the first conference the merit is ascribed to B. Ross who organized the *First Conference on Fractional Calculus and its Applications* at the University of New Haven in June 1974, and edited the proceedings, see [1]. For the first monograph the merit is ascribed to K.B. Oldham and J. Spanier, see [2], who, after a joint collaboration started in 1968, published a book devoted to fractional calculus in 1974. Nowadays, the list of texts and proceedings devoted solely or partly to fractional calculus and its applications includes about a dozen of titles [1-14], among which the encyclopaedic treatise by Samko, Kilbas & Marichev [5] is the most prominent. Furthermore, we recall the attention to the treatises by Davis [15], Erdélyi [16], Gel'fand & Shilov [17], Djrbashian [18, 22], Caputo [19], Babenko [20], Gorenflo & Vessella [21], which contain a detailed analysis of some mathematical aspects and/or physical applications of fractional calculus, although without explicit mention in their titles.

In recent years considerable interest in fractional calculus has been stimulated by the applications that this calculus finds in numerical analysis and different areas of physics and engineering, possibly including fractal phenomena. In this respect A. Carpinteri and F. Mainardi have edited the present book of lecture notes and entitled it as *Fractals and Fractional Calculus in Continuum Mechanics*. For the topic of fractional calculus, in addition to this joint article of introduction, we have contributed also with two single articles, one by Gorenflo [23], devoted to numerical methods, and one by Mainardi [24], concerning applications in mechanics.

1.2 *The Fractional Integral*

According to the Riemann-Liouville approach to fractional calculus the notion of fractional integral of order α ($\alpha > 0$) is a natural consequence of the well known formula (usually attributed to Cauchy), that reduces the calculation of the $n-$fold primitive of a function $f(t)$ to a single integral of convolution type. In our notation the Cauchy formula reads

$$J^n f(t) := f_n(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau)\, d\tau\,, \quad t > 0\,, \quad n \in \mathbb{N}\,, \tag{1.1}$$

where $\mathbb{N}$ is the set of positive integers. From this definition we note that $f_n(t)$ vanishes at $t = 0$ with its derivatives of order $1, 2, \dots, n-1$. For convention we require that $f(t)$ and henceforth $f_n(t)$ be a *causal* function, *i.e.* identically vanishing for $t < 0$.

In a natural way one is led to extend the above formula from positive integer values of the index to any positive real values by using the Gamma function. Indeed, noting that $(n-1)! = \Gamma(n)$, and introducing the arbitrary *positive* real number α, one defines the <u>*Fractional Integral of order $\alpha > 0$*</u> :

$$J^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau\,, \quad t > 0 \quad \alpha \in \mathbb{R}^+\,, \tag{1.2}$$

where $\mathbb{R}^+$ is the set of positive real numbers. For complementation we define $J^0 := I$ (Identity operator), *i.e.* we mean $J^0 f(t) = f(t)$. Furthermore, by $J^\alpha f(0^+)$ we mean the limit (if it exists) of $J^\alpha f(t)$ for $t \to 0^+$; this limit may be infinite.

<u>Remark 1</u> :

Here, and in all our following treatment, the integrals are intended in the *generalized* Riemann sense, so that any function is required to be *locally* absolutely integrable in $\mathbb{R}^+$. However, we will not bother to give descriptions of sets of admissible functions and will not hesitate, when necessary, to use *formal* expressions with generalized functions (*distributions*), which, as far as possible, will be re-interpreted in the framework of classical functions. The reader interested in the strict mathematical rigor is referred to [5], where the fractional calculus is treated in the framework of Lebesgue spaces of summable functions and Sobolev spaces of generalized functions.

<u>Remark 2</u> :

In order to remain in accordance with the standard notation I for the Identity operator we use the character J for the integral operator and its power J^α. If one likes to denote by I^α the integral operators, he would adopt a different notation for the Identity, *e.g.* $\mathbb{I}$, to avoid a possible confusion.

We note the *semigroup property*

$$J^\alpha J^\beta = J^{\alpha+\beta}\,, \quad \alpha\,,\,\beta \ge 0\,, \tag{1.3}$$

which implies the *commutative property* $J^\beta J^\alpha = J^\alpha J^\beta$, and the effect of our operators J^α on the power functions

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}\, t^{\gamma+\alpha}\,, \quad \alpha > 0\,, \quad \gamma > -1\,, \quad t > 0\,. \tag{1.4}$$

The properties (1.3-4) are of course a natural generalization of those known when the order is a positive integer. The proofs, see *e.g.* [2], [5] or [10], are based on the properties of the two Eulerian integrals, *i.e.* the *Gamma* and *Beta* functions,

$$\Gamma(z) := \int_0^\infty \mathrm{e}^{-u}\, u^{z-1}\, du\,, \quad \Gamma(z+1) = z\,\Gamma(z)\,, \quad \mathrm{Re}\,\{z\} > 0\,, \tag{1.5}$$

$$B(p,q) := \int_0^1 (1-u)^{p-1}\, u^{q-1}\, du \;=\; \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} = B(q,p)\,, \quad \mathrm{Re}\,\{p\,,\,q\} > 0\,. \tag{1.6}$$

It may be convenient to introduce the following causal function

$$\Phi_\alpha(t) := \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}\,, \quad \alpha > 0\,, \tag{1.7}$$

where the suffix $+$ is just denoting that the function is vanishing for $t<0$. Being $\alpha > 0$, this function turns out to be *locally* absolutely integrable in $\mathbb{R}^+$. Let us now recall the notion of *Laplace convolution*, *i.e.* the convolution integral with two causal functions, which reads in a standard notation $f(t) * g(t) := \int_0^t f(t-\tau)\, g(\tau)\, d\tau = g(t) * f(t)$.

Then we note from (1.2) and (1.7) that the fractional integral of order $\alpha > 0$ can be considered as the Laplace convolution between $\Phi_\alpha(t)$ and $f(t)$, *i.e.*

$$J^\alpha\, f(t) = \Phi_\alpha(t)\, *\, f(t)\,, \quad \alpha > 0\,. \tag{1.8}$$

Furthermore, based on the Eulerian integrals, one proves the *composition rule*

$$\Phi_\alpha(t)\, *\, \Phi_\beta(t) = \Phi_{\alpha+\beta}(t)\,, \quad \alpha\,,\,\beta > 0\,, \tag{1.9}$$

which can be used to re-obtain (1.3) and (1.4).

Introducing the Laplace transform by the notation $\mathcal{L}\,\{f(t)\} := \int_0^\infty \mathrm{e}^{-st}\, f(t)\, dt = \widetilde{f}(s)\,,\ s \in \mathbb{C}$, and using the sign $\div$ to denote a Laplace transform pair, *i.e.* $f(t) \div \widetilde{f}(s)$, we note the following rule for the Laplace transform of the fractional integral,

$$J^\alpha\, f(t) \div \frac{\widetilde{f}(s)}{s^\alpha}\,, \quad \alpha > 0\,, \tag{1.10}$$

which is the straightforward generalization of the case with an n-fold repeated integral ($\alpha = n$). For the proof it is sufficient to recall the convolution theorem for Laplace transforms and note the pair $\Phi_\alpha(t) \div 1/s^\alpha$, with $\alpha > 0$, see *e.g.* Doetsch [25].

1.3 *The Fractional Derivative*

After the notion of fractional integral, that of fractional derivative of order α ($\alpha > 0$) becomes a natural requirement and one is attempted to substitute α with $-\alpha$ in the above formulas. However, this generalization needs some care in order to guarantee the convergence of the integrals and preserve the well known properties of the ordinary derivative of integer order.

Denoting by D^n with $n \in \mathbb{N}$, the operator of the derivative of order n, we first note that

$$D^n\, J^n = I\,, \quad J^n\, D^n \neq I\,, \quad n \in \mathbb{N}\,, \tag{1.11}$$

i.e. D^n is left-inverse (and not right-inverse) to the corresponding integral operator J^n. In fact we easily recognize from (1.1) that

$$J^n\, D^n\, f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\, \frac{t^k}{k!}\,, \quad t > 0\,. \tag{1.12}$$

As a consequence we expect that D^α is defined as left-inverse to J^α. For this purpose, introducing the positive integer m such that $m-1 < \alpha \le m$, one defines the <u>*Fractional Derivative of order* $\alpha > 0$</u> : $D^\alpha\, f(t) := D^m\, J^{m-\alpha}\, f(t)$, namely

$$D^\alpha\, f(t) := \begin{cases} \dfrac{d^m}{dt^m}\left[\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau\right], & m-1 < \alpha < m\,, \\[2ex] \dfrac{d^m}{dt^m} f(t)\,, & \alpha = m\,. \end{cases} \tag{1.13}$$

Defining for complementation $D^0 = J^0 = I$, then we easily recognize that

$$D^\alpha\, J^\alpha = I\,, \quad \alpha \ge 0\,, \tag{1.14}$$

and

$$D^\alpha\, t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}\, t^{\gamma-\alpha}\,, \quad \alpha > 0\,, \quad \gamma > -1\,, \quad t > 0\,. \tag{1.15}$$

Of course, the properties (1.14-15) are a natural generalization of those known when the order is a positive integer. Since in (1.15) the argument of the Gamma function in the denominator can be negative, we need to consider the analytical continuation of $\Gamma(z)$ in (1.5) to the left half-plane, see *e.g.* Henrici [26].

Note the remarkable fact that the fractional derivative $D^\alpha f$ is not zero for the constant function $f(t) \equiv 1$ if $\alpha \notin \mathbb{N}$. In fact, (1.15) with $\gamma = 0$ teaches us that

$$D^\alpha 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,, \quad \alpha \ge 0\,, \quad t > 0\,. \tag{1.16}$$

This, of course, is $\equiv 0$ for $\alpha \in \mathbb{N}$, due to the poles of the gamma function in the points $0, -1, -2, \ldots$.

We now observe that an alternative definition of fractional derivative, originally introduced by Caputo [19], [27] in the late sixties and adopted by Caputo and Mainardi [28] in the framework of the theory of *Linear Viscoelasticity* (see a review in [24]), is the so-called *Caputo Fractional Derivative of order* $\alpha > 0$: $D_*^\alpha \, f(t) := J^{m-\alpha} \, D^m \, f(t)$ with $m-1 < \alpha \le m$, namely

$$D_*^\alpha \, f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} \, d\tau \,, & m-1 < \alpha < m \,, \\ \dfrac{d^m}{dt^m} f(t) \,, & \alpha = m \,. \end{cases} \tag{1.17}$$

This definition is of course more restrictive than (1.13), in that requires the absolute integrability of the derivative of order m. Whenever we use the operator D_*^α we (tacitly) assume that this condition is met. We easily recognize that in general

$$D^\alpha \, f(t) := D^m \, J^{m-\alpha} \, f(t) \neq J^{m-\alpha} \, D^m \, f(t) := D_*^\alpha \, f(t) \,, \tag{1.18}$$

unless the function $f(t)$ along with its first $m-1$ derivatives vanishes at $t = 0^+$. In fact, assuming that the passage of the m-derivative under the integral is legitimate, one recognizes that, for $m-1 < \alpha < m$ and $t > 0$,

$$D^\alpha \, f(t) = D_*^\alpha \, f(t) + \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} \, f^{(k)}(0^+) \,, \tag{1.19}$$

and therefore, recalling the fractional derivative of the power functions (1.15),

$$D^\alpha \left(f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} \, f^{(k)}(0^+) \right) = D_*^\alpha \, f(t) \,. \tag{1.20}$$

The alternative definition (1.17) for the fractional derivative thus incorporates the initial values of the function and of its integer derivatives of lower order. The subtraction of the Taylor polynomial of degree $m-1$ at $t = 0^+$ from $f(t)$ means a sort of regularization of the fractional derivative. In particular, according to this definition, the relevant property for which the fractional derivative of a constant is still zero, *i.e.*

$$D_*^\alpha 1 \equiv 0 \,, \quad \alpha > 0 \,. \tag{1.21}$$

can be easily recognized.

We now explore the most relevant differences between the two fractional derivatives (1.13) and (1.17). We agree to denote (1.17) as the *Caputo fractional derivative* to distinguish it from the standard Riemann-Liouville fractional derivative (1.13). We observe, again by looking at (1.15), that

$$D^\alpha t^{\alpha-1} \equiv 0 \,, \quad \alpha > 0 \,, \quad t > 0 \,. \tag{1.22}$$

From (1.22) and (1.21) we thus recognize the following statements about functions which for $t > 0$ admit the same fractional derivative of order α, with $m-1 < \alpha \le m$, $m \in \mathbb{N}$,

$$D^{\alpha}\, f(t) = D^{\alpha}\, g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{\alpha-j}\,, \tag{1.23}$$

$$D_*^{\alpha}\, f(t) = D_*^{\alpha}\, g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{m-j}\,. \tag{1.24}$$

In these formulas the coefficients c_j are arbitrary constants.

Incidentally, we note that (1.22) provides an instructive example to show how D^{α} is not right-inverse to J^{α}, since

$$J^{\alpha} D^{\alpha} t^{\alpha-1} \equiv 0, \quad \text{but} \quad D^{\alpha} J^{\alpha} t^{\alpha-1} = t^{\alpha-1}\,, \quad \alpha > 0\,, \quad t > 0,. \tag{1.25}$$

For the two definitions we also note a difference with respect to the *formal* limit as $\alpha \to (m-1)^{+}$. From (1.13) and (1.17) we obtain respectively,

$$\alpha \to (m-1)^{+} \implies D^{\alpha}\, f(t) \to D^{m}\, J\, f(t) = D^{m-1}\, f(t)\,; \tag{1.26}$$

$$\alpha \to (m-1)^{+} \implies D_*^{\alpha}\, f(t) \to J\, D^{m}\, f(t) = D^{m-1}\, f(t) - f^{(m-1)}(0^{+})\,. \tag{1.27}$$

We now consider the *Laplace transform* of the two fractional derivatives. For the standard fractional derivative D^{α} the Laplace transform, assumed to exist, requires the knowledge of the (bounded) initial values of the fractional integral $J^{m-\alpha}$ and of its integer derivatives of order $k = 1, 2, m-1$, as we learn from [2], [5], [10]. The corresponding rule reads, in our notation,

$$D^{\alpha}\, f(t) \div s^{\alpha}\, \widetilde{f}(s) - \sum_{k=0}^{m-1} D^{k}\, J^{(m-\alpha)}\, f(0^{+})\, s^{m-1-k}\,, \quad m-1 < \alpha \le m\,. \tag{1.28}$$

The *Caputo fractional derivative* appears more suitable to be treated by the Laplace transform technique in that it requires the knowledge of the (bounded) initial values of the function and of its integer derivatives of order $k = 1, 2, m-1$, in analogy with the case when $\alpha = m$. In fact, by using (1.10) and noting that

$$J^{\alpha}\, D_*^{\alpha}\, f(t) = J^{\alpha}\, J^{m-\alpha}\, D^{m}\, f(t) = J^{m}\, D^{m}\, f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^{+})\, \frac{t^{k}}{k!}\,. \tag{1.29}$$

we easily prove the following rule for the Laplace transform,

$$D_*^{\alpha}\, f(t) \div s^{\alpha}\, \widetilde{f}(s) - \sum_{k=0}^{m-1} f^{(k)}(0^{+})\, s^{\alpha-1-k}\,, \quad m-1 < \alpha \le m\,, \tag{1.30}$$

Indeed, the result (1.30), first stated by Caputo [19] by using the Fubini-Tonelli theorem, appears as the most "natural" generalization of the corresponding result well known for $\alpha = m$.

We now show how both the definitions (1.13) and (1.17) for the fractional derivative of $f(t)$ can be derived, at least *formally*, by the convolution of $\Phi_{-\alpha}(t)$ with $f(t)$, in a sort of analogy with (1.8) for the fractional integral. For this purpose we need to recall from the treatise on generalized functions by Gel'fand and Shilov [16] that (with proper interpretation of the quotient as a limit if $t=0$)

$$\Phi_{-n}(t) := \frac{t_+^{-n-1}}{\Gamma(-n)} = \delta^{(n)}(t)\,, \quad n = 0\,,1\,,\ldots \tag{1.31}$$

where $\delta^{(n)}(t)$ denotes the generalized derivative of order n of the Dirac delta distribution. Here, we assume that the reader has some minimal knowledge concerning these distributions, sufficient for handling classical problems in physics and engineering.

The equation (1.31) provides an interesting (not so well known) representation of $\delta^{(n)}(t)$, which is useful in our following treatment of fractional derivatives. In fact, we note that the derivative of order n of a causal function $f(t)$ can be obtained *formally* by the (generalized) convolution between Φ_{-n} and f,

$$\frac{d^n}{dt^n}\,f(t) = f^{(n)}(t) = \Phi_{-n}(t) \,*\, f(t) = \int_{0^-}^{t^+} f(\tau)\,\delta^{(n)}(t-\tau)\,d\tau\,, \quad t>0\,, \tag{1.32}$$

based on the well known properties

$$\int_{0^-}^{t^+} f(\tau)\,\delta^{(n)}(\tau-t)\,d\tau = (-1)^n\,f^{(n)}(t)\,, \quad \delta^{(n)}(t-\tau) = (-1)^n\,\delta^{(n)}(\tau-t)\,. \tag{1.33}$$

According to a usual convention, in (1.32-33) the limits of integration are extended to take into account for the possibility of impulse functions centred at the extremes.

Then, a formal definition of the fractional derivative of order α could be

$$\Phi_{-\alpha}(t) \,*\, f(t) = \frac{1}{\Gamma(-\alpha)} \int_{0^-}^{t^+} \frac{f(\tau)}{(t-\tau)^{1+\alpha}}\,d\tau\,, \quad \alpha \in \mathbb{R}^+\,.$$

The formal character is evident in that the kernel $\Phi_{-\alpha}(t)$ turns out to be not locally absolutely integrable and consequently the integral is in general divergent. In order to obtain a definition that is still valid for classical functions, we need to *regularize* the divergent integral in some way. For this purpose let us consider the integer $m \in \mathbb{N}$ such that $m-1<\alpha<m$ and write $-\alpha = -m + (m-\alpha)$ or $-\alpha = (m-\alpha) - m$. We then obtain

$$[\Phi_{-m}(t) \,*\, \Phi_{m-\alpha}(t)] \,*\, f(t) = \Phi_{-m}(t) \,*\, [\Phi_{m-\alpha}(t) \,*\, f(t)] = D^m\,J^{m-\alpha}\,f(t)\,, \tag{1.34}$$

or

$$[\Phi_{m-\alpha}(t) \,*\, \Phi_{-m}(t)] \,*\, f(t) = \Phi_{m-\alpha}(t) \,*\, [\Phi_{-m}(t) \,*\, f(t)] = J^{m-\alpha}\,D^m\,f(t)\,. \tag{1.35}$$

As a consequence we derive two alternative definitions for the fractional derivative, corresponding to (1.13) and (1.17), respectively. The singular behaviour of $\Phi_{-m}(t)$ is reflected in the non-commutativity of convolution in these formulas.

