Heisenberg's inequality for Fourier transform

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Abstract

In this paper, we prove the Heisenberg's inequality using the Fourier transform. Then we show that the equality holds for the Gaussian and the strict inequality holds for the function $e^{-|t|}$.

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1 Fourier transform

Definition 1. Let $f \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$. The Fourier transform of $f(t)$ is defined by

$$F[f(t); \omega] = \widehat{f}(\omega) = A \int_{-\infty}^{+\infty} e^{\pm i\omega t} f(t)\, dt \qquad \omega \in \mathbb{R} \tag{1}$$

and the inverse Fourier transform by

$$f(t) = B \int_{-\infty}^{+\infty} e^{\mp i\omega t} \widehat{f}(\omega)\, d\omega \qquad t \in \mathbb{R} \tag{2}$$

where $AB = \frac{1}{2\pi}$.

Remark 1.1. Since there are different definitions of Fourier transform, in order to include most of them, in (1) and (2) we have used the symbol $\pm$ and the generic constants A and B, that can be chosen in three ways:

1. $A = 1$ and $B = \frac{1}{2\pi}$

2. $A = B = \frac{1}{\sqrt{2\pi}}$

3. $A = \frac{1}{2\pi}$ and $B = 1$.

As we will point out in the sequel, each choice of A and B is suitably adopted in order to simplify some formulas.

We recall some properties of the Fourier transform that will be useful to prove the Heisenberg's inequality.

Proposition 1.1. *If $f \in L^2(\mathbb{R})$ then $\widehat{f} \in L^2(\mathbb{R})$.*

Theorem 1.1 (Convolution)**.** *Let $f, g \in L^2(\mathbb{R})$. We define the* convolution of f and g *as*

$$(f * g)(t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau \qquad t \in \mathbb{R}$$

Then

1. $(f * g)(t) \in L^2(\mathbb{R})$

2. $F[(f * g)(t);\, \omega] = \frac{1}{A}\widehat{f}(\omega)\widehat{g}(\omega)$.

Remark 1.2. In this case we should choose $A = 1$, so the above equation become

$$F[(f * g)(t);\, \omega] = \widehat{f}(\omega)\widehat{g}(\omega).$$

The previous theorem is necessary to prove the next fundamental equality:

Theorem 1.2 (Parseval)**.** *Let $f \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$. Then*

$$A\int_{-\infty}^{+\infty} |f(t)|^2\, dt = B\int_{-\infty}^{+\infty} |\widehat{f}(\omega)|^2\, d\omega \tag{3}$$

called Parseval's formula.

Remark 1.3. In this case we should choose $A = B = \frac{1}{\sqrt{2\pi}}$, so that (3) become

$$\int_{-\infty}^{+\infty} |f(t)|^2\, dt = \int_{-\infty}^{+\infty} |\widehat{f}(\omega)|^2\, d\omega.$$

The above equation can be read as a preservation of the L^2-norm: recalling that

$$\|f\|_{L^2}^2 = \|f\|_2^2 = \int_{-\infty}^{+\infty} |f(t)|^2 dt$$

we obtain that

$$\|f\|_2 = \left\|\widehat{f}\right\|_2 .$$

Remark 1.4. As a consequence of the Parseval's formula we find that

$$\begin{aligned}\frac{A}{B}\left(\int_{-\infty}^{+\infty} |f(t)|^2\, dt\right)^2 &= \frac{A}{B}\left(\frac{B}{A}\int_{-\infty}^{+\infty} |\widehat{f}(\omega)|^2\, d\omega\right)^2 \\ &= \frac{B