1.4 *Other Definitions and Notations*

Up to now we have considered the approach to fractional calculus usually referred to Riemann and Liouville. However, while Riemann (1847) had generalized the integral Cauchy formula with starting point $t = 0$ as reported in (1.1), originally Liouville (1832) had chosen $t = -\infty$. In this case we define

$$J^{\alpha}_{-\infty} f(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-\tau)^{\alpha-1} f(\tau)\, d\tau\,, \quad \alpha \in \mathbb{R}^{+}\,, \tag{1.36}$$

and consequently, for $m-1 < \alpha \le m\,,\ m \in \mathbb{N}\,,\ D^{\alpha}_{-\infty} f(t) := D^m\, J^{m-\alpha}_{-\infty} f(t)\,,$ namely

$$D^{\alpha}_{-\infty} f(t) := \begin{cases} \dfrac{d^m}{dt^m} \left[\dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_{-\infty}^{t} \dfrac{f(\tau)\, d\tau}{(t-\tau)^{\alpha+1-m}} \right], & m-1 < \alpha < m\,, \\ \dfrac{d^m}{dt^m} f(t)\,, & \alpha = m\,. \end{cases} \tag{1.37}$$

In this case, assuming $f(t)$ to vanish as $t \to -\infty$ along with its first $m-1$ derivatives, we have the identity

$$D^m\, J^{m-\alpha}_{-\infty} f(t) = J^{m-\alpha}_{-\infty}\, D^m f(t)\,, \tag{1.38}$$

in contrast with (1.18).

While for the fractional integral (1.2) a sufficient condition that the integral converge is that

$$f(t) = O\left(t^{\epsilon-1}\right)\,, \quad \epsilon > 0\,, \quad t \to 0^{+}\,, \tag{1.39}$$

a sufficient condition that (1.36) converge is that

$$f(t) = O\left(|t|^{-\alpha-\epsilon}\right)\,, \quad \epsilon > 0\,, \quad t \to -\infty\,. \tag{1.40}$$

Integrable functions satisfying the properties (1.39) and (1.40) are sometimes referred to as functions of Riemann class and Liouville class, respectively [10]. For example power functions t^{γ} with $\gamma > -1$ and $t > 0$ (and hence also constants) are of Riemann class, while $|t|^{-\delta}$ with $\delta > \alpha > 0$ and $t < 0$ and $\exp(ct)$ with $c > 0$ are of Liouville class. For the above functions we obtain (as real versions of the formulas given in [10])

$$J^{\alpha}_{-\infty}\, |t|^{-\delta} = \frac{\Gamma(\delta-\alpha)}{\Gamma(\delta)}\, |t|^{-\delta+\alpha}\,, \quad D^{\alpha}_{-\infty}\, |t|^{-\delta} = \frac{\Gamma(\delta+\alpha)}{\Gamma(\delta)}\, |t|^{-\delta-\alpha}\,, \tag{1.41}$$

and

$$J^{\alpha}_{-\infty}\, \mathrm{e}^{\,ct} = c^{-\alpha}\, \mathrm{e}^{\,ct}\,, \quad D^{\alpha}_{-\infty}\, \mathrm{e}^{\,ct} = c^{\,\alpha}\, \mathrm{e}^{\,ct}\,. \tag{1.42}$$

Causal functions can be considered in the above integrals with the due care. In fact, in view of the possible jump discontinuities of the integrands at $t = 0$, in this case it is worthwhile to write

$$\int_{-\infty}^{t} (\ldots)\, d\tau = \int_{0^-}^{t} (\ldots)\, d\tau\,.$$

As an example we consider for $0 < \alpha < 1$ the chain of identities

$$\begin{aligned}\frac{1}{\Gamma(1-\alpha)} \int_{0^-}^{t} \frac{f'(\tau)}{(t-\tau)^{\alpha}}\, d\tau &= \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\, f(0^+) + \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} \frac{f'(\tau)}{(t-\tau)^{\alpha}}\, d\tau \\ &= \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\, f(0^+) + D_*^{\alpha}\, f(t) = D^{\alpha}\, f(t)\,,\end{aligned} \tag{1.43}$$

where we have used (1.19) with $m = 1$.

In recent years it has become customary to use in place of (1.36) the Weyl fractional integral

$$W_{\infty}^{\alpha}\, f(t) := \frac{1}{\Gamma(\alpha)} \int_{t}^{\infty} (\tau - t)^{\alpha-1}\, f(\tau)\, d\tau\,, \quad \alpha \in \mathbb{R}^+\,, \tag{1.44}$$

based on a definition of Weyl (1917). For $t > 0$ it is a sort of complementary integral with respect to the usual Riemann-Liouville integral (1.2). The relation between (1.36) and (1.44) can be readily obtained by noting that, see *e.g.* [10],

$$\begin{aligned}J_{-\infty}^{\alpha}\, f(t) &= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-\tau)^{\alpha-1}\, f(\tau)\, d\tau = -\frac{1}{\Gamma(\alpha)} \int_{\infty}^{-t} (t+\tau')^{\alpha-1}\, f(-\tau')\, d\tau' \\ &= \frac{1}{\Gamma(\alpha)} \int_{t'}^{\infty} (\tau' - t')^{\alpha-1}\, f(-\tau')\, d\tau' = W_{\infty}^{\alpha}\, g(t')\,,\end{aligned} \tag{1.45}$$

with $g(t') = f(-t')$, $t' = -t$. In the above passages we have carried out the changes of variable $\tau \to \tau' = -\tau$ and $t \to t' = -t$.

For convenience of the reader, let us recall that exhaustive tables of Riemann-Liouville and Weyl fractional integrals are available in the second volume of the Bateman Project collection of Integral Transforms [16], in the chapter $XIII$ devoted to *fractional integrals.*

Last but not the least, let us consider the question of notation. The present authors oppose to the use of the notation $D^{-\alpha}$ for denoting the fractional integral, since it is misleading, even if it is used in distinguished treatises as [2], [10], [15]. It is well known that derivation and integration operators are not inverse to each other, even if their order is integer, and therefore such unification of symbols, present only in the framework of the fractional calculus, appears not justified. Furthermore, we have to keep in mind that for fractional order the derivative is yet an *integral* operator, so that, perhaps, it would be less disturbing to denote our D^{α} as $J^{-\alpha}$, than our J^{α} as $D^{-\alpha}$.

1.5 *The Law of Exponents*

In the ordinary calculus the properties of the operators of integration and differentiation with respect to the laws of commutation and additivity of their (integer) exponents are well known. Using our notation, the (trivial) laws

$$J^m\, J^n = J^n\, J^m = J^{m+n}\,, \qquad D^m\, D^n = D^n\, D^m = D^{m+n}\,, \tag{1.46}$$

where $m, n = 0, 1, 2, \ldots$, can be referred to as the *Law of Exponents* for the operators of integration and differentiation of integer order, respectively. Of course, for any positive integer order, the operators D^m and J^n do not commute, see (1.11-12).

In the fractional calculus the *Law of Exponents* is known to be generally true for the operators of *fractional integration* thanks to their semigroup property (1.3). In general, both the operators of fractional differentiation, D^α and D_*^α , do not satisfy either the semigroup property, or the (weaker) commutative property. To show how the *Law of Exponents* does not necessarily hold for the standard fractional derivative, we provide two simple examples (with power functions) for which

$$\begin{cases} \text{(a)} \quad D^\alpha D^\beta\, f(t) = D^\beta D^\alpha\, f(t) \neq D^{\alpha+\beta}\, f(t)\,, \\ \text{(b)} \quad D^\alpha D^\beta\, g(t) \neq D^\beta D^\alpha\, g(t) = D^{\alpha+\beta}\, g(t)\,. \end{cases} \tag{1.47}$$

In the example (a) let us take $f(t) = t^{-1/2}$ and $\alpha = \beta = 1/2$. Then, using (1.15), we get $D^{1/2}\, f(t) \equiv 0$, $D^{1/2}D^{1/2}\, f(t) \equiv 0$, but $D^{1/2+1/2}\, f(t) = D\, f(t) = -t^{-3/2}/2$. In the example (b) let us take $g(t) = t^{1/2}$ and $\alpha = 1/2\,, \beta = 3/2$. Then, again using (1.15), we get $D^{1/2}\, g(t) = \sqrt{\pi}/2$, $D^{3/2}\, g(t) \equiv 0$, but $D^{1/2}D^{3/2}\, g(t) \equiv 0$, $D^{3/2}D^{1/2}\, g(t) = -t^{3/2}/4$ and $D^{1/2+3/2}\, g(t) = D^2\, g(t) = -t^{3/2}/4$.

Although modern mathematicians would seek the conditions to justify the *Law of Exponents* when the order of differentiation and integration are composed together, we resist the temptation to dive into the delicate details of the matter, but rather refer the interested reader to §IV.6 ("The Law of Exponents") in the book by Miller and Ross [10]. Let us, however, extract (in our notation, writing J^α in place of $D^{-\alpha}$ for $\alpha > 0$) three important cases, contained in their Theorem 3: *If* $f(t) = t^\lambda\, \eta(t)$ *or* $f(t) = t^\lambda \ln t\, \eta(t)$, *where* $\lambda > -1$ *and* $\eta(t) = \sum_{n=0}^\infty a_n\, t^n$ *having a positive radius* R *of convergence, then for* $0 \le t < R$, *the following three formulas are valid:*

$$\begin{cases} \mu \ge 0 \quad \text{and} \quad 0 \le \nu \le \mu \Longrightarrow D^\nu J^\mu\, f(t) = J^{\mu-\nu}\, f(t)\,, \\ \mu \ge 0 \quad \text{and} \quad \nu > \mu \Longrightarrow D^\nu J^\mu\, f(t) = D^{\mu-\nu}\, f(t)\,, \\ 0 \le \mu < \lambda + 1 \quad \text{and} \quad \nu \ge 0 \Longrightarrow D^\nu D^\mu\, f(t) = D^{\mu+\nu}\, f(t)\,. \end{cases} \tag{1.48}$$

At least in the case of $f(t)$ without the factor $\ln t$, the proof of these formulas is straightforward. Use the definitions (1.2) and (1.13) of fractional integration and differentiation, the semigroup property (1.3) of fractional integration, and apply the formulas (1.4) and (1.15) termwise to the infinite series you meet in the course of calculations. Of course, the condition that the function $\eta(t)$ be analytic can be considerably relaxed; it only need be "sufficiently" smooth."

The lack of commutativity and the non-validity of the law of exponents has led to the notion of *sequential fractional differentiation* in which the order in which fractional differentiation operators D^{α_1}, $D^{\alpha_2}, \dots, D^{\alpha_k}$ are concatenated is crucial. For this and the related field of fractional differential equations we refer again to Miller and Ross [10]. Furthermore, Podlubny [29] has also given formulas for the Laplace transforms of sequential fractional derivatives.

In order to give an impression on the strange effects to be expected in use of sequential fractional derivatives we consider for a function $f(t)$ continuous for $t \ge 0$ and for positive numbers α and β with $\alpha+\beta = 1$ the three problems (a), (b), (c) with the respective general solutions $u\,, v\,, w$ in the set of locally integrable functions,

$$\begin{cases} (a)\; D^\alpha D^\beta\, u(t) = f(t) \;\Rightarrow\; u(t) = J\,f(t) + a_1 + a_2\, t^{\beta-1}\,, \\ (b)\; D^\beta D^\alpha\, v(t) = f(t) \;\Rightarrow\; v(t) = J\,f(t) + b_1 + b_2\, t^{\alpha-1}\,, \\ (c)\; D\,u(t) = f(t) \;\Rightarrow\; w(t) = J\,f(t) + c\,, \end{cases} \tag{1.49}$$

where $a_1\,, a_2\,, b_1\,, b_2\,, c$ are arbitrary constants. Whereas the result for (c) is obvious, in order to obtain the final results for (a) [or (b)] we need to apply first the operator J^α [or J^β] and then the operator J^β [or J^α]. The additional terms must be taken into account because $D^\gamma\, t^{\gamma-1} \equiv 0\,,\; J^{1-\gamma}\, t^{\gamma-1} = \Gamma(\gamma)\,,\; \gamma = \alpha, \beta\,.$ We observe that, whereas the general solution of (c) contains *one* arbitrary constant, that of (a) and likewise of (b) contains *two* arbitrary constants, even though $\alpha+\beta = 1\,.$ In case $\alpha \ne \beta$ the singular behaviour of $u(t)$ at $t = 0^+$ is distinct from that of $v(t)\,.$

From above we can conclude in rough words: sufficiently fine sequentialization increases the number of free constants in the general solution of a fractional differential equation, hence the number of conditions that must be imposed to make the solution unique. For an example see Bagley's treatment of a composite fractional oscillation equation [30]; there the highest order of derivative is 2, but four conditions are required to achieve uniqueness.

In the present lectures we shall avoid the above troubles since we shall consider only differential equations containing *single* fractional derivatives. Furthermore we shall adopt the *Caputo fractional derivative* in order to meet the usual physical requirements for which the initial conditions are expressed in terms of a given number of bounded values assumed by the field variable and its derivatives of integer order, see (1.24) and (1.30).

2. FRACTIONAL INTEGRAL EQUATIONS

In this section we shall consider the most simple integral equations of fractional order, namely the Abel integral equations of the first and the second kind. The former investigations on such equations are due to Abel (1823-26), after whom they are named, for the first kind, and to Hille and Tamarkin (1930) for the second kind. The interested reader is referred to [5], [21] and [31-33] for historical notes and detailed analysis with applications. Here we limit ourselves to put some emphasis on the method of the Laplace transforms, that makes easier and more comprehensible the treatment of such fractional integral equations, and provide some applications.

2.1 Abel integral equation of the first kind

Let us consider the Abel integral equation of the first kind

$$\frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau = f(t)\,, \quad 0 < \alpha < 1\,, \tag{2.1}$$

where $f(t)$ is a given function. We easily recognize that this equation can be expressed in terms of a fractional integral, *i.e.*

$$J^\alpha\, u(t) = f(t)\,, \tag{2.2}$$

and consequently solved in terms of a fractional derivative, according to

$$u(t) = D^\alpha\, f(t)\,. \tag{2.3}$$

To this end we need to recall the definition (1.2) and the property (1.14) $D^\alpha\, J^\alpha = I$.

Let us now solve (2.1) using the Laplace transform. Noting from (1.7-8) and (1.10) that $J^\alpha\, u(t) = \Phi_\alpha(t) * u(t) \div \tilde{u}(s)/s^\alpha$, we then obtain

$$\frac{\tilde{u}(s)}{s^\alpha} = \tilde{f}(s) \Longrightarrow \tilde{u}(s) = s^\alpha\, \tilde{f}(s)\,. \tag{2.4}$$

Now we can choose two different ways to get the inverse Laplace transform from (2.4), according to the standard rules. Writing (2.4) as

$$\tilde{u}(s) = s\left[\frac{\tilde{f}(s)}{s^{1-\alpha}}\right], \tag{2.4a}$$

we obtain

$$u(t) = \frac{1}{\Gamma(1-\alpha)}\, \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha}\, d\tau\,. \tag{2.5a}$$

On the other hand, writing (2.4) as

$$\tilde{u}(s) = \frac{1}{s^{1-\alpha}}\, [s\, \tilde{f}(s) - f(0^+)] + \frac{f(0^+)}{s^{1-\alpha}}\,, \tag{2.4b}$$

we obtain

$$u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha}\, d\tau + f(0^+)\, \frac{t^{-\alpha}}{\Gamma(1-a)}\,. \tag{2.5b}$$

Thus, the solutions (2.5a) and (2.5b) are expressed in terms of the fractional derivatives D^α and D_*^α, respectively, according to (1.13), (1.17) and (1.19) with $m = 1$.

The way *b*) requires that $f(t)$ be differentiable with $\mathcal{L}$-transformable derivative; consequently $0 \le |f(0^+)| < \infty$. Then it turns out from (2.5b) that $u(0^+)$ can be infinite if $f(0^+) \ne 0$, being $u(t) = O(t^{-\alpha})$, as $t \to 0^+$. The way *a*) requires weaker conditions in that the integral at the R.H.S. of (2.5a) must vanish as $t \to 0^+$; consequently $f(0^+)$ could be infinite but with $f(t) = O(t^{-\nu})$, $0 < \nu < 1 - \alpha$ as $t \to 0^+$. To this end keep in mind that $\Phi_{1-\alpha} * \Phi_{1-\nu} = \Phi_{2-\alpha-\nu}$. Then it turns out from (2.5a) that $u(0^+)$ can be infinite if $f(0^+)$ is infinite, being $u(t) = O(t^{-(\alpha+\nu)})$, as $t \to 0^+$.

Finally, let us remark that we can analogously treat the case of equation (2.1) with $0 < \alpha < 1$ replaced by $\alpha > 0$. If $m - 1 < \alpha \le m$ with $m \in \mathbb{N}$, then again we have (2.2), now with $D^\alpha f(t)$ given by the formula (1.13) which can also be obtained by the Laplace transform method.

2.2 Abel integral equation of the second kind

Let us now consider the Abel equation of the second kind

$$u(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau = f(t)\,, \quad \alpha > 0\,, \quad \lambda \in \mathbb{C}\,. \tag{2.6}$$

In terms of the fractional integral operator such equation reads

$$(1 + \lambda\, J^\alpha)\, u(t) = f(t)\,, \tag{2.7}$$

and consequently can be formally solved as follows:

$$u(t) = (1 + \lambda J^\alpha)^{-1}\, f(t) = \left(1 + \sum_{n=1}^{\infty} (-\lambda)^n\, J^{\alpha n}\right) f(t)\,. \tag{2.8}$$

Noting by (1.7-8) that

$$J^{\alpha n}\, f(t) = \Phi_{\alpha n}(t) * f(t) = \frac{t_+^{\alpha n - 1}}{\Gamma(\alpha n)} * f(t)$$

the formal solution reads

$$u(t) = f(t) + \left(\sum_{n=1}^{\infty} (-\lambda)^n\, \frac{t_+^{\alpha n-1}}{\Gamma(\alpha n)}\right) * f(t)\,. \tag{2.9}$$

Recalling from the Appendix the definition of the function,

$$e_\alpha(t;\lambda) := E_\alpha(-\lambda\, t^\alpha) = \sum_{n=0}^{\infty} \frac{(-\lambda\, t^\alpha)^n}{\Gamma(\alpha n + 1)}\,, \quad t > 0\,, \quad \alpha > 0\,, \quad \lambda \in \mathbb{C}\,, \tag{2.10}$$

where E_α denotes the Mittag-Leffler function of order α, we note that

$$\sum_{n=1}^{\infty} (-\lambda)^n\, \frac{t_+^{\alpha n-1}}{\Gamma(\alpha n)} = \frac{d}{dt} E_\alpha(-\lambda t^\alpha) = e'_\alpha(t;\lambda)\,, \quad t > 0\,. \tag{2.11}$$

Finally, the solution reads

$$u(t) = f(t) + e'_\alpha(t;\lambda) \, * \, f(t)\,. \tag{2.12}$$

Of course the above formal proof can be made rigorous. Simply observe that because of the rapid growth of the gamma function the infinite series in (2.9) and (2.11) are uniformly convergent in every bounded interval of the variable t so that term-wise integrations and differentiations are allowed. However, we prefer to use the alternative technique of Laplace transforms, which will allow us to obtain the solution in different forms, including the result (2.12).

Applying the Laplace transform to (2.6) we obtain

$$\left[1 + \frac{\lambda}{s^\alpha}\right] \tilde{u}(s) = \tilde{f}(s) \Longrightarrow \tilde{u}(s) = \frac{s^\alpha}{s^\alpha + \lambda}\, \tilde{f}(s)\,. \tag{2.13}$$

Now, let us proceed to obtain the inverse Laplace transform of (2.13) using the following Laplace transform pair (see Appendix)

$$e_\alpha(t;\lambda) := E_\alpha(-\lambda\, t^\alpha) \div \frac{s^{\alpha-1}}{s^\alpha + \lambda}\,. \tag{2.14}$$

As for the Abel equation of the first kind, we can choose two different ways to get the inverse Laplace transforms from (2.13), according to the standard rules. Writing (2.13) as

$$\tilde{u}(s) = s\left[\frac{s^{\alpha-1}}{s^\alpha + \lambda}\, \tilde{f}(s)\right], \tag{2.13a}$$

we obtain

$$u(t) = \frac{d}{dt} \int_0^t f(t-\tau)\, e_\alpha(\tau;\lambda)\, d\tau\,. \tag{2.15a}$$

If we write (2.13) as

$$\tilde{u}(s) = \frac{s^{\alpha-1}}{s^\alpha + \lambda}\, [s\,\tilde{f}(s) - f(0^+)] + f(0^+)\, \frac{s^{\alpha-1}}{s^\alpha + \lambda}\,, \tag{2.13b}$$

we obtain

$$u(t) = \int_0^t f'(t-\tau)\, e_\alpha(\tau;\lambda)\, d\tau + f(0^+)\, e_\alpha(t;\lambda)\,. \tag{2.15b}$$

We also note that, $e_\alpha(t;\lambda)$ being a function differentiable with respect to t with $e_\alpha(0^+;\lambda) = E_\alpha(0^+) = 1$, there exists another possibility to re-write (2.13), namely

$$\tilde{u}(s) = \left[s\, \frac{s^{\alpha-1}}{s^\alpha + \lambda} - 1\right] \tilde{f}(s) + \tilde{f}(s)\,. \tag{2.13c}$$

Then we obtain

$$u(t) = \int_0^t f(t-\tau)\, e'_\alpha(\tau;\lambda)\, d\tau + f(t)\,, \tag{2.15c}$$

in agreement with (2.12). We see that the way b) is more restrictive than the ways a) and c) since it requires that $f(t)$ be differentiable with $\mathcal{L}$-transformable derivative.

2.3 Some applications of Abel integral equations

It is well known that Niels Henrik Abel was led to his famous equation by the mechanical problem of the *tautochrone*, that is by the problem of determining the shape of a curve in the vertical plane such that the time required for a particle to slide down the curve to its lowest point is equal to a given function of its initial height (which is considered as a variable in an interval $[0, H]$). After appropriate changes of variables he obtained his famous integral equation of first kind with $\alpha = 1/2$. He did, however, solve the general case $0 < \alpha < 1$. See Tamarkin's translation [1)] of and comments to Abel's short paper [2)]. As a special case Abel discussed the problem of the *isochrone*, in which it is required that the time of sliding down is independent of the initial height. Already in his earlier publication [3)] he recognized the solution as derivative of non-integer order.

We point out that integral equations of Abel type, including the simplest (2.1) and (2.6), have found so many applications in diverse fields that it is almost impossible to provide an exhaustive list of them.

Abel integral equations occur in many situations where physical measurements are to be evaluated. In many of these the independent variable is the radius of a circle or a sphere and only after a change of variables the integral operator has the form J^α, usually with $\alpha = 1/2$, and the equation is of first kind. Applications are, *e.g.* , in evaluation of spectroscopic measurements of cylindrical gas discharges, the study of the solar or a planetary atmosphere, the investigation of star densities in a globular cluster, the inversion of travel times of seismic waves for determination of terrestrial sub-surface structure, spherical stereology. Descriptions and analysis of several problems of this kind can be found in the books by Gorenflo and Vessella [21] and by Craig and Brown [31], see also [32]. Equations of first and of second kind, depending on the arrangement of the measurements, arise in spherical stereology. See [33] where an analysis of the basic problems and many references to previous literature are given.

---

1) Abel, N. H.: Solution of a mechanical problem. Translated from the German. In: D. E. Smith, editor: A Source Book in Mathematics, pp. 656-662. Dover Publications, New York, 1959.

2) Abel, N. H.: Aufloesung einer mechanischen Aufgabe. Journal für die reine und angewandte Mathematik (Crelle), Vol. I (1826), pp. 153-157.

3) Abel, N. H.: Solution de quelques problèmes à l'aide d'intégrales définies. Translated from the Norwegian original, published in Magazin for Naturvidenskaberne. Aargang 1, Bind 2, Christiana 1823. French Translation in Oeuvres Complètes, Vol I, pp. 11-18. Nouvelle èdition par L. Sylow et S. Lie, 1881.

Another field in which Abel integral equations or integral equations with more general weakly singular kernels are important is that of *inverse boundary value problems* in partial differential equations, in particular parabolic ones in which naturally the independent variable has the meaning of time. We are going to describe in detail the occurrence of Abel integral equations of first and of second kind in the problem of heating (or cooling) of a semi-infinite rod by influx (or efflux) of heat across the boundary into (or from) its interior. Consider the *equation of heat flow*

$$u_t - u_{xx} = 0\,, \quad u = u(x,t)\,, \tag{2.16}$$

in the semi-infinite intervals $0 < x < \infty$ and $0 < t < \infty$ of space and time, respectively. In this dimensionless equation $u = u(x,t)$ means temperature. Assume vanishing initial temperature, *i.e.* $u(x,0) = 0$ for $0 < x < \infty$ and given influx across the boundary $x = 0$ from $x < 0$ to $x > 0$,

$$-u_x(0,t) = p(t)\,. \tag{2.17}$$

Then, under appropriate regularity conditions, $u(x,t)$ is given by the formula, see *e.g.* [34],

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{p(\tau)}{\sqrt{t-\tau}}\, \mathrm{e}^{-x^2/[4(t-\tau)]}\, d\tau\,, \quad x > 0\,,\ t > 0\,. \tag{2.18}$$

We turn our special interest to the interior boundary temperature $\phi(t) := u(0^+,t)$, $t > 0$, which by (2.18) is represented as

$$\frac{1}{\sqrt{\pi}} \int_0^t \frac{p(\tau)}{\sqrt{t-\tau}}\, d\tau = J^{1/2}\, p(t) = \phi(t)\,, \quad t > 0\,. \tag{2.19}$$

We recognize (2.19) as an Abel integral equation of *first kind* for *determination of an unknown influx $p(t)$ if the interior boundary temperature $\phi(t)$ is given* by measurements, or intended to be achieved by controlling the influx. Its solution is given by formula (1.13) with $m = 1$, $\alpha = 1/2$, as

$$p(t) = D^{1/2}\, \phi(t) = \frac{1}{\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{\phi(\tau)}{\sqrt{t-\tau}}\, d\tau\,. \tag{2.20}$$

It may be illuminating to consider the following special cases,

$$\begin{cases} (i) \quad \phi(t) = t \Longrightarrow p(t) = \dfrac{1}{2}\sqrt{\pi\, t}\,, \\[2ex] (ii) \quad \phi(t) = 1 \Longrightarrow p(t) = \dfrac{1}{\sqrt{\pi\, t}}\,, \end{cases} \tag{2.21}$$

where we have used formula (1.15). So, for linear increase of interior boundary temperature the required influx is continuous and increasing from 0 towards ∞ (with unbounded derivative at $t = 0^+$), whereas for instantaneous jump-like increase from 0 to 1 the required influx decreases from ∞ at $t = 0^+$ to 0 as $t \to \infty$.

We now modify our problem to obtain an Abel integral equation of *second kind.* Assume that the rod $x > 0$ is bordered at $x = 0$ by a bath of liquid in $x < 0$ with controlled exterior boundary temperature $u(0^-, t) := \psi(t)$.

Assuming Newton's radiation law we have an influx of heat from 0^- to 0^+ proportional to the difference of exterior and interior temperature,

$$p(t) = \lambda\,[\psi(t) - \phi(t)]\,, \quad \lambda > 0\,. \tag{2.22}$$

Inserting (2.22) into (2.19) we obtain

$$\phi(t) = \frac{\lambda}{\sqrt{\pi}} \int_0^t \frac{\psi(\tau) - \phi(\tau)}{\sqrt{t-\tau}}\, d\tau\,,$$

namely, in operator notation,

$$\left(1 + \lambda\, J^{1/2}\right)\, \phi(t) = \lambda\, J^{1/2}\, \psi(t)\,. \tag{2.23}$$

If we *now assume the exterior boundary temperature* $\psi(t)$ *as given and the evolution in time of the interior boundary temperature* $\phi(t)$ *as unknown*, then (2.23) is an Abel integral equation of *second kind* for determination of $\phi(t)$.

With $\alpha = 1/2$ the equation (2.23) is of the form (2.7), and by (2.8) its solution is

$$\phi(t) = \lambda\, \left(1 + \lambda J^{1/2}\right)^{-1}\, J^{1/2}\, \psi(t) = -\sum_{m=0}^{\infty} (-\lambda)^{m+1}\, J^{(m+1)/2}\, \psi(t)\,. \tag{2.24}$$

Let us investigate the very special case of constant exterior boundary temperature

$$\psi(t) = 1\,. \tag{2.25}$$

Then, by (1.4) with $\gamma = 0$,

$$J^{(m+1)/2}\, \psi(t) = \frac{t^{(m+1)/2}}{\Gamma\,[(m+1)/2 + 1]}\,,$$

hence

$$\phi(t) = -\sum_{m=0}^{\infty} (-\lambda)^{m+1}\, \frac{t^{(m+1)/2}}{\Gamma\,[(m+1)/2+1]} = 1 - \sum_{n=0}^{\infty} (-\lambda)^n\, \frac{t^{n/2}}{\Gamma\,(n/2+1)}\,,$$

so that

$$\phi(t) = 1 - E_{1/2}\left(-\lambda t^{1/2}\right) = 1 - e_{1/2}(t; \lambda)\,. \tag{2.26}$$

Observe that $\phi(t)$ is creep function, increasing strictly monotonically from 0 towards 1 as t runs from 0 to ∞.

For more or less distinct treatments of this problem of "Newtonian heating" the reader may consult [21], and [35-37]. In [37] a formulation in terms of fractional differential equations is derived and, furthermore, the analogous problem of "Newtonian cooling" is discussed.

3. FRACTIONAL DIFFERENTIAL EQUATIONS: 1-st PART

We now analyse the most simple differential equations of fractional order. For this purpose, following our recent works [37-42], we choose the examples which, by means of fractional derivatives, generalize the well-known ordinary differential equations related to relaxation and oscillation phenomena. In this section we treat the simplest types, which we refer to as the *simple fractional relaxation and oscillation equations.* In the next section we shall consider the types, somewhat more cumbersome, which we refer to as the *composite fractional relaxation and oscillation equations.*

3.1 The simple fractional relaxation and oscillation equations

The classical phenomena of relaxation and oscillations in their simplest form are known to be governed by linear ordinary differential equations, of order one and two respectively, that hereafter we recall with the corresponding solutions. Let us denote by $u = u(t)$ the field variable and by $q(t)$ a given continuous function, with $t \geq 0$.

The *relaxation* differential equation reads

$$u'(t) = -u(t) + q(t)\,, \tag{3.1}$$

whose solution, under the initial condition $u(0^+) = c_0$, is

$$u(t) = c_0\,\mathrm{e}^{-t} + \int_0^t q(t-\tau)\,\mathrm{e}^{-\tau}\,d\tau\,. \tag{3.2}$$

The *oscillation* differential equation reads

$$u''(t) = -u(t) + q(t)\,, \tag{3.3}$$

whose solution, under the initial conditions $u(0^+) = c_0$ and $u'(0^+) = c_1$, is

$$u(t) = c_0\,\cos t + c_1\,\sin t + \int_0^t q(t-\tau)\,\sin\tau\,d\tau\,. \tag{3.4}$$

From the point of view of the fractional calculus a natural generalization of equations (3.1) and (3.3) is obtained by replacing the ordinary derivative with a fractional one of order α. In order to preserve the type of initial conditions required in the classical phenomena, we agree to replace the first and second derivative in (3.1) and (3.3) with a Caputo fractional derivative of order α with $0 < \alpha < 1$ and $1 < \alpha < 2$, respectively. We agree to refer to the corresponding equations as the *simple fractional relaxation equation* and the *simple fractional oscillation equation.*

Generally speaking, we consider the following differential equation of fractional order $\alpha > 0$,

$$D_*^\alpha\, u(t) = D^\alpha \left(u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!}\, u^{(k)}(0^+) \right) = -u(t) + q(t)\,, \quad t > 0\,. \tag{3.5}$$

Here m is a positive integer uniquely defined by $m-1 < \alpha \le m$, which provides the number of the prescribed initial values $u^{(k)}(0^+) = c_k\,,\ k = 0, 1, 2, \ldots, m-1$. Implicit in the form of (3.5) is our desire to obtain solutions $u(t)$ for which the $u^{(k)}(t)$ are continuous for $t \ge 0,\ k = 0, 1, \ldots, m-1$. In particular, the cases of *fractional relaxation* and *fractional oscillation* are obtained for $m = 1$ and $m = 2$, respectively

We note that when α is the integer m the equation (3.5) reduces to an ordinary differential equation whose solution can be expressed in terms of m linearly independent solutions of the *homogeneous* equation and of one particular solution of the *inhomogeneous* equation. We summarize the well-known result as follows

$$u(t) = \sum_{k=0}^{m-1} c_k u_k(t) + \int_0^t q(t-\tau)\, u_\delta(\tau)\, d\tau\,. \tag{3.6}$$

$$u_k(t) = J^k\, u_0(t)\,, \quad u_k^{(h)}(0^+) = \delta_{k\,h}\,, \quad h, k = 0, 1, \ldots, m-1\,, \tag{3.7}$$

$$u_\delta(t) = -\, u_0'(t)\,. \tag{3.8}$$

Thus, the m functions $u_k(t)$ represent the *fundamental solutions* of the differential equation of order m, namely those linearly independent solutions of the *homogeneous* equation which satisfy the initial conditions in (3.7). The function $u_\delta(t)$, with which the free term $q(t)$ appears convoluted, represents the so called *impulse-response solution*, namely the particular solution of the *inhomogeneous* equation with all $c_k \equiv 0\,,\ k = 0, 1, \ldots, m-1$, and with $q(t) = \delta(t)$. In the cases of ordinary relaxation and oscillation we recognize that $u_0(t) = \mathrm{e}^{-t} = u_\delta(t)$ and $u_0(t) = \cos t$, $u_1(t) = J\, u_0(t) = \sin t = \cos(t - \pi/2) = u_\delta(t)$, respectively.

Remark 1 : The more general equation

$$D^\alpha \left(u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!}\, u^{(k)}(0^+) \right) = -\rho^\alpha\, u(t) + q(t)\,, \quad \rho > 0\,, \quad t > 0\,, \tag{3.5$'$}$$

can be reduced to (3.5) by a change of scale $t \to t/\rho$. We prefer, for ease of notation, to discuss the "dimensionless" form (3.5).

Let us now solve (3.5) by the method of Laplace transforms. For this purpose we can use directly the Caputo formula (1.30) or, alternatively, reduce (3.5) with the prescribed initial conditions as an equivalent (fractional) integral equation and then treat the integral equation by the Laplace transform method. Here we prefer to follow the second way. Then, applying the operator J^α to both sides of (3.5) we obtain

$$u(t) = \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} - J^\alpha u(t) + J^\alpha q(t)\,. \tag{3.9}$$

The application of the Laplace transform yields

$$\tilde{u}(s) = \sum_{k=0}^{m-1} \frac{c_k}{s^{k+1}} - \frac{1}{s^\alpha} \tilde{u}(s) + \frac{1}{s^\alpha} \tilde{q}(s)\,,$$

hence

$$\tilde{u}(s) = \sum_{k=0}^{m-1} c_k \frac{s^{\alpha-k-1}}{s^\alpha+1} + \frac{1}{s^\alpha+1} \tilde{q}(s)\,. \tag{3.10}$$

Introducing the Mittag-Leffler type functions

$$e_\alpha(t) \equiv e_\alpha(t;1) := E_\alpha(-t^\alpha) \div \frac{s^{\alpha-1}}{s^\alpha+1}\,, \tag{3.11}$$

$$u_k(t) := J^k e_\alpha(t) \div \frac{s^{\alpha-k-1}}{s^\alpha+1}\,, \quad k = 0, 1, \ldots, m-1\,, \tag{3.12}$$

we find, from inversion of the Laplace transforms in (3.10),

$$u(t) = \sum_{k=0}^{m-1} c_k\, u_k(t) - \int_0^t q(t-\tau)\, u_0'(\tau)\, d\tau\,. \tag{3.13}$$

For finding the last term in the R.H.S. of (3.13), we have used the well-known rule for the Laplace transform of the derivative noting that $u_0(0^+) = e_\alpha(0^+) = 1$, and

$$\frac{1}{s^\alpha+1} = -\left(s\, \frac{s^{\alpha-1}}{s^\alpha+1} - 1 \right) \div -u_0'(t) = -e_\alpha'(t)\,. \tag{3.14}$$

The formula (3.13) encompasses the solutions (3.2) and (3.4) found for $\alpha = 1\,, 2\,,$ respectively.

When α is not integer, namely for $m-1 < \alpha < m$, we note that $m-1$ represents the integer part of α (usually denoted by $[\alpha]$) and m the number of initial conditions necessary and sufficient to ensure the uniqueness of the solution $u(t)$. Thus the m functions $u_k(t) = J^k e_\alpha(t)$ with $k = 0, 1, \ldots, m-1$ represent those particular solutions of the *homogeneous* equation which satisfy the initial conditions

$$u_k^{(h)}(0^+) = \delta_{k\,h}\,, \quad h, k = 0, 1, \ldots, m-1\,, \tag{3.15}$$

and therefore they represent the *fundamental solutions* of the fractional equation (3.5), in analogy with the case $\alpha = m$. Furthermore, the function $u_\delta(t) = -e_\alpha'(t)$ represents the *impulse-response solution.* Hereafter, we are going to compute and exhibit the *fundamental solutions* and the *impulse-response solution* for the cases (a) $0 < \alpha < 1$ and (b) $1 < \alpha < 2$, pointing out the comparison with the corresponding solutions obtained when $\alpha = 1$ and $\alpha = 2$.

Remark 2 : The reader is invited to verify that the solution (3.13) has continuous derivatives $u^{(k)}(t)$ for $k = 0, 1, 2, \ldots, m-1$, which fulfill the m initial conditions $u^{(k)}(0^+) = c_k$. In fact, looking back at (3.9), one must recognize the smoothing power of the operator J^α. However, the so called *impulse-response solution* of our equation (3.5), $u_\delta(t)$, is expected to be not so regular like the ordinary solution (3.13). In fact, from (3.10) and (3.13-14), one obtains

$$u_\delta(t) = -u_0'(t) \div \frac{1}{s^\alpha + 1}\,, \tag{3.16}$$

and therefore, using the limiting theorem for Laplace transforms, one can recognize that, being $m-1 < \alpha < m$,

$$u_\delta^{(h)}(0^+) = 0\,,\ h = 0, 1, \ldots, m-2\,; \quad u_\delta^{(m-1)}(0^+) = \infty\,. \tag{3.17}$$

We now prefer to derive the relevant properties of the basic functions $e_\alpha(t)$ directly from their representation as a Laplace inverse integral

$$e_\alpha(t) = \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{\,st}\,\frac{s^{\alpha-1}}{s^\alpha + 1}\,ds\,, \tag{3.18}$$

in detail for $0 < \alpha \le 2$, without detouring on the general theory of Mittag-Leffler functions in the complex plane. In (3.18) Br denotes the Bromwich path, *i.e.* a line $\mathrm{Re}\,\{s\} = \sigma$ with a value $\sigma \ge 1$, and $\mathrm{Im}\,\{s\}$ running from $-\infty$ to $+\infty$.

For transparency reasons, we separately discuss the cases

$$\text{(a)}\quad 0 < \alpha < 1 \quad \text{and} \quad \text{(b)}\quad 1 < \alpha < 2\,,$$

recalling that in the limiting cases $\alpha = 1\,,\,2$, we know $e_\alpha(t)$ as elementary function, namely $e_1(t) = \mathrm{e}^{\,-t}$ and $e_2(t) = \cos t$. For α not integer the power function s^α is uniquely defined as $s^\alpha = |s|^\alpha\,\mathrm{e}^{i\arg s}$, with $-\pi < \arg s < \pi$, that is in the complex s-plane cut along the negative real axis.

The essential step consists in decomposing $e_\alpha(t)$ into two parts according to $e_\alpha(t) = f_\alpha(t) + g_\alpha(t)$, as indicated below. In case (a) the function $f_\alpha(t)$, in case (b) the function $-f_\alpha(t)$ is *completely monotone*; in both cases $f_\alpha(t)$ tends to zero as t tends to infinity, from above in case (a), from below in case (b). The other part, $g_\alpha(t)$, is identically vanishing in case (a), but of *oscillatory* character with exponentially decreasing amplitude in case (b).

In order to obtain the desired decomposition of e_α we bend the Bromwich path of integration Br into the equivalent Hankel path $Ha(1^+)$, a loop which starts from $-\infty$ along the lower side of the negative real axis, encircles the circular disc $|s| = 1$ in the positive sense and ends at $-\infty$ along the upper side of the negative real axis.

One obtains

$$e_\alpha(t) = f_\alpha(t) + g_\alpha(t)\,, \quad t \ge 0\,, \tag{3.19}$$

with

$$f_\alpha(t) := \frac{1}{2\pi i} \int_{Ha(\epsilon)} \mathrm{e}^{\,st}\, \frac{s^{\alpha-1}}{s^\alpha + 1}\, ds\,, \tag{3.20}$$

where now the Hankel path $Ha(\epsilon)$ denotes a loop constituted by a small circle $|s| = \epsilon$ with $\epsilon \to 0$ and by the two borders of the cut negative real axis, and

$$g_\alpha(t) := \sum_h \mathrm{e}^{\,s'_h\, t}\, Res\, \left[\frac{s^{\alpha-1}}{s^\alpha + 1}\right]_{s'_h} = \frac{1}{\alpha} \sum_h \mathrm{e}^{\,s'_h\, t}\,, \tag{3.21}$$

where s'_h are the relevant poles of $s^{\alpha-1}/(s^\alpha + 1)$. In fact the poles turn out to be $s_h = \exp\left[i(2h+1)\pi/\alpha\right]$ with unitary modulus; they are all simple but relevant are only those situated in the main Riemann sheet, *i.e.* the poles s'_h with argument such that $-\pi < \arg\, s'_h < \pi$.

If $0 < \alpha < 1$, there are no such poles, since for all integers h we have $|\arg\, s_h| = |2h+1|\,\pi/\alpha > \pi$; as a consequence,

$$g_\alpha(t) \equiv 0\,, \quad \text{hence} \quad e_\alpha(t) = f_\alpha(t)\,, \quad \text{if} \quad 0 < \alpha < 1\,. \tag{3.22}$$

If $1 < \alpha < 2$, then there exist precisely two relevant poles, namely $s'_0 = \exp(i\pi/\alpha)$ and $s'_{-1} = \exp(-i\pi/\alpha) = \overline{s_0}'$, which are located in the left half plane. Then one obtains

$$g_\alpha(t) = \frac{2}{\alpha}\, \mathrm{e}^{t\, \cos(\pi/\alpha)}\, \cos\left[t\, \sin\left(\frac{\pi}{\alpha}\right)\right]\,, \quad \text{if} \quad 1 < \alpha < 2\,. \tag{3.23}$$

We note that this function exhibits oscillations with circular frequency $\omega(\alpha) = \sin(\pi/\alpha)$ and with an exponentially decaying amplitude with rate $\lambda(\alpha) = |\cos(\pi/\alpha)|$.

<u>Remark 3</u> : One easily recognizes that (3.23) is valid also for $2 \le \alpha < 3$. In the classical case $\alpha = 2$ the two poles are purely imaginary (coinciding with $\pm i$) so that we recover the sinusoidal behaviour with unitary frequency. In the case $2 < \alpha < 3$, however, the two poles are located in the right half plane, so providing amplified oscillations. This instability, which is common to the case $\alpha = 3$, is the reason why we limit ourselves to consider α in the range $0 < \alpha \le 2$.

It is now an exercise in complex analysis to show that the contribution from the Hankel path $Ha(\epsilon)$ as $\epsilon \to 0$ is provided by

$$f_\alpha(t) := \int_0^\infty \mathrm{e}^{-rt}\, K_\alpha(r)\, dr\,, \tag{3.24}$$

with

$$K_\alpha(r) = -\,\frac{1}{\pi}\, \mathrm{Im}\, \left\{ \left.\frac{s^{\alpha-1}}{s^\alpha + 1}\right|_{s = r\, \mathrm{e}^{i\pi}} \right\} = \frac{1}{\pi}\, \frac{r^{\alpha-1}\, \sin(\alpha\pi)}{r^{2\alpha} + 2\, r^\alpha\, \cos(\alpha\pi) + 1}\,. \tag{3.25}$$

This function $K_\alpha(r)$ vanishes identically if α is an integer, it is positive for all r if $0 < \alpha < 1$, negative for all r if $1 < \alpha < 2$. In fact in (3.25) the denominator is, for α not integer, always positive being $> (r^\alpha - 1)^2 \ge 0$. Hence $f_\alpha(t)$ has the aforementioned monotonicity properties, decreasing towards zero in case (a), increasing towards zero in case (b). We also note that, in order to satisfy the initial condition $e_\alpha(0^+) = 1$, we find $\int_0^\infty K_\alpha(r)\,dr = 1$ if $0 < \alpha < 1$, $\int_0^\infty K_\alpha(r)\,dr = 1 - 2/\alpha$ if $1 < \alpha < 2$. In Fig. 1 we show the plots of the *spectral functions* $K_\alpha(r)$ for some values of α in the intervals (a) $0 < \alpha < 1$, (b) $1 < \alpha < 2$.

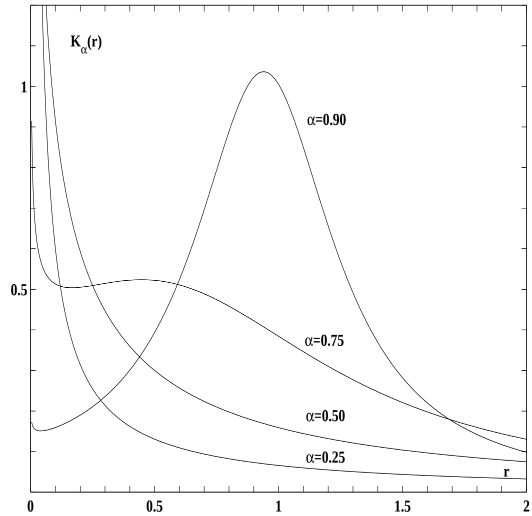

Fig. 1a – Plots of the *basic spectral function* $K_\alpha(r)$ for $0 < \alpha < 1$

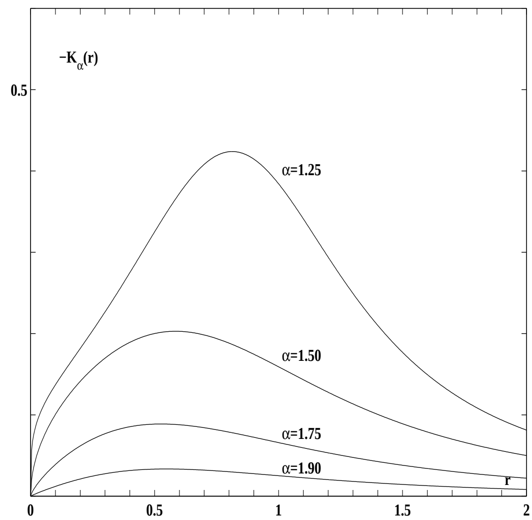

Fig. 1b – Plots of the *basic spectral function* $-K_\alpha(r)$ for $1 < \alpha < 2$

In addition to the basic fundamental solutions, $u_0(t) = e_\alpha(t)$ we need to compute the impulse-response solutions $u_\delta(t) = -D^1\, e_\alpha(t)$ for cases (a) and (b) and, only in case (b), the second fundamental solution $u_1(t) = J^1\, e_\alpha(t)$. For this purpose we note that in general it turns out that

$$J^k\, f_\alpha(t) = \int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,k}(r)\, dr\,, \tag{3.26}$$

with

$$K_{\alpha,k}(r) := (-1)^k\, r^{-k}\, K_\alpha(r) = \frac{(-1)^k}{\pi}\, \frac{r^{\alpha-1-k}\, \sin(\alpha\pi)}{r^{2\alpha} + 2\, r^\alpha\, \cos(\alpha\pi) + 1}\,, \tag{3.27}$$

where $K_\alpha(r) = K_{\alpha,0}(r)$, and

$$J^k g_\alpha(t) = \frac{2}{\alpha}\, \mathrm{e}^{t\, \cos(\pi/\alpha)}\, \cos\left[t\, \sin\left(\frac{\pi}{\alpha}\right) - k\frac{\pi}{\alpha}\right]\,. \tag{3.27}$$

This can be done in direct analogy to the computation of the functions $e_\alpha(t)$, the Laplace transform of $J^k e_\alpha(t)$ being given by (3.12). For the impulse-response solution we note that the effect of the differential operator D^1 is the same as that of the virtual operator J^{-1} .

In conclusion we can resume the solutions for the fractional relaxation and oscillation equations as follows:

(a) $0 < \alpha < 1$,

$$u(t) = c_0\, u_0(t) + \int_0^t q(t-\tau)\, u_\delta(\tau)\, d\tau\,, \tag{3.28a}$$

where

$$\begin{cases} u_0(t) = \displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,0}(r)\, dr\,, \\ u_\delta(t) = -\displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,-1}(r)\, dr\,, \end{cases} \tag{3.29a}$$

with $u_0(0^+) = 1\,,\ u_\delta(0^+) = \infty$;

(b) $1 < \alpha < 2$,

$$u(t) = c_0\, u_0(t) + c_1\, u_1(t) + \int_0^t q(t-\tau)\, u_\delta(\tau)\, d\tau\,, \tag{3.28b}$$

where

$$\begin{cases} u_0(t) = \displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,0}(r)\, dr + \frac{2}{\alpha}\, \mathrm{e}^{t\, \cos(\pi/\alpha)}\, \cos\left[t\, \sin\left(\frac{\pi}{\alpha}\right)\right]\,, \\ u_1(t) = \displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,1}(r)\, dr + \frac{2}{\alpha}\, \mathrm{e}^{t\, \cos(\pi/\alpha)}\, \cos\left[t\, \sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}\right]\,, \\ u_\delta(t) = -\displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,-1}(r)\, dr - \frac{2}{\alpha}\, \mathrm{e}^{t\, \cos(\pi/\alpha)}\, \cos\left[t\, \sin\left(\frac{\pi}{\alpha}\right) + \frac{\pi}{\alpha}\right]\,, \end{cases} \tag{3.29b}$$

with $u_0(0^+) = 1$, $u_0'(0^+) = 0$, $u_1(0^+) = 0$, $u_1'(0^+) = 1$, $u_\delta(0^+) = 0$, $u_\delta'(0^+) = +\infty$.

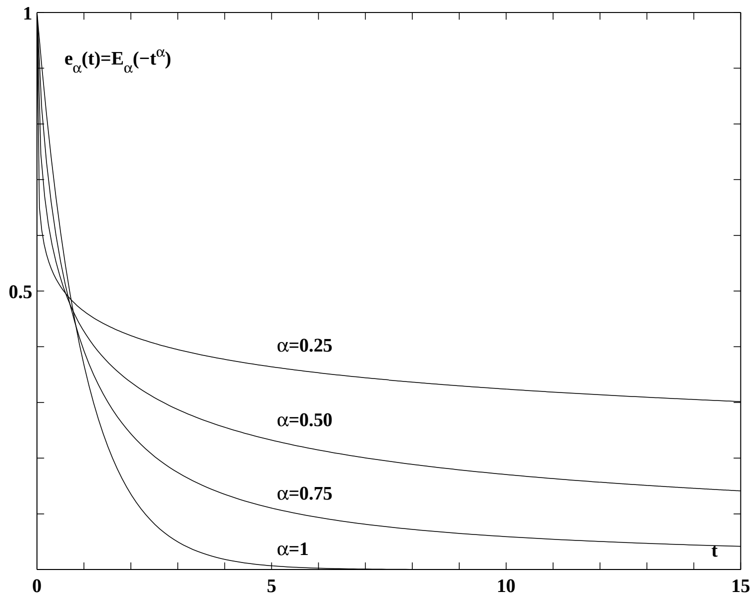

Fig. 2a – Plots of the *basic fundamental solution* $u_0(t) = e_\alpha(t)$ for $0 < \alpha \le 1$

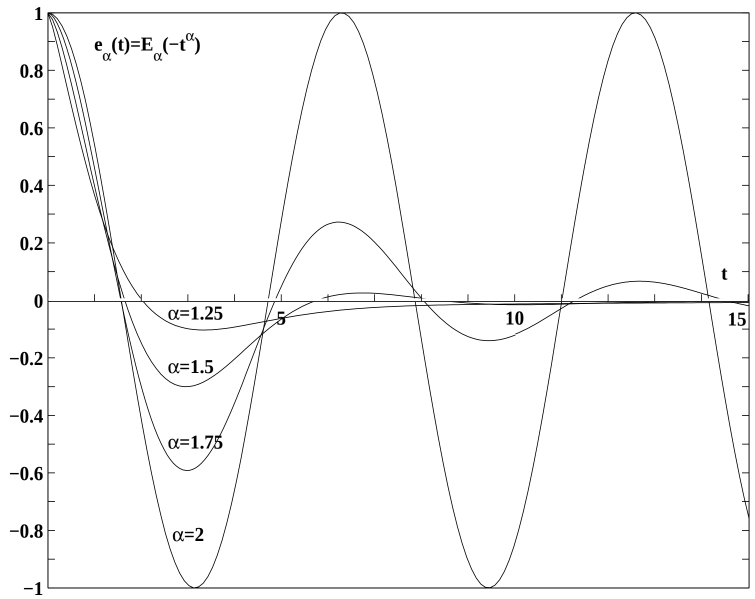

Fig. 2b – Plots of the *basic fundamental solution* $u_0(t) = e_\alpha(t)$ for $1 < \alpha \le 2$:

In Fig. 2 we quote the plots of the basic fundamental solution for the following cases : (a) $\alpha = 0.25\,, 0.50\,, 0.75\,, 1\,,$ and (b) $\alpha = 1.25\,, 1.50\,, 1.75\,, 2\,,$ obtained from the first formula in (3.29a) and (3.29b), respectively. We have verified that our present results confirm those obtained by Blank [43] by a numerical treatment and those obtained by Mainardi [39] by an analytical treatment, valid when α is a rational number, see §$A2$ of the Appendix. Of particular interest is the case $\alpha = 1/2$ where we recover a well-known formula of the Laplace transform theory, see (A.34),

$$e_{1/2}(t) := E_{1/2}(-\sqrt{t}) = \mathrm{e}^{\,t}\,\mathrm{erfc}(\sqrt{t}) \div \frac{1}{s^{1/2}\,(s^{1/2}+1)}\,, \tag{3.30}$$

where erfc denotes the *complementary error* function.

We now desire to point out that in both the cases (a) and (b) (in which α is just not integer) *i.e.* for *fractional relaxation* and *fractional oscillation*, all the fundamental and impulse-response solutions exhibit an *algebraic decay* as $t \to \infty$, as discussed below. Let us start with the asymptotic behaviour of $u_0(t)$. To this purpose we first derive an asymptotic series for the function $f_\alpha(t)$, valid for $t \to \infty$. Using the identity

$$\frac{1}{s^\alpha+1} = 1 - s^\alpha + s^{2\alpha} - s^{3\alpha} + \ldots + (-1)^{N-1}\,s^{(N-1)\alpha} + (-1)^N\,\frac{s^{N\alpha}}{s^\alpha+1}\,,$$

in formula (3.20) and the Hankel representation of the reciprocal Gamma function, we (formally) obtain the asymptotic expansion (for α non integer)

$$f_\alpha(t) = \sum_{n=1}^{N}(-1)^{n-1}\,\frac{t^{-n\alpha}}{\Gamma(1-n\alpha)} + O\left(t^{-(N+1)\alpha}\right)\,,\quad \text{as}\quad t\to\infty\,. \tag{3.31}$$

The validity of this asymptotic expansion can be established rigorously using the (generalized) Watson lemma, see [44]. We also can start from the spectral representation (3.24-25) and expand the spectral function for small r. Then the (ordinary) Watson lemma yields (3.31). We note that this asymptotic expansion coincides with that for $u_0(t) = e_\alpha(t)$, having assumed $0 < \alpha < 2$ $(\alpha \neq 1)$. In fact the contribution of $g_\alpha(t)$ is identically zero if $0 < \alpha < 1$ and exponentially small as $t \to \infty$ if $1 < \alpha < 2$.

The asymptotic expansions of the solutions $u_1(t)$ and $u_\delta(t)$ are obtained from (3.31) integrating or differentiating term by term with respect to t. In particular, taking the leading term in (3.31), we obtain the asymptotic representations

$$u_0(t) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,,\quad u_1(t) \sim \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\,,\quad u_\delta(t) \sim -\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}\,,\quad \text{as}\quad t\to\infty\,, \tag{3.32}$$

that point out the algebraic decay of the fundamental and impulse-response solutions.

In Fig. 3 we show some plots of the *basic fundamental solution* $u_0(t) = e_\alpha(t)$ for $\alpha = 1.25\,, 1.50\,, 1.75\,.$ Here the algebraic decay of the fractional oscillation can be recognized and compared with the two contributions provided by f_α (monotonic behaviour) and $g_\alpha(t)$ (exponentially damped oscillation).

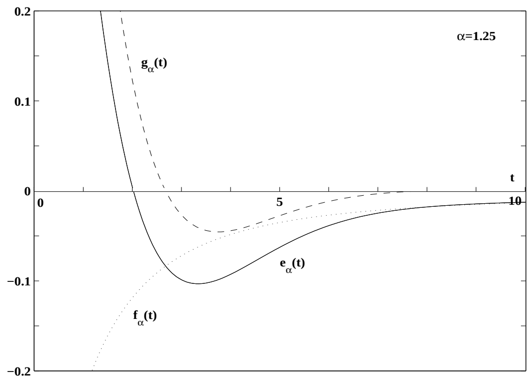

Fig. 3a – Decay of the *basic fundamental solution* $u_0(t) = e_\alpha(t)$ for $\alpha = 1.25$

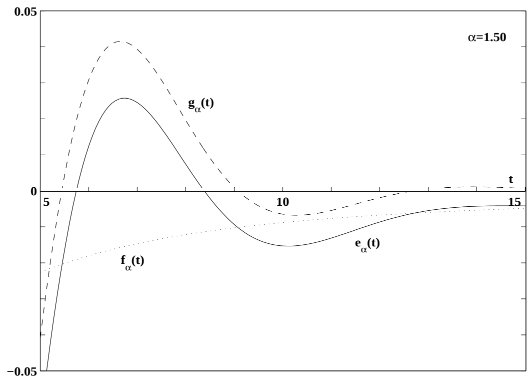

Fig. 3b – Decay of the *basic fundamental solution* $u_0(t) = e_\alpha(t)$ for $\alpha = 1.50$

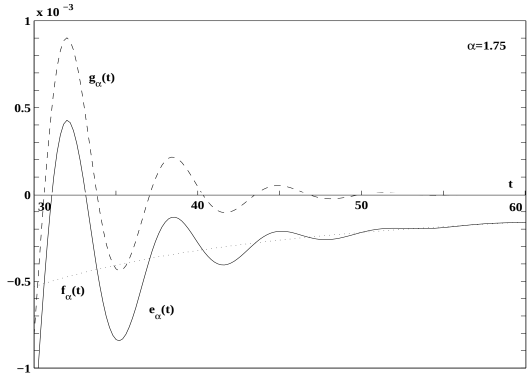

Fig. 3c – Decay of the *basic fundamental solution* $u_0(t) = e_\alpha(t)$ for $\alpha = 1.75$

3.2 The zeros of the solutions of the fractional oscillation equation

Now we find it interesting to carry out some investigations about the zeros of the basic fundamental solution $u_0(t) = e_\alpha(t)$ in the case (b) of fractional oscillations. For the second fundamental solution and the impulse-response solution the analysis of the zeros can be easily carried out analogously.

Recalling the first equation in (3.29b), the required zeros of $e_\alpha(t)$ are the solutions of the equation

$$e_\alpha(t) = f_\alpha(t) + \frac{2}{\alpha}\,\mathrm{e}^{t\,\cos(\pi/\alpha)}\,\cos\left[t\,\sin\left(\frac{\pi}{\alpha}\right)\right] = 0\,. \tag{3.33}$$

We first note that the function $e_\alpha(t)$ exhibits an *odd* number of zeros, in that $e_\alpha(0) = 1$, and, for sufficiently large t, $e_\alpha(t)$ turns out to be permanently negative, as shown in (3.32) by the sign of $\Gamma(1-\alpha)$. The smallest zero lies in the first positivity interval of $\cos[t\sin(\pi/\alpha)]$, hence in the interval $0 < t < \pi/[2\,\sin(\pi/\alpha)]$; all other zeros can only lie in the succeeding positivity intervals of $\cos[t\,\sin(\pi/\alpha)]$, in each of these two zeros are present as long as

$$\frac{2}{\alpha}\,\mathrm{e}^{t\,\cos(\pi/\alpha)} \ge |f_\alpha(t)|\,. \tag{3.34}$$

When t is sufficiently large the zeros are expected to be found approximately from the equation

$$\frac{2}{\alpha}\,\mathrm{e}^{t\,\cos(\pi/\alpha)} \approx \frac{t^{-\alpha}}{|\Gamma(1-\alpha)|}\,, \tag{3.35}$$

obtained from (3.33) by ignoring the oscillation factor of $g_\alpha(t)$ [see (3.23)] and taking the first term in the asymptotic expansion of $f_\alpha(t)$ [see (3.31-32)]. As we have shown in [40], such approximate equation turns out to be useful when $\alpha \to 1^+$ and $\alpha \to 2^-$.

For $\alpha \to 1^+$, only one zero is present, which is expected to be very far from the origin in view of the large period of the function $\cos[t\,\sin(\pi/\alpha)]$. In fact, since there is no zero for $\alpha = 1$, and by increasing α more and more zeros arise, we are sure that only one zero exists for α sufficiently close to 1. Putting $\alpha = 1+\epsilon$ the asymptotic position T_* of this zero can be found from the relation (3.35) in the limit $\epsilon \to 0^+$. Assuming in this limit the first-order approximation, we get

$$T_* \sim \log\left(\frac{2}{\epsilon}\right)\,, \tag{3.36}$$

which shows that T_* tends to infinity slower than $1/\epsilon$, as $\epsilon \to 0$. For details see [40].

For $\alpha \to 2^-$, there is an increasing number of zeros up to infinity since $e_2(t) = \cos t$ has infinitely many zeros [in $t_n^* = (n+1/2)\pi$, $n = 0, 1, \ldots$]. Putting now $\alpha = 2 - \delta$ the asymptotic position T_* for the largest zero can be found again from (3.35) in the limit $\delta \to 0^+$. Assuming in this limit the first-order approximation, we get

$$T_* \sim \frac{12}{\pi\,\delta}\,\log\left(\frac{1}{\delta}\right) . \tag{3.37}$$

For details see [40]. Now, for $\delta \to 0^+$ the length of the positivity intervals of $g_\alpha(t)$ tends to π and, as long as $t \le T_*$, there are two zeros in each positivity interval. Hence, in the limit $\delta \to 0^+$, there is in average one zero per interval of length π, so we expect that $N_* \sim T_*/\pi$.

Remark 4 : For the above considerations we got inspiration from an interesting paper by Wiman [45] who at the beginning of our century, after having treated the Mittag-Leffler function in the complex plane, considered the position of the zeros of the function on the negative real axis (without providing any detail). Our expressions of T_* are in disagreement with those by Wiman for numerical factors; however, the results of our numerical studies carried out in [40] confirm and illustrate the validity of our analysis.

Here, we find it interesting to analyse the phenomenon of the transition of the (odd) number of zeros as $1.4 \le \alpha \le 1.8$. For this purpose, in Table I we report the intervals of amplitude $\Delta\alpha = 0.01$ where these transitions occur, and the location T_* (evaluated within a relative error of 0.1%) of the largest zeros found at the two extreme values of the above intervals. We recognize that the transition from 1 to 3 zeros occurs as $1.40 \le \alpha \le 1.41$, that one from 3 to 5 zeros occurs as $1.56 \le \alpha \le 1.57$, and so on. The last transition in the considered range of α is from 15 to 17 zeros, and it just occurs as $1.79 \le \alpha \le 1.80$.

N_*	α	T_*
$1 \div 3$	$1.40 \div 1.41$	$1.730 \div 5.726$
$3 \div 5$	$1.56 \div 1.57$	$8.366 \div 13.48$
$5 \div 7$	$1.64 \div 1.65$	$14.61 \div 20.00$
$7 \div 9$	$1.69 \div 1.70$	$20.80 \div 26.33$
$9 \div 11$	$1.72 \div 1.73$	$27.03 \div 32.83$
$11 \div 13$	$1.75 \div 1.76$	$33.11 \div 38.81$
$13 \div 15$	$1.78 \div 1.79$	$39.49 \div 45.51$
$15 \div 17$	$1.79 \div 1.80$	$45.51 \div 51.46$

Table I

N_* = number of zeros, α = fractional order, T_* location of the largest zero.

4. FRACTIONAL DIFFERENTIAL EQUATIONS: 2-nd PART

In this section we shall consider the following fractional differential equations for $t \geq 0$, equipped with the necessary initial conditions,

$$\frac{du}{dt} + a\,\frac{d^\alpha u}{dt^\alpha} + u(t) = q(t)\,, \quad u(0^+) = c_0\,, \quad 0 < \alpha < 1\,, \tag{4.1}$$

$$\frac{d^2 v}{dt^2} + a\,\frac{d^\alpha v}{dt^\alpha} + v(t) = q(t)\,, \quad v(0^+) = c_0\,, \quad v'(0^+) = c_1\,, \quad 0 < \alpha < 2\,, \tag{4.2}$$

where a is a positive constant. The unknown functions $u(t)$ and $v(t)$ (the field variables) are required to be sufficiently well behaved to be treated with their derivatives $u'(t)$ and $v'(t)\,,\ v''(t)$ by the technique of Laplace transform. The given function $q(t)$ is supposed to be continuous. In the above equations the fractional derivative of order α is assumed to be provided by the operator D_*^α, the *Caputo derivative*, see (1.17), in agreement with our choice followed in the previous section. Note that in (4.2) we must distinguish the cases (a) $0 < \alpha < 1$, (b) $1 < \alpha < 2$ and $\alpha = 1$.

The equations (4.1) and(4.2) will be referred to as the *composite fractional relaxation equation* and the *composite fractional oscillation equation*, respectively, to be distinguished from the corresponding *simple* fractional equations treated in §3.

The fractional differential equation in (4.1) with $\alpha = 1/2$ corresponds to the *Basset problem*, a classical problem in fluid dynamics concerning the unsteady motion of a particle accelerating in a viscous fluid under the action of the gravity, see [24].

The fractional differential equation in (4.2) with $0 < \alpha < 2$ models an oscillation process with fractional damping term. It was formerly treated by Caputo [19], who provided a preliminary analysis by the Laplace transform. The special cases $\alpha = 1/2$ and $\alpha = 3/2$, but with the standard definition D^α for the fractional derivative, have been discussed by Bagley [30]. Recently, Beyer and Kempfle [46] discussed (4.2) for $-\infty < t < +\infty$ to investigate the uniqueness and *causality* of the solutions. As they let t running in all of $\mathbb{R}$, they used Fourier transforms and characterized the fractional derivative D^α by its properties in frequency space, thereby requiring that for non-integer α the principal branch of $(i\omega)^\alpha$ should be taken. Under the global condition that the solution is square summable, they showed that the system described by (4.2) is *causal iff* $a > 0$.

Also here we shall apply the method of Laplace transform to solve the fractional differential equations and get some insight into their *fundamental* and *impulse-response solutions*. However, in contrast with the previous section, we now find it more convenient to apply directly the formula (1.30) for the Laplace transform of fractional and integer derivatives, than reduce the equations with the prescribed initial conditions as equivalent (fractional) integral equations to be treated by the Laplace transform.

4.1 The composite fractional relaxation equation

Let us apply the Laplace transform to the fractional relaxation equation (4.1). Using the rule (1.30) we are led to the transformed algebraic equation

$$\widetilde{u}(s) = c_0 \frac{1 + a\, s^{\alpha-1}}{w_1(s)} + \frac{\widetilde{q}(s)}{w_1(s)}, \quad 0 < \alpha < 1\,, \tag{4.3}$$

where

$$w_1(s) := s + a\, s^{\alpha} + 1\,, \tag{4.4}$$

and $a > 0$. Putting

$$u_0(t) \div \widetilde{u}_0(s) := \frac{1 + a\, s^{\alpha-1}}{w_1(s)}, \quad u_\delta(t) \div \widetilde{u}_\delta(s) := \frac{1}{w_1(s)}, \tag{4.5}$$

and recognizing that

$$u_0(0^+) = \lim_{s\to\infty} s\, \widetilde{u}_0(s) = 1\,, \quad \widetilde{u}_\delta(s) = -\left[s\, \widetilde{u}_0(s) - 1\right], \tag{4.6}$$

we can conclude that

$$u(t) = c_0\, u_0(t) + \int_0^t q(t-\tau)\, u_\delta(\tau)\, d\tau\,, \quad u_\delta(t) = -\, u_0'(t)\,. \tag{4.7}$$

We thus recognize that $u_0(t)$ and $u_\delta(t)$ are the *fundamental solution* and *impulse-response solution* for the equation (4.1), respectively.

Let us first consider the problem to get $u_0(t)$ as the inverse Laplace transform of $\widetilde{u}_0(s)$. We easily see that the function $w_1(s)$ has no zero in the main sheet of the Riemann surface including its boundaries on the cut (simply show that Im $\{w_1(s)\}$ does not vanish if s is not a real positive number), so that the inversion of the Laplace transform $\widetilde{u}_0(s)$ can be carried out by deforming the original Bromwich path into the Hankel path $Ha(\epsilon)$ introduced in the previous section, *i.e.* into the loop constituted by a small circle $|s| = \epsilon$ with $\epsilon \to 0$ and by the two borders of the cut negative real axis. As a consequence we write

$$u_0(t) = \frac{1}{2\pi i} \int_{Ha(\epsilon)} \mathrm{e}^{\,st}\, \frac{1 + a s^{\alpha-1}}{s + a\, s^{\alpha} + 1}\, ds\,. \tag{4.8}$$

It is now an exercise in complex analysis to show that the contribution from the Hankel path $Ha(\epsilon)$ as $\epsilon \to 0$ is provided by

$$u_0(t) = \int_0^\infty \mathrm{e}^{-rt}\, H^{(1)}_{\alpha,0}(r;a)\, dr\,, \tag{4.9}$$

with

$$H^{(1)}_{\alpha,0}(r;a) = -\frac{1}{\pi}\,\mathrm{Im}\left\{\left.\frac{1+as^{\alpha-1}}{w_1(s)}\right|_{s=r\,\mathrm{e}^{i\pi}}\right\} \\ = \frac{1}{\pi}\,\frac{a\,r^{\alpha-1}\,\sin(\alpha\pi)}{(1-r)^2+a^2\,r^{2\alpha}+2\,(1-r)\,a\,r^{\alpha}\,\cos(\alpha\pi)}\,. \tag{4.10}$$

For $a>0$ and $0<\alpha<1$ the function $H^{(1)}_{\alpha,0}(r;a)$ is positive for all $r>0$ since it has the sign of the numerator; in fact in (4.10) the denominator is strictly positive being equal to $|w_1(s)|^2$ as $s=r\,\mathrm{e}^{\pm i\pi}$. Hence, the *fundamental solution* $u_0(t)$ has the peculiar property to be *completely monotone*, and $H^{(1)}_{\alpha,0}(r;a)$ is its *spectral function.*

Now the determination of $u_\delta(t) = -u_0'(t)$ is straightforward. We see that also the *impulse-response solution* $u_\delta(t)$ is *completely monotone* since it can be represented by

$$u_\delta(t) = \int_0^\infty \mathrm{e}^{-rt}\,H^{(1)}_{\alpha,-1}(r;a)\,dr\,, \tag{4.11}$$

with *spectral function*

$$H^{(1)}_{\alpha,-1}(r;a) = r\,H^{(1)}_{\alpha,0}(r;a) = \frac{1}{\pi}\,\frac{a\,r^{\alpha}\,\sin(\alpha\pi)}{(1-r)^2+a^2\,r^{2\alpha}+2\,(1-r)\,a\,r^{\alpha}\,\cos(\alpha\pi)}\,. \tag{4.12}$$

We recognize that both the solutions $u_0(t)$ and $u_\delta(t)$ turn out to be strictly decreasing from 1 towards 0 as t runs from 0 to ∞. Their behaviour as $t\to 0^+$ and $t\to\infty$ can be inspected by means of a proper asymptotic analysis.

The behaviour of the solutions as $t\to 0^+$ can be determined from the behaviour of their Laplace transforms as $\mathrm{Re}\,\{s\}\to+\infty$ as well known from the theory of the Laplace transform, see *e.g.* [25]. We obtain as $\mathrm{Re}\,\{s\}\to+\infty$,

$$\widetilde{u}_0(s) = s^{-1} - s^{-2} + O\left(s^{-3+\alpha}\right)\,, \quad \widetilde{u}_\delta(s) = s^{-1} - a\,s^{-(2-\alpha)} + O\left(s^{-2}\right)\,, \tag{4.13}$$

so that

$$u_0(t) = 1 - t + O\left(t^{2-\alpha}\right)\,, \quad u_\delta(t) = 1 - a\,\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + O\left(t\right)\,, \quad \text{as} \quad t\to 0^+\,. \tag{4.14}$$

The spectral representations (4.9) and (4.11) are suitable to obtain the asymptotic behaviour of $u_0(t)$ and $u_\delta(t)$ as $t\to+\infty$, by using the Watson lemma. In fact, expanding the spectral functions for small r and taking the dominant term in the corresponding asymptotic series, we obtain

$$u_0(t) \sim a\,\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,, \quad u_\delta(t) \sim -a\,\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}\,, \quad \text{as} \quad t\to\infty\,. \tag{4.15}$$

We note that the limiting case $\alpha=1$ can be easily treated extending the validity of eqs (4.3-7) to $\alpha=1$, as it is legitimate. In this case we obtain

$$u_0(t) = \mathrm{e}^{-t/(1+a)}\,, \quad u_\delta(t) = \frac{1}{1+a}\,\mathrm{e}^{-t/(1+a)}\,, \quad \alpha = 1\,. \tag{4.16}$$

Of course, in the case $a\equiv 0$ we recover the standard solutions $u_0(t) = u_\delta(t) = \mathrm{e}^{-t}$.

We conclude this sub-section with some considerations on the solutions when the order α is just a rational number. If we take $\alpha = p/q$, where $p, q \in \mathbb{N}$ are assumed (for convenience) to be relatively prime, a factorization in (4.4) is possible by using the procedure indicated by Miller and Ross [10]. In these cases the solutions can be expressed in terms of a linear combination of q Mittag-Leffler functions of fractional order $1/q$, which, on their turn can be expressed in terms of incomplete gamma functions, see (A.14) of the Appendix.

Here we shall illustrate the factorization in the simplest case $\alpha = 1/2$ and provide the solutions $u_0(t)$ and $u_\delta(t)$ in terms of the functions $e_\alpha(t;\lambda)$ (with $\alpha = 1/2$), introduced in the previous section. In this case, in view of the application to the *Basset problem*, see [24], the equation (4.1) deserves a particular attention. For $\alpha = 1/2$ we can write

$$w_1(s) = s + a\, s^{1/2} + 1 = (s^{1/2} - \lambda_+)\,(s^{1/2} - \lambda_-)\,, \quad \lambda_\pm = -a/2 \pm (a^2/4 - 1)^{1/2}\,. \tag{4.17}$$

Here $\lambda_\pm$ denote the two roots (real or conjugate complex) of the second degree polynomial with positive coefficients $z^2 + az + 1$, which, in particular, satisfy the following binary relations

$$\lambda_+ \cdot \lambda_- = 1\,, \quad \lambda_+ + \lambda_- = -a\,, \quad \lambda_+ - \lambda_- = 2(a^2/4 - 1)^{1/2} = (a^2 - 4)^{1/2}\,. \tag{4.18}$$

We recognize that we must treat separately the following two cases

$$i)\quad 0 < a < 2\,, \quad \text{or} \quad a > 2\,, \quad \text{and} \quad ii)\quad a = 2\,,$$

which correspond to two distinct roots ($\lambda_+ \neq \lambda_-$), or two coincident roots ($\lambda_+ \equiv \lambda_- = -1$), respectively. For this purpose, using the notation introduced in [24], we write

$$\widetilde{M}(s) := \frac{1 + a\, s^{-1/2}}{s + a\, s^{1/2} + 1} = \begin{cases} i) \ \dfrac{A_-}{s^{1/2}\,(s^{1/2} - \lambda_+)} + \dfrac{A_+}{s^{1/2}(s^{1/2} - \lambda_-)}\,, \\ ii) \ \dfrac{1}{(s^{1/2} + 1)^2} + \dfrac{2}{s^{1/2}(s^{1/2} + 1)^2}\,, \end{cases} \tag{4.19}$$

and

$$\widetilde{N}(s) := \frac{1}{s + a\, s^{1/2} + 1} = \begin{cases} i) \ \dfrac{A_+}{s^{1/2}\,(s^{1/2} - \lambda_+)} + \dfrac{A_-}{s^{1/2}(s^{1/2} - \lambda_-)}\,, \\ ii) \ \dfrac{1}{(s^{1/2} + 1)^2}\,, \end{cases} \tag{4.20}$$

where

$$A_\pm = \pm \frac{\lambda_\pm}{\lambda_+ - \lambda_-}\,. \tag{4.21}$$

Using (4.18) we note that

$$A_+ + A_- = 1\,, \quad A_+\,\lambda_- + A_-\,\lambda_+ = 0\,, \quad A_+\,\lambda_+ + A_-\,\lambda_- = -\,a\,. \tag{4.22}$$

Recalling the Laplace transform pairs (A.34), (A.36) and (A.37) in Appendix, we obtain

$$u_0(t) = M(t) := \begin{cases} i)\ A_-\, E_{1/2}\,(\lambda_+\,\sqrt{t}) + A_+\, E_{1/2}\,(\lambda_-\,\sqrt{t})\,, \\ ii)\ (1-2t)\, E_{1/2}\,(-\sqrt{t}) + 2\,\sqrt{t/\pi}\,, \end{cases} \tag{4.23}$$

and

$$u_\delta(t) = N(t) := \begin{cases} i)\ A_+\, E_{1/2}(\lambda_+\,\sqrt{t}) + A_-\, E_{1/2}(\lambda_-\,\sqrt{t})\,, \\ ii)\ (1+2t)\, E_{1/2}(-\sqrt{t}) - 2\,\sqrt{t/\pi}\,. \end{cases} \tag{4.24}$$

We thus recognize in (4.23-24) the presence of the functions $e_{1/2}(t;-\lambda_\pm) = E_{1/2}(\lambda_\pm\,\sqrt{t})$ and $e_{1/2}(t) = e_{1/2}(t;1) = E_{1/2}(-\sqrt{t})$.

In particular, the solution of the *Basset problem* can be easily obtained from (4.7) with $q(t) = q_0$ by using (4.23-24) and noting that $\int_0^t N(\tau)\, d\tau = 1 - M(t)$. Denoting this solution by $u_B(t)$ we get

$$u_B(t) = q_0 - (q_0 - c_0)\, M(t)\,. \tag{4.25}$$

When $a \equiv 0$, *i.e.* in the absence of term containing the fractional derivative (due to the Basset force), we recover the classical Stokes solution, that we denote by $u_S(t)$,

$$u_S(t) = q_0 - (q_0 - c_0)\, \mathrm{e}^{\,-t}\,.$$

In the particular case $q_0 = c_0$, we get the steady-state solution $u_B(t) = u_S(t) \equiv q_0$.

For vanishing initial condition $c_0 = 0$, we have the creep-like solutions

$$u_B(t) = q_0\,\left[1 - M(t)\right]\,, \quad u_S(t) = q_0\,\left[1 - \mathrm{e}^{\,-t}\right]\,,$$

that we compare in the normalized plots of Fig. 5 of [24]. In this case it is instructive to compare the behaviours of the two solutions as $t \to 0^+$ and $t \to \infty$. Recalling the general asymptotic expressions of $u_0(t) = M(t)$ in (4.14) and (4.15) with $\alpha = 1/2$, we recognize that

$$u_B(t) = q_0\,\left[t + O\left(t^{3/2}\right)\right]\,, \quad u_S(t) = q_0\,\left[t + O\left(t^2\right)\right]\,, \quad \text{as} \quad t \to 0^+\,,$$

and

$$u_B(t) \sim q_0\,\left[1 - a/\sqrt{\pi\, t}\,\right]\,, \quad u_S(t) \sim q_0\,\left[1 - EST\,\right]\,, \quad \text{as} \quad t \to \infty\,,$$

where EST denotes *exponentially small terms.* In particular we note that the normalized plot of $u_B(t)/q_0$ remains under that of $u_S(t)/q_0$ as t runs from 0 to ∞ .

The reader is invited to convince himself of the following fact. In the general case $0 < \alpha < 1$ the solution $u(t)$ has the particular property of being equal to 1 for all $t \ge 0$ if $q(t)$ has this property and $u(0^+) = 1$, whereas $q(t) = 1$ for all $t \ge 0$ and $u(0^+) = 0$ implies that $u(t)$ is a creep function tending to 1 as $t \to \infty$.

4.2 The composite fractional oscillation equation

Let us now apply the Laplace transform to the fractional oscillation equation (4.2). Using the rule (1.30) we are led to the transformed algebraic equations

$$\text{(a)}\quad \widetilde{v}(s) = c_0 \frac{s + a\, s^{\alpha-1}}{w_2(s)} + c_1 \frac{1}{w_2(s)} + \frac{\widetilde{q}(s)}{w_2(s)}\,, \quad 0 < \alpha < 1\,, \tag{4.26a}$$

or

$$\text{(b)}\quad \widetilde{v}(s) = c_0 \frac{s + a\, s^{\alpha-1}}{w_2(s)} + c_1 \frac{1 + a\, s^{\alpha-2}}{w_2(s)} + \frac{\widetilde{q}(s)}{w_2(s)}\,, \quad 1 < \alpha < 2\,, \tag{4.26b}$$

where

$$w_2(s) := s^2 + a\, s^\alpha + 1\,, \tag{4.27}$$

and $a > 0$. Putting

$$\widetilde{v}_0(s) := \frac{s + a\, s^{\alpha-1}}{w_2(s)}\,, \quad 0 < \alpha < 2\,, \tag{4.28}$$

we recognize that

$$v_0(0^+) = \lim_{s\to\infty} s\, \widetilde{v}_0(s) = 1\,, \quad \frac{1}{w_2(s)} = -\,[s\, \widetilde{v}_0(s) - 1] \div -v_0'(t)\,, \tag{4.29}$$

and

$$\frac{1 + a\, s^{\alpha-2}}{w_2(s)} = \frac{\widetilde{v}_0(s)}{s} \div \int_0^t v_0(\tau)\, d\tau\,. \tag{4.30}$$

Thus we can conclude that

$$\text{(a)}\quad v(t) = c_0\, v_0(t) - c_1\, v_0'(t) - \int_0^t q(t-\tau)\, v_0'(\tau)\, d\tau\,, \quad 0 < \alpha < 1\,, \tag{4.31a}$$

or

$$\text{(b)}\quad v(t) = c_0\, v_0(t) + c_1 \int_0^t v_0(\tau)\, d\tau - \int_0^t q(t-\tau)\, v_0'(\tau)\, d\tau\,, \quad 1 < \alpha < 2\,. \tag{4.31b}$$

In both of the above equations the term $-v_0'(t)$ represents the *impulse-response solution* $v_\delta(t)$ for the *composite fractional oscillation equation* (4.2), namely the particular solution of the inhomogeneous equation with $c_0 = c_1 = 0$ and with $q(t) = \delta(t)$. For the *fundamental solutions* of (4.2) we recognize from eqs (4.31) that we have two distinct couples of solutions according to the case (a) and (b) which read

$$\text{(a)}\quad \{v_0(t)\,,\, v_{1\,\mathrm{a}}(t) = -v_0'(t)\}\,, \qquad \text{(b)}\quad \{v_0(t)\,,\, v_{1\,\mathrm{b}}(t) = \int_0^t v_0(\tau)\, d\tau\}\,. \tag{4.32}$$

We first consider the particular case $\alpha = 1$ for which the fundamental and impulse response solutions are known in terms of elementary functions. This limiting case can also be treated by extending the validity of eqs (4.26a) and (4.31a) to $\alpha = 1$, as it is legitimate. From

$$\widetilde{v}_0(s) = \frac{s+a}{s^2 + a\,s + 1} = \frac{s + a/2}{(s+a/2)^2 + (1 - a^2/4)} - \frac{a/2}{(s+a/2)^2 + (1-a^2/4)}\,, \quad (4.33)$$

we obtain the *basic fundamental solution*

$$v_0(t) = \begin{cases} \mathrm{e}^{-at/2}\left[\cos(\omega t) + \dfrac{a}{2\omega}\,\sin(\omega t)\right] & \text{if} \quad 0 < a < 2\,, \\ \mathrm{e}^{-t}\,(1-t) \qquad \text{if} \quad a = 2\,, \\ \mathrm{e}^{-at/2}\left[\cosh(\chi t) + \dfrac{a}{2\chi}\,\sinh(\chi t)\right] & \text{if} \quad a > 2\,, \end{cases} \quad (4.34)$$

where

$$\omega = \sqrt{1 - a^2/4}\,, \quad \chi = \sqrt{a^2/4 - 1}\,. \quad (4.35)$$

By a differentiation of (4.34) we easily obtain the *second fundamental solution* $v_{1\,a}(t)$ and the *impulse-response solution* $v_\delta(t)$ since $v_{1\,\mathrm{a}}(t) = v_\delta(t) = -v_0'(t)$. We point out that all the solutions exhibit an *exponential decay* as $t \to \infty$.

Let us now consider the problem to get $v_0(t)$ as the inverse Laplace transform of $\widetilde{v}_0(s)$, as given by (4.26-27),

$$v_0(t) = \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{\,st}\,\frac{s + a\,s^{\alpha-1}}{w_2(s)}\,ds\,, \quad (4.36)$$

where Br denotes the usual Bromwich path. Using a result by Beyer and Kempfle [46] we know that the function $w_2(s)$ (for $a > 0$ and $0 < \alpha < 2$, $\alpha \neq 1$) has exactly *two simple, conjugate complex zeros on the principal branch in the open left half-plane*, cut along the negative real axis, say $s_+ = \rho\,\mathrm{e}^{+i\gamma}$ and $s_- = \rho\,\mathrm{e}^{-i\gamma}$ with $\rho > 0$ and $\pi/2 < \gamma < \pi$. This enables us to repeat the considerations carried out for the simple fractional oscillation equation to decompose the basic fundamental solution $v_0(t)$ into two parts according to $v_0(t) = f_\alpha(t;a) + g_\alpha(t;a)$. In fact, the evaluation of the Bromwich integral (4.36) can be achieved by adding the contribution $f_\alpha(t;a)$ from the Hankel path $Ha(\epsilon)$ as $\epsilon \to 0$, to the residual contribution $g_\alpha(t;a)$ from the two poles $s_\pm$.

As an exercise in complex analysis we obtain

$$f_\alpha(t;a) = \int_0^\infty \mathrm{e}^{\,-rt}\,H^{(2)}_{\alpha,0}(r;a)\,dr\,, \quad (4.37)$$

with *spectral function*

$$
\begin{aligned}
H^{(2)}_{\alpha,0}(r;a) &= -\frac{1}{\pi}\,\mathrm{Im}\left\{ \left.\frac{s+as^{\alpha-1}}{w_2(s)}\right|_{s=r\,\mathrm{e}^{i\pi}} \right\} \\
&= \frac{1}{\pi}\,\frac{a\,r^{\alpha-1}\,\sin(\alpha\pi)}{(r^2+1)^2+a^2\,r^{2\alpha}+2\,(r^2+1)\,a\,r^\alpha\,\cos(\alpha\pi)}\,.
\end{aligned}
\tag{4.38}
$$

Since in (4.38) the denominator is strictly positive being equal to $|w_2(s)|^2$ as $s = r\,\mathrm{e}^{\pm i\pi}$, the *spectral function* $H^{(2)}_{\alpha,0}(r;a)$ turns out to be positive for all $r > 0$ for $0 < \alpha < 1$ and negative for all $r > 0$ for $1 < \alpha < 2$. Hence, in case (a) the function $f_\alpha(t)$, in case (b) the function $-f_\alpha(t)$ is *completely monotone*; in both cases $f_\alpha(t)$ tends to zero as $t \to \infty$, from above in case (a), from below in case (b), according to the asymptotic behaviour

$$
f_\alpha(t;a) \sim a\,\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,, \quad \text{as} \quad t\to\infty\,, \quad 0<\alpha<1\,, \quad 1<\alpha<2\,,
\tag{4.39}
$$

as derived by applying the Watson lemma in (4.37) and considering (4.38).

The other part, $g_\alpha(t;a)$, is obtained as

$$
\begin{aligned}
g_\alpha(t;a) &= \mathrm{e}^{s_+\,t}\,Res\left[\frac{s+a\,s^{\alpha-1}}{w_2(s)}\right]_{s_+} + \quad \text{conjugate complex} \\
&= 2\,\mathrm{Re}\left\{ \frac{s_+ + a\,s_+^{\alpha-1}}{2\,s_+ + a\,\alpha\,s_+^{\alpha-1}}\,\mathrm{e}^{s_+\,t} \right\}.
\end{aligned}
\tag{4.40}
$$

Thus this term exhibits an *oscillatory* character with exponentially decreasing amplitude like $\exp(-\rho\,t\,|\cos\gamma|)$.

Then we recognize that the basic fundamental solution $v_0(t)$ exhibits a *finite* number of zeros and that, for sufficiently large t, it turns out to be permanently positive if $0 < \alpha < 1$ and permanently negative if $1 < \alpha < 2$ with an *algebraic decay* provided by (4.39).

For the second fundamental solutions $v_{1\,\mathrm{a}}(t)$, $v_{1\,\mathrm{b}}(t)$ and for the impulse-response solution $v_\delta(t)$, the corresponding analysis is straightforward in view of their connection with $v_0(t)$, pointed out in (4.31-32). The *algebraic decay* of all the solutions as $t \to \infty$, for $0 < \alpha < 1$ and $1 < \alpha < 2$, is henceforth resumed in the relations

$$
v_0(t) \sim a\,\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,, \quad v_{1\,\mathrm{a}}(t) = v_\delta(t) \sim -a\,\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}\,, \quad v_{1\,\mathrm{b}}(t) \sim a\,\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\,.
\tag{4.41}
$$

In conclusion, except in the particular case $\alpha = 1$, all the present solutions of the composite fractional oscillation equation exhibit similar characteristics with the corresponding solutions of the simple fractional oscillation equation, namely a *finite number of damped oscillations* followed by a *monotonic algebraic decay* as $t \to \infty$.

5. CONCLUSIONS

Starting from the classical *Riemann-Liouville* definitions of the fractional integration operator and its left-inverse, the fractional differentiation operator, and using the powerful tool of the Laplace transform method, we have described the basic *analytical* theory of fractional integral and differential equations. For a *numerical* treatment we refer to Gorenflo [23] and to the references there quoted.

For the *fractional integral equations* we have considered the basic examples provided by the linear *Abel equations of first and second kind*. For both the kinds we have given the solution in different forms and discussed an interesting application to inverse heat conduction problems.

Then we have analyzed in detail the scale of *fractional ordinary differential equations* ($FODE$), see (3.5), $D_*^\alpha\, u(t) + u(t) = q(t),\ t > 0\,,\ 0 < \alpha \le 2$, with a modified fractional differentiation D_*^α, the *Caputo fractional derivative*, that takes account of given initial values $u(0^+)$ if $0 < \alpha < 1$, the case of *fractional relaxation*, $u(0^+)$ and $u'(0^+)$ if $1 < \alpha < 2$, the case of *fractional oscillation*.

We have investigated in depth the properties of the *fundamental* and the *impulse-response solutions*. All these solutions can be explicitly written down in terms of *Mittag-Leffler functions*. They tend to zero like powers $t^{-\beta}$ (with appropriate choices of β), monotonically if $0 < \alpha < 1$, but exhibiting finitely many oscillations around zero if $1 < \alpha < 2$ (the more of these the nearer α is to the limiting value 2). If $1 < \alpha < 2$ these equations are able to model processes intermediate between exponential decay ($\alpha = 1$) and pure sinusoidal oscillation ($\alpha = 2$). We have found these qualitative properties essentially by bending the Bromwich integration path of the Laplace inversion formula into the Hankel path, thus for each of these functions obtaining an integral representation as the Laplace transform of a function that nowhere changes its sign, augmented if $1 < \alpha < 2$ by an oscillatory contribution resulting from a pair of conjugate complex poles lying in the left half-plane.

By quite analogous methods we have studied the *composite equations*, see (4.1-2), $(D + aD_*^\alpha + 1)\, u(t) = q(t)\,,\ 0 < \alpha < 1$, and $(D^2 + aD_*^\alpha + 1)\, v(t) = q(t)\,,\ 0 < \alpha < 2$, with $a > 0$, which model processes of relaxation and of oscillation, respectively. We have obtained similar properties of the *fundamental* and *impulse-response solutions* with respect to monotonicity and oscillatory behaviour.

Let us stress the fact that our adoption of the *Caputo fractional derivative* D_*^α with $m-1 < \alpha \le m\,,\ m \in \mathbb{N}$, and the consequent prescription of the initial values in analogy with the ordinary differential equations of integer order m, stands in contrast to the majority of the treatments of fractional differential equations, where the standard fractional derivative D^α is used, see *e.g.* [5], [10]. As pointed in §1.3 the adoption of D^α requires the prescription of certain fractional integrals as $t \to 0^+$.

In our opinion, the different prescription of the the initial data points out the major difference between the two definitions for the fractional derivative. The analogy with the cases of integer order would induce one to adopt the *Caputo derivative* in the treatment of differential equations of fractional order for *physical applications*. In fact, in physical problems, the initial conditions are usually expressed in terms of a given number of bounded values assumed by the field variable and its derivatives of integer order, no matter if the governing evolution equation may be a generic integro-differential equation and therefore, in particular, a fractional differential equation.

The liveliness of the field of fractional integral and differential equations, both in applications and in pure theory, is underlined by several papers and some books that have appeared recently. We would like to conclude the present lectures with brief hints to recent investigations, which have not been quoted explicitly up to now since not strictly related to our results, but have attracted our attention. Naturally, this listing is far from exhaustive. The interested reader can find more on problems and aspects in several papers recently published or in press in some conference volumes and specialized magazines.

We first like to quote the most recent book by Rubin [14], who starting from one-dimensional fractional calculus develops the theory of multidimensional weakly singular integral equations of first kind, making heavy use of the Marchaud approach. We thus recognize that all existing books on fractional calculus vary widely from each other in their character concerning problems treated and methods applied. We quote also the Ph.D. thesis by Michalski [47], who treats linear and nonlinear problems of fractional calculus (in one and in several dimensions) in a very elegant way.

The importance of using fractional methods in physics for describing *slow decay processes* and *processes intermediate between relaxation and oscillation* was stressed by Nigmatullin [48] in 1984. Nonnenmacher and associates published a series of papers (of which we quote [49-50]) discussing various physical aspects of *fractional relaxation*. Fractional relaxation is overall a peculiarity of a class of *viscoelastic bodies* which are extensively treated by Mainardi [24], to which we refer for details and additional bibliography. The fractional calculus finds important applications in different areas of applied science including *electrochemistry*, see *e.g.* [51-54], *electromagnetism*, see *e.g.* [55-56], *radiation physics*, see *e.g.* [57-59], and *control theory*, see *e.g.* [60-64].

Yet another field of applications of fractional calculus is that of *fractional partial differential equations* ($FPDE$), including certain equations of *fractional diffusion*, introduced to explain the phenomena of anomalous diffusion in complex or fractal systems. We refer again to Mainardi [24] for a mathematical treatment of a relevant $FPDE$, referred to as the *time fractional diffusion-wave equation*, with some applications and related references.

APPENDIX: THE MITTAG-LEFFLER TYPE FUNCTIONS

In this Appendix we shall consider the Mittag-Leffler function and some of the related functions which are relevant for their connection with fractional calculus. It is our purpose to provide a review of the main properties of these functions including their Laplace transforms.

A.1 The Mittag-Leffler functions $E_\alpha(z)$, $E_{\alpha,\beta}(z)$

The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is so named from the great Swedish mathematician who introduced it at the beginning of this century in a sequence of five notes, see [65-69]. The function is defined by the following series representation, valid in the whole complex plane,

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \quad \alpha > 0, \quad z \in \mathbb{C}. \qquad (A.1)$$

It turns out that $E_\alpha(z)$ is an *entire function* of order $\rho = 1/\alpha$ and type 1. This property is still valid but with $\rho = 1/\mathrm{Re}\{\alpha\}$, if $\alpha \in \mathbb{C}$ with *positive real part*, as formerly noted by Mittag-Leffler himself in [68].

In the limit for $\alpha \to 0^+$ the analyticity in the whole complex plane is lost since

$$E_0(z) := \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad |z| < 1. \qquad (A.2)$$

The Mittag-Leffler function provides a simple generalization of the exponential function because of the substitution of $n! = \Gamma(n+1)$ with $(\alpha n)! = \Gamma(\alpha n+1)$. Particular cases of (A.1), from which elementary functions are recovered, are

$$E_2\left(+z^2\right) = \cosh z, \qquad E_2\left(-z^2\right) = \cos z, \qquad z \in \mathbb{C}, \qquad (A.3)$$

and

$$E_{1/2}(\pm z^{1/2}) = \mathrm{e}^z \left[1 + \mathrm{erf}\,(\pm z^{1/2})\right] = \mathrm{e}^z\, \mathrm{erfc}\,(\mp z^{1/2}), \quad z \in \mathbb{C}, \qquad (A.4)$$

where erf (erfc) denotes the (complementary) error function defined as

$$\mathrm{erf}\,(z) := \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-u^2}\, du, \quad \mathrm{erfc}\,(z) := 1 - \mathrm{erf}\,(z), \quad z \in \mathbb{C}.$$

In (A.4) for $z^{1/2}$ we mean the principal value of the square root of z in the complex plane cut along the the negative real axis. With this choice $\pm z^{1/2}$ turns out to be positive/negative for $z \in \mathbb{R}^+$.

Since the identities in (A.3) are trivial, we present the proof only for (A.4). Avoiding the inessential polidromy with the substitution $\pm z^{1/2} \to z$, we write

$$E_{1/2}(z) = \sum_{m=0}^{\infty} \frac{z^{2m}}{\Gamma(m+1)} + \sum_{m=0}^{\infty} \frac{z^{2m+1}}{\Gamma(m+3/2)} = u(z) + v(z). \qquad (A.5)$$

Whereas the even part is easily recognized to be $u(z) = \exp(z^2)$, only after some manipulations the odd part can be proved to be $v(z) = \exp(z^2)\,\mathrm{erf}(z)$. To this end we need to recall the following series representation for the error function, see *e.g.* the handbook of the Bateman Project [70] or that by Abramowitz and Stegun [71],

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\,\mathrm{e}^{-z^2} \sum_{m=0}^{\infty} \frac{2^m}{(2m+1)!!}\, z^{2m+1}\,, \quad z \in \mathbb{C}$$

and note that $(2m+1)!! := 1 \cdot 3 \cdot 5 \ldots \cdot (2m+1) = 2^{m+1}\,\Gamma(m+3/2)/\sqrt{\pi}$. An alternative proof is obtained by recognizing, after a term-wise differentiation of the series representation in (A.5), that $v(z)$ satisfies the differential equation in $\mathbb{C}$,

$$v'(z) = 2\left[\frac{1}{\sqrt{\pi}} + z\,v(z)\right], \quad v(0) = 0\,,$$

whose solution can immediately be checked to be

$$v(z) = \frac{2}{\sqrt{\pi}}\,\mathrm{e}^{z^2} \int_0^z \mathrm{e}^{-u^2}\,du = \mathrm{e}^{z^2}\,\mathrm{erf}(z)\,.$$

A straightforward generalization of the Mittag-Leffler function, originally due to Agarwal in 1953 based on a note by Humbert, see [72-74], is obtained by replacing the additive constant 1 in the argument of the Gamma function in (A.1) by an arbitrary complex parameter β. Later, when we shall deal with Laplace transform pairs, the parameter β is required to be positive as α. For the new function we agree to use the following notation

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}\,, \quad \alpha > 0\,,\ \beta \in \mathbb{C}\,, \quad z \in \mathbb{C}\,. \qquad (A.6)$$

Particular simple cases are

$$E_{1,2}(z) = \frac{\mathrm{e}^z - 1}{z}\,, \quad E_{2,2}(z) = \frac{\sinh(z^{1/2})}{z^{1/2}}\,. \qquad (A.7)$$

We note that $E_{\alpha,\beta}(z)$ is still an entire function of order $\rho = 1/\alpha$ and type 1.

In these lectures we have preferred to use only the original Mittag-Leffler function (A.1) since our problems depend on only a single parameter α, the order of fractional integration of differentiation. However, for completeness, we list hereafter the general functional relations for the generalized Mittag-Leffler function (A.6), which involve both the two parameters α, β, see [18] and [70],

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + z\,E_{\alpha,\beta+\alpha}(z)\,, \qquad (A.8)$$

$$E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z\,\frac{d}{dz}\,E_{\alpha,\beta+1}(z)\,, \qquad (A.9)$$

$$\left(\frac{d}{dz}\right)^p \left[z^{\beta-1} E_{\alpha,\beta}(z^\alpha)\right] = z^{\beta-p-1} E_{\alpha,\beta-p}(z^\alpha)\,, \quad p \in \mathbb{N}\,. \qquad (A.10)$$

A2. The Mittag-Leffler functions of rational order

Let us now consider the Mittag-Leffler functions of rational order $\alpha = p/q$ with $p\,,\,q \in \mathbb{N}$ relatively prime. The relevant functional relations, that we quote from [18], [70], turn out to be

$$\left(\frac{d}{dz}\right)^p E_p\left(z^p\right) = E_p\left(z^p\right), \tag{A.11}$$

$$\frac{d^p}{dz^p} E_{p/q}\left(z^{p/q}\right) = E_{p/q}\left(z^{p/q}\right) + \sum_{k=1}^{q-1} \frac{z^{-k\,p/q}}{\Gamma(1-k\,p/q)}\,, \quad q = 2, 3, \ldots\,, \tag{A.12}$$

$$E_{p/q}\left(z\right) = \frac{1}{p} \sum_{h=0}^{p-1} E_{1/q}\left(z^{1/p} e^{i2\pi h/p}\right), \tag{A.13}$$

and

$$E_{1/q}\left(z^{1/q}\right) = \mathrm{e}^z \left[1 + \sum_{k=1}^{q-1} \frac{\gamma(1-k/q, z)}{\Gamma(1-k/q)}\right], \quad q = 2, 3, \ldots\,, \tag{A.14}$$

where $\gamma(a, z) := \int_0^z \mathrm{e}^{-u}\, u^{\,a-1}\, du$ denotes the *incomplete gamma function.* Let us now sketch the proof for the above functional relations.

One easily recognizes that the relations (A.11) and (A.12) are immediate consequences of the definition (A.1).

In order to prove the relation (A.13) we need to recall the identity

$$\sum_{h=0}^{p-1} \mathrm{e}^{\,i2\pi hk/p} = \begin{cases} p & \text{if } k \equiv 0 \pmod{p}\,, \\ 0 & \text{if } k \not\equiv 0 \pmod{p}\,. \end{cases} \tag{A.15}$$

In fact, using this identity and the definition (A.1), we have

$$\sum_{h=0}^{p-1} E_\alpha(z\, \mathrm{e}^{i2\pi h/p}) = p\, E_{\alpha p}(z^p)\,, \quad p \in \mathbb{N}\,. \tag{A.16}$$

Substituting in the above relation α/p instead of α and $z^{1/p}$ instead of z, we obtain

$$E_\alpha(z) = \frac{1}{p} \sum_{h=0}^{p-1} E_{\alpha/p}\left(z^{1/p} e^{i2\pi h/p}\right)\,, \quad p \in \mathbb{N}\,. \tag{A.17}$$

Setting above $\alpha = p/q$, we finally obtain (A.13).

To prove the relation (A.14) we consider (A.12) for $p = 1$. Multiplying both sides by e^{-z}, we obtain

$$\frac{d}{dz}\left[\mathrm{e}^{-z}\, E_{1/q}\left(z^{1/q}\right)\right] = \mathrm{e}^{-z} \sum_{k=1}^{q-1} \frac{z^{-k/q}}{\Gamma(1-k/q)}\,. \tag{A.18}$$

Then, upon integration of this and recalling the definition of the incomplete gamma function, we arrive at (A.14).

The relation (A.14) shows how the Mittag-Leffler functions of rational order can be expressed in terms of exponentials and incomplete gamma functions. In particular, taking in (A.14) $q = 2$, we now can verify again the relation (A.4). In fact, from (A.14) we obtain

$$E_{1/2}(z^{1/2}) = \mathrm{e}^z \left[1 + \frac{1}{\sqrt{\pi}}\, \gamma(1/2\,,\, z)\right], \tag{A.19}$$

which is equivalent to (A.4) if we use the relation $\mathrm{erf}\,(z) = \gamma(1/2, z^2)/\sqrt{\pi}$, see *e.g.* [70-71].

A3. The Mittag-Leffler integral representation and asymptotic expansions

Many of the most important properties of $E_\alpha(z)$ follow from Mittag-Leffler's *integral representation*

$$E_\alpha(z) = \frac{1}{2\pi i} \int_{Ha} \frac{\zeta^{\alpha-1}\, \mathrm{e}^{\zeta}}{\zeta^\alpha - z}\, d\zeta\,, \quad \alpha > 0\,, \quad z \in \mathbb{C}\,, \tag{A.20}$$

where the path of integration Ha (the *Hankel path*) is a loop which starts and ends at $-\infty$ and encircles the circular disk $|\zeta| \le |z|^{1/\alpha}$ in the positive sense: $-\pi \le \arg \zeta \le \pi$ on Ha. To prove (A.20), expand the integrand in powers of ζ, integrate term-by-term, and use Hankel's integral for the reciprocal of the Gamma function.

The integrand in (A.20) has a branch-point at $\zeta = 0$. The complex ζ-plane is cut along the negative real axis, and in the cut plane the integrand is single-valued: the principal branch of ζ^α is taken in the cut plane. The integrand has poles at the points $\zeta_m = z^{1/\alpha}\, \mathrm{e}^{2\pi\, i\, m/\alpha}$, m integer, but only those of the poles lie in the cut plane for which $-\alpha\,\pi < \arg z + 2\pi\, m < \alpha\,\pi$. Thus, the number of the poles inside Ha is either $[\alpha]$ or $[\alpha + 1]$, according to the value of $\arg z$.

The integral representation of the generalized Mittag-Leffler function turns out to be

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{Ha} \frac{\zeta^{\alpha-\beta}\, \mathrm{e}^{\zeta}}{\zeta^\alpha - z}\, d\zeta\,, \quad \alpha\,,\, \beta > 0\,, \quad z \in \mathbb{C}\,. \tag{A.21}$$

The most interesting properties of the Mittag-Leffler function are associated with its asymptotic developments as $z \to \infty$ in various sectors of the complex plane. These properties can be summarized as follows.

For the case $0 < \alpha < 2$ we have

$$E_\alpha(z) \sim \frac{1}{\alpha} \exp(z^{1/\alpha}) - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}\,, \qquad |z| \to \infty\,, \quad |\arg z| < \alpha\pi/2\,, \tag{A.22}$$

$$E_\alpha(z) \sim -\sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}\,, \qquad |z| \to \infty\,, \quad \alpha\pi/2 < \arg z < 2\pi - \alpha\pi/2\,. \tag{A.23}$$

For the case $\alpha \geq 2$ we have

$$E_\alpha(z) \sim \frac{1}{\alpha} \sum_m \exp\left(z^{1/\alpha} \mathrm{e}^{2\pi\, i\, m/\alpha}\right) - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}\,, \qquad |z| \to \infty\,, \tag{A.24}$$

where m takes all integer values such that $-\alpha\pi/2 < \arg\, z + 2\pi\, m < \alpha\pi/2$, and $\arg\, z$ can assume any value between $-\pi$ and $+\pi$ inclusive.

From the asymptotic properties (A.22-24) and the definition of the order of an entire function, we infer that the Mittag-Leffler function is an *entire function of order* $1/\alpha$ for $\alpha > 0$; in a certain sense each $E_\alpha(z)$ is the simplest entire function of its order, see Phragmén [75]. The Mittag-Leffler function also furnishes examples and counter-examples for the growth and other properties of entire functions of finite order, see Buhl [76].

A4. The Laplace transform pairs related to the Mittag-Leffler functions

The Mittag-Leffler functions are connected to the Laplace integral through the equation

$$\int_0^\infty \mathrm{e}^{-u}\, E_\alpha\left(u^\alpha\, z\right)\, du = \frac{1}{1-z} = \int_0^\infty \mathrm{e}^{-u}\, u^{\beta-1}\, E_{\alpha,\beta}\left(u^\alpha\, z\right)\, du\,, \quad \alpha\,,\, \beta > 0\,. \tag{A.25}$$

The integral at the L.H.S. was evaluated by Mittag-Leffler who showed that the region of its convergence contains the unit circle and is bounded by the line $\mathrm{Re}\, z^{1/\alpha} = 1$.

The above integral is fundamental in the evaluation of the Laplace transform of $E_\alpha\left(-\lambda\, t^\alpha\right)$ and $E_{\alpha,\beta}\left(-\lambda\, t^\alpha\right)$ with $\alpha\,,\, \beta > 0$ and $\lambda \in \mathbb{C}$. Since these functions turn out to play a key role in problems of fractional calculus, we shall introduce a special notation for them.

Putting in (A.25) $u = st$ and $u^\alpha\, z = -\lambda\, t^\alpha$ with $t \geq 0$ and $\lambda \in \mathbb{C}$, and using the sign $\div$ for the juxtaposition of a function depending on t with its Laplace transform depending on s, we get the following Laplace transform pairs

$$e_\alpha(t;\lambda) := E_\alpha\left(-\lambda\, t^\alpha\right) \div \frac{s^{\alpha-1}}{s^\alpha + \lambda}\,, \quad Re\, s > |\lambda|^{1/\alpha}\,, \tag{A.26}$$

and

$$e_{\alpha,\beta}(t;\lambda) := t^{\beta-1}\, E_{\alpha,\beta}\left(-\lambda\, t^\alpha\right) \div \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}\,, \quad Re\, s > |\lambda|^{1/\alpha}\,. \tag{A.27}$$

We note that the results (A.26-27), but with a different notation, were used by Humbert and Agarwal [72-74] to obtain a number of functional relations satisfied by $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$. Of course the results (A.26-27) can also be obtained formally by Laplace transforming term by term the series (A.1) and (A.6) with $z = -\lambda\, t^\alpha$, respectively, and summing the resulting series.

We find worthwhile to list the following relations for the functions $e_{\alpha,\beta}$ easily obtained from (A.8-9) :

$$e_{\alpha,\beta}(t;\lambda) = \frac{t^{\beta-1}}{\Gamma(\alpha)} - \lambda\, e_{\alpha,\beta+\alpha}(t;\lambda)\,, \tag{A.28}$$

and

$$\frac{d}{dt}\, e_{\alpha,\beta+1}(t;\lambda) = e_{\alpha,\beta}(t;\lambda)\,. \tag{A.29}$$

A remarkable property satisfied by the functions $e_\alpha(t;\lambda)$, $e_{\alpha,\beta}(t;\lambda)$ when λ *is positive* and $0 < \alpha \le 1$, $0 < \alpha \le \beta \le 1$, respectively, is to be *completely monotone* for $t > 0$.

We recall that a function $f(t)$ is told to be *completely monotone* for $t > 0$ if $(-1)^n f^{(n)}(t) \ge 0$ for all $n = 0, 1, 2, \dots$ and for all $t > 0$, and that a sufficient condition for this is the existence of a nonnegative locally integrable function $K(r)$, $r > 0$, referred to as the *spectral function*, with which $f(t) = \int_0^\infty \mathrm{e}^{-rt}\, K(r)\, dr$. For more details see *e.g.* the book by Berg & Forst [77].

Excluding the trivial case $\alpha = \beta = 1$ for which $e_1(t;\lambda) = e_{1,1}(t;\lambda) = \mathrm{e}^{-\lambda\, t}$, we can prove the existence of the corresponding spectral functions using the complex Bromwich formula to invert the Laplace transform in (A.26-27) and bending the Bromwich path into the Hankel path, as we have already shown in the special case $e_\alpha(t) := e_\alpha(t;1)$ in §3. As an exercise in complex analysis (we kindly invite the reader to carry it out) we obtain the integral representations [A.30-33],

$$e_\alpha(t;\lambda) := \int_0^\infty \mathrm{e}^{-rt}\, K_\alpha(r;\lambda)\, dr\,, \quad 0 < \alpha < 1\,, \quad \lambda > 0\,, \tag{A.30}$$

with *spectral function*

$$K_\alpha(r;\lambda) = \frac{1}{\pi}\, \frac{\lambda\, r^{\alpha-1}\, \sin(\alpha\pi)}{r^{2\alpha} + 2\,\lambda\, r^\alpha\, \cos(\alpha\pi) + \lambda^2} \ge 0\,, \tag{A.31}$$

and

$$e_{\alpha,\beta}(t;\lambda) := \int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,\beta}(r;\lambda)\, dr\,, \quad 0 < \alpha \le \beta < 1\,, \quad \lambda > 0\,, \tag{A.32}$$

with *spectral function*

$$K_{\alpha,\beta}(r;\lambda) = \frac{1}{\pi}\, \frac{\lambda\, \sin[(\beta-\alpha)\pi] + r^\alpha\, \sin(\beta\pi)}{r^{2\alpha} + 2\,\lambda\, r^\alpha\, \cos(\alpha\pi) + \lambda^2}\, r^{\,\alpha\,-\,\beta} \ge 0\,. \tag{A.33}$$

Historically, the complete monotonicity of the Mittag-Leffler function in the negative real axis, *i.e.* of $E_\alpha(-x)$, for $x \in \mathbb{R}^+$, when $0 < \alpha < 1$, was first conjectured by Feller using probabilistic methods and rigorously proved by Pollard in 1948 [78].

Only recently, Schneider [79] has proved a theorem for the complete monotonicity of the generalized Mittag-Leffler function in the negative real axis. He proved that $E_{\alpha,\beta}(-x)$, for $x \in \mathbb{R}^+$, is completely monotone *iff* $0 < \alpha \le 1$ and $\beta \ge \alpha$. Our conditions for $e_{\alpha,\beta}(t,\lambda)$ to be completely monotone appear more restrictive than those by Schneider for $E_{\alpha,\beta}(-x)$; however, we must note that in our case (A.27) the factor $t^{\beta-1}$ precedes the generalized Mittag-Leffler function.

We note that, up to our knowledge, in the handbooks containing tables for the Laplace transforms, the Mittag-Leffler function is ignored so that the transform pairs (A.26-27) do not appear if not in the special cases $\alpha = 1/2$ and $\beta = 1$, $1/2$, written however in terms of the error and complementary error functions, see *e.g.* [71]. In fact, in these cases we can use (A.4) and (A.28) and recover from (A.26-27) the two Laplace transform pairs

$$\frac{1}{s^{1/2}\,(s^{1/2} \pm \lambda)} \div e_{1/2}\,(t;\pm\lambda) = \mathrm{e}^{\lambda^2 t}\,\mathrm{erfc}\,(\pm\lambda\,\sqrt{t})\,, \quad \lambda \in \mathbb{C}\,, \tag{A.34}$$

$$\frac{1}{s^{1/2} \pm \lambda} \div e_{1/2,1/2}\,(t;\pm\lambda) = \frac{1}{\sqrt{\pi\,t}} \mp \lambda\, e_{1/2}\,(t;\pm\lambda)\,, \quad \lambda \in \mathbb{C}\,. \tag{A.35}$$

We also obtain the related pairs

$$\frac{1}{s^{1/2}\,(s^{1/2} \pm \lambda)^2} \div 2\sqrt{\frac{t}{\pi}} \mp 2\,\lambda\,t\,e_{1/2}(t;\pm\lambda)\,, \quad \lambda \in \mathbb{C}\,, \tag{A.36}$$

$$\frac{1}{(s^{1/2} \pm \lambda)^2} \div \mp 2\,\lambda\sqrt{\frac{t}{\pi}} + (1 + 2\,\lambda^2\,t)\,e_{1/2}(t;\pm\lambda)\,, \quad \lambda \in \mathbb{C}\,, \tag{A.37}$$

In the pair (A.36) we have used the properties

$$\frac{1}{s^{1/2}\,(s^{1/2} \pm \lambda)^2} = -2\,\frac{d}{ds}\left(\frac{1}{s^{1/2} \pm \lambda}\right)\,, \qquad \frac{d^n}{ds^n}\,\tilde{f}(s) \div (-t)^n\,f(t)\,.$$

The pair (A.37) is easily obtained by noting that

$$\frac{1}{(s^{1/2} \pm \lambda)^2} = \frac{1}{s^{1/2}\,(s^{1/2} \pm \lambda)} \mp \frac{\lambda}{s^{1/2}\,(s^{1/2} \pm \lambda)^2}\,.$$

A.5 Additional references for the Mittag-Leffler type functions

We note that the Mittag-Leffler type functions are unknown to the majority of scientists, because they are ignored in the common books on special functions. Thanks to our suggestion the new *2000 Mathematics Subject Classification* has included these functions, see the item *33E12: Mittag-Leffler functions and generalizations.*

A description of the most important properties of these functions with relevant references can be found in the third volume of the Bateman Project [70], in the chapter *XVIII* devoted to *miscellaneous functions.* The specialized treatises where great attention is devoted to the Mittag-Leffler type functions are those by Dzherbashyan [18], [22]. For the interested readers we also recommend the classical treatise on complex functions by Sansone & Gerretsen [80], where a sufficiently detailed treatment of the original Mittag-Leffler function is given. Since the times of Mittag-Leffler several scientists have recognized the importance of the Mittag-Leffler type functions, providing interesting results and applications, which unfortunately are not much known. As pioneering works of mathematical nature in the field of fractional integral and differential equations, we like to quote those by Hille & Tamarkin and by Barret. In 1930 Hille & Tamarkin [81] have provided the solution of the Abel integral equation of the second kind in terms of a Mittag-Leffler function, whereas in 1956 Barret [82] has expressed the general solution of the linear fractional differential equation with constant coefficients in terms of Mittag-Leffler functions.

As former applications in physics we like to quote the contributions by K.S. Cole (1933), quoted by H.T. Davis [15, p. 287] in connection with nerve conduction, and by F.M. de Oliveira Castro (1939) [83] and B. Gross (1947) [84] in connection with dielectrical and mechanical relaxation, respectively. Subsequently, in 1971, Caputo & Mainardi [28] have proved that the Mittag-Leffler function is present whenever derivatives of fractional order are introduced in the constitutive equations of a linear viscoelastic body. Since then, several other authors have pointed out the relevance of the Mittag-Leffler function for fractional viscoelastic models, see Mainardi [24].

In recent times the attention of mathematicians towards the Mittag-Leffler type functions has increased from both the analytical and numerical point of view, overall because of their relation with the fractional calculus. In addition to the books and papers already quoted in the text, here we would like to draw the reader's attention to the most recent papers on the Mittag-Leffler type functions, *e.g.* Al Saqabi & Tuan [85], Kilbas & Saigo [86], Gorenflo, Luchko & Rogozin [87] and Mainardi & Gorenflo [88]. Since the fractional calculus has actually recalled a wide interest for its applications in different areas of physics and engineering, we expect that soon the Mittag-Leffler function will exit from its isolated life as *Cinderella* (using the term coined by F.G. Tricomi in the 1950s for the *incomplete Gamma* function).

REFERENCES

1. Ross, B. (Editor): *Fractional Calculus and its Applications*, Lecture Notes in Mathematics # 457, Springer Verlag, Berlin 1975. [Proc. Int. Conf. held at Univ. of New Haven, USA, 1974]
2. Oldham, K.B. and J. Spanier: *The Fractional Calculus*, Academic Press, New York 1974.
3. McBride, A.C.: *Fractional Calculus and Integral Transforms of Generalized Functions*, Pitman Research Notes in Mathematics # 31, Pitman, London 1979.
4. McBride, A.C. and G.F. Roach (Editors): *Fractional Calculus*, Pitman Research Notes in Mathematics # 138, Pitman, London 1985. [Proc. Int. Workshop. held at Univ. of Strathclyde, UK, 1984]
5. Samko S.G., Kilbas, A.A. and O.I. Marichev: *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Amsterdam 1993. [Engl. Transl. from Russian, *Integrals and Derivatives of Fractional Order and Some of Their Applications*, Nauka i Tekhnika, Minsk 1987]
6. H.M. Srivastava and S. Owa (Editors): *Univalent Functions, Fractional Calculus, and their Applications*, Ellis Horwood, Chichester 1989.
7. Nishimoto, K. (Editor): *Fractional Calculus and its Applications*, Nihon University, Tokyo 1990. [Proc. Int. Conf. held at Nihon Univ., Tokyo 1989]
8. Nishimoto, K.: *An Essence of Nishimoto's Fractional Calculus*, Descartes Press, Koriyama 1991.
9. Kalia, R.N. (Editor): *Recent Advances in Fractional Calculus*, Global Publ., Sauk Rapids, Minnesota 1993.
10. Miller, K.S. and B. Ross: *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York 1993.
11. Kiryakova, V.: *Generalized Fractional Calculus and Applications*, Pitman Research Notes in Mathematics # 301, Longman, Harlow 1994.
12. Rusev, P., Dimovski, I and V. Kiryakova (Editors): *Transform Methods and Special Functions, Sofia 1994*, Science Culture Technology, Singapore 1995. [Proc. Int. Workshop, Sofia, Bulgaria, 12-17 August 1994]
13. Kilbas, A.A. (Editor): *Boundary Value Problems, Special Functions and Fractional Calculus*, Byelorussian State University, Minsk 1996. (ISBN 985-6144-40-X) [Proc. Int. Conf., 90-th Birth Anniversary of Academician F.D. Gakhov, Minsk, Byelorussia, 16-20 February 1996]
14. Rubin, B.: *Fractional Integrals and Potentials*, Pitman Monographs and Surveys in Pure and Applied Mathematics #82, Addison Wesley Longman, Harlow 1996.

15. Davis, H.T.:*The Theory of Linear Operators*, The Principia Press, Bloomington, Indiana 1936.

16. Erdélyi, A. (Editor): *Tables of Integral Transforms*, Bateman Project, Vols. 1-2, McGraw-Hill, New York 1953-1954.

17. Gel'fand, I.M. and G.E. Shilov: *Generalized Functions*, Vol. 1, Academic Press, New York 1964.

18. Dzherbashian, M.M.: *Integral Transforms and Representations of Functions in the Complex Plane*, Nauka, Moscow 1966. [in Russian]

19. Caputo, M.: *Elasticità e Dissipazione*, Zanichelli, Bologna 1969. [in Italian]

20. Babenko, Yu.I.: *Heat and Mass Transfer*, Chimia, Leningrad 1986. [in Russian]

21. Gorenflo, R. and S. Vessella: *Abel Integral Equations: Analysis and Applications*, Lecture Notes in Mathematics # 1461, Springer-Verlag, Berlin 1991.

22. Dzherbashian, M.M.: *Harmonic Analysis and Boundary Value Problems in the Complex Domain*, Birkhäuser Verlag, Basel 1993.

23. Gorenflo, R.: Fractional calculus: some numerical methods, in: *Fractals and Fractional Calculus in Continuum Mechanics* (Eds. A. Carpinteri and F. Mainardi), Springer Verlag, Wien 1997, 277-290. (this book)

24. Mainardi, F.: Fractional calculus: some basic problems in continuum and statistical mechanics, in: *Fractals and Fractional Calculus in Continuum Mechanics* (Eds. A. Carpinteri and F. Mainardi), Springer Verlag, Wien 1997, 291-348.

25. Doetsch, G.: *Introduction to the Theory and Application of the Laplace Transformation*, Springer Verlag, Berlin 1974.

26. Henrici, P.: *Applied and Computational Complex Analysis*, Vol. 2, Wiley, New York 1977.

27. Caputo, M.: Linear models of dissipation whose Q is almost frequency independent, Part II., *Geophys. J. R. Astr. Soc.*, **13** (1967), 529-539.

28. Caputo, M. and F. Mainardi: Linear models of dissipation in anelastic solids, *Riv. Nuovo Cimento* (Ser. II), **1** (1971), 161-198.

29. Podlubny, I.: Solutions of linear fractional differential equations with constant coefficients, in: *Transform Methods and Special Functions, Sofia 1994* (Eds. P. Rusev, I. Dimovski and V. Kiryakova), Science Culture Technology, Singapore 1995, 227-237. [Proc. Int. Workshop, Sofia, Bulgaria, 12-17 August 1994]

30. Bagley, R. L.: On the fractional order initial value problem and its engineering applications, in: *Fractional Calculus and Its Applications* (Ed. K. Nishimoto), College of Engineering, Nihon University, Tokyo 1990, pp. 12-20. [Proc. Int. Conf. held at Nihon Univ., Tokyo, 1989]

31. Craig, J. D. and J. C. Brown: *Inverse Problems in Astronomy* Adam Hilger Ltd, Bristol 1986.

32. Gorenflo, R.: *Abel Integral Equations with special Emphasis on Applications*, Lectures in Mathematical Sciences Vol. 13, The University of Tokyo, Graduate School of Mathematical Sciences, 1996. (ISSN 0919-8180)

33. Gorenflo, R.: The tomato salad problem in spherical stereology, in *Transform Methods and Special Functions, Varna 1996* (Eds. Rusev, P., Dimovski, I and V. Kiryakova), Science Culture Technology, Singapore 1997 (*in press*). [Proc. Int. Workshop, Varna, Bulgaria, 23-30 August 1996] An extended version is available as Pre-print A-25/96, Fachbereich Mathematik und Informatik, Freie Universität, Berlin 1996, via Internet: <http://www.math.fu-berlin.de/publ/index.html>

34. Duff, G. F. D. and D. Naylor: *Differential Equations of Applied Mathematics*, Wiley & Sons, New York 1966.

35. Mann, W. R. and F. Wolf: *Heat transfer between solids and gases under nonlinear boundary conditions*, Quart. Appl. Math., **9** (1951), 163 - 184.

36. Lebedev, N. N.: *Special Functions and Their Applications*, Dover, New York 1965.

37. Gorenflo, R. and R. Rutman: On ultraslow and intermediate processes, in: *Transform Methods and Special Functions, Sofia 1994* (Eds. Rusev, P., Dimovski, I and V. Kiryakova), Science Culture Technology, Singapore 1995, 61-81. [Proc. Int. Workshop, Sofia, Bulgaria, 12-17 August 1994]

38. Mainardi, F.: Fractional relaxation and fractional diffusion equations, mathematical aspects, in: *Proceedings 12-th IMACS World Congress* (Ed. W.F. Ames), GeorgiaTech, Atlanta 1994, Vol. 1, 329-332.

39. Mainardi, F.: Fractional relaxation-oscillation and fractional diffusion-wave phenomena, *Chaos, Solitons & Fractals*, **7** (1996), 1461-1477.

40. Gorenflo, R. and F. Mainardi: Fractional oscillations and Mittag-Leffler functions, Pre-print A-14/96, Fachbereich Mathematik und Informatik, Freie Universität, Berlin 1996, Internet: <http://www.math.fu-berlin.de/publ/index.html>

41. Mainardi, F. and R. Gorenflo: The Mittag-Leffler function in the Riemann-Liouville fractional calculus, in: *Boundary Value Problems, Special Functions and Fractional Calculus* (Ed. A.A.Kilbas), Byelorussian State University, Minsk 1996, 215-225. (ISBN 985-6144-40-X) [Proc. Int. Conf., 90-th Birth Anniversary of Academician F.D. Gakhov, Minsk, Byelorussia, 16-20 February 1996]

42. Gorenflo, R. and F. Mainardi: Fractional relaxation and oscillations in linear causal systems, Pre-print, Department of Physics, University of Bologna, 1996.

43. Blank, L.: Numerical treatment of differential equations of fractional order,

MCCM Numerical Analysis Report No. 287, The University of Manchester 1996, Internet: <http://www.ma.man.ac.uk/MCCM/MCCM.html>

44. Bleistein, N. and R. A. Handelsman: *Asymptotic Expansions of Integrals*, Ch. 4, p. 162, Dover, New York 1986.
45. Wiman, A.: Über die Nullstellen der Funktionen $E_\alpha(x)$, *Acta Math.*, **29** (1905), 217-234.
46. Beyer, H. and S. Kempfle: Definition of physically consistent damping laws with fractional derivatives, *ZAMM*, **75** (1995), 623-635.
47. Michalski, M.W.: *Derivatives of Noninteger Order and their Applications*, Dissertationes Mathematicae, Polska Akademia Nauk, Instytut Matematyczny, Warsaw (1993).
48. Nigmatullin, R.: On the theory of relaxation with "remnant" memory, *Phys. Stat. Sol. B*, **124** (1984), 389-393. Translated from the Russian.
49. Nonnenmacher, T.F. and W.G. Glöckle: A fractional model for mechanical stress relaxation, *Phil. Mag. Lett.*, **64** (1991), 89-93.
50. Nonnenmacher, T.F. and R. Metzler: On the Riemann-Liouville fractional calculus and some recent applications, *Fractals*, **3** (1995), 557-566.
51. Oldham, K.B. and C.G. Zoski: Analogue instrumentation for processing polorographic data, *J. Electroanalytical Chemistry*, **157** (1983), 27-51.
52. Macdonald, J.R. and L.D. Potter Jr.: A flexible procedure for analyzing impedance spectroscopy results; descriptions and illustrations, *Solid State Ionics*, **23** (1987), 61-79.
53. Mulder, W.H. and J.H. Sluyters: An explanation of depressed semi-circular arcs in impedance plots for irreversible electrode reactions, *Electrochemica Acta*, **33** (1988), 303-310.
54. Keddam, M. and H. Takenouti: Impedance of fractional interfaces; new data on the Von Koch model, *Electrochemica Acta*, **33** (1988), 3445-448.
55. Engheta, N.: On the role of non-integral (fractional) calculus in electrodynamics, *Digest of the 1992 IEEE AP-S/URSI International Symposium*, Chicago, July 17-20, 1992, Vol. URSI Digest, 163-175.
56. Engheta, N.: Fractional differintegrals and electrostatic fields and potentials near sharp conducting edges, Lectures presented at the International Summer School: *Fractals and Hyperbolic Geometries, Fractional and Fractal Derivatives in Engineering, Applied Physics and Economics* (Eds. LeMéhauté and A. Oustaloup), Bordeaux, July 3-8, 1994, 23 pages.
57. Kalla, S.L., Al Saqabi, B. and S. Conde: Some results related to radiation-field problems, *Hadronic Journal*, **10** (1987), 221-230.

58. Gabutti, B., Kalla, S.L. and J.H. Hubbell: Some expansions related to the Hubbell rectangular-source integral, *J. Comp. Appl. Maths*, **37** (1991), 273-285.

59. Kalla, S.L.: The Hubbell rectangular source integral and its generalizations, *Radiation Physics and Chemistry*, **41** (1993), 775-781.

60. Rutman, R.S.: On physical interpretations of fractional integration and differentiation, *Theor. and Math. Physics*, **105** (1995), 1509-1519. Translated from the Russian.

61. Podlubny, I.: *Fractional-order systems and fractional-order controllers*, Report UEF-03-94, Slovak Academy of Sciences, Institute of Experimental Physics, Kosice, Slovakia, November 1994, 18 pages.

62. Oustaloup, A.: *La dérivation non entière: théorie, synthèse, applications*, série Automatique, Éditions Hermès (1995).

63. Matignon, D.: Stability results for fractional differential equations with applications to control processing, Proceedings *Computational Engineering in Systems and Application multiconference*, IMACS, IEEE-SMC, Lille, France, July 1966, pp. 963-968.

64. Fliess, M. and R. Hotzel: Sur les systèmes linéaires à dérivation non entière, *C. R. Acad. Sci. Paris*, XXX, Automatique, Preprint 1996, 6 pages.

65. Mittag-Leffler, G.M.: Sur l'intégrale de Laplace-Abel, *C.R. Acad. Sci. Paris*, (ser. II) **136** (1902), 937-939.

66. Mittag-Leffler, G.M.: Une généralisation de l'intégrale de Laplace-Abel, *C.R. Acad. Sci. Paris*, (ser. II) **137** (1903), 537-539.

67. Mittag-Leffler, G.M.: Sur la nouvelle fonction $E_\alpha(x)$, *C.R. Acad. Sci. Paris*, (ser. II) **137** (1903), 554-558.

68. Mittag-Leffler, G.M: Sopra la funzione $E_\alpha(x)$, *R. Accad. Lincei, Rend.*, (ser. V) (1904) **13**, 3-5.

69. Mittag-Leffler, G.M.: Sur la représentation analytique d'une branche uniforme d'une fonction monogène, *Acta Math.*, **29** (1905), 101-181.

70. Erdélyi, A. (Ed.): *Higher Transcendental Functions*, Bateman Project, Vols. 1-3, McGraw-Hill, New York 1953-1955.

71. Abramowitz, M. and I.A. Stegun: *Handbook of Mathematical Functions*, Dover, New York 1965.

72. Humbert, P.: Quelques résultats relatifs à la fonction de Mittag-Leffler, *C.R. Acad. Sci. Paris*, **236** (1953), 1467-1468.

73. Agarwal, R.P.: A propos d'une note de M. Pierre Humbert, *C.R. Acad. Sci. Paris*, **236** (1953), 2031-2032.

74. Humbert, P. and R.P. Agarwal: Sur la fonction de Mittag-Leffler et quelques-unes de ses généralisations, *Bull. Sci. Math* (Ser. II), **77** (1953), 180-185.

75. Phragmén, E.: Sur une extension d'un théoreme classique de la théorie des fonctions, *Acta Math.*, **28** (1904), 351-368.

76. Buhl, A. : *Séries Analytiques. Sommabilité*, Mémorial des Sciences Mathématiques, Acad. Sci. Paris, Fasc. VII, Gauthier-Villars, Paris 1925, Ch. 3.

77. Berg, C. and G. Forst: *Potential Theory on Locally Compact Abelian Groups*, Springer Verlag, Berlin 1975, §9., pp. 61-72.

78. Pollard, H.: The completely monotonic character of the Mittag-Leffler function $E_\alpha(-x)$, *Bull. Amer. Math. Soc.*, **54** (1948), 1115-1116.

79. Schneider, W.R.: Complete monotone generalized Mittag-Leffler functions, *Expositiones Mathematicae*, **14** (1996), 3-16.

80. Sansone, G. and J. Gerretsen: *Lectures on the Theory of Functions of a Complex Variable*, Vol. I. *Holomorphic Functions*, Nordhoff, Groningen 1960, pp. 345-349.

81. Hille E. and J. D. Tamarkin: On the theory of linear integral equations, *Ann. Math.*, **31** (1930), 479-528.

82. Barret, J. H.: Differential equations of non-integer order, *Canad. J. Math.*, **6** (1954), 529-541.

83. de Oliveira Castro, F. M.: Zur Theorie der dielektrischen Nachwirkung, *Zeits. f. Physik*, **114** (1939), 116–126.

84. Gross, B: On creep and relaxation, *J. Appl. Phys.*, **18** (1947), 212–221.

85. Al Saqabi, B. N. and Vu Kim Tuan: Solution of a fractional differintegral equation, *Integral Transforms and Special Functions*, **4** (1996), 321-326.

86. Kilbas, A. A. and M. Saigo: On Mittag-Leffler type functions, fractional calculus operators and solution of integral equations, *Integral Transforms and Special Functions*, **4** (1996), 355-370.

87. Gorenflo, R., Luchko, Yu. and S. Rogozin: Mittag-Leffler type functions : notes on growth properties and distribution of zeros, Pre-print A-04/97, Fachbereich Mathematik und Informatik, Freie Universität, Berlin 1997, available via Internet: $<$http://www.math.fu-berlin.de/publ/index.html$>$

88. Mainardi F. and R. Gorenflo: On Mittag-Leffler-type functions in fractional evolution processes, *J. Comput. and Appl. Mathematics*, **118** No 1-2 (2000), 283-299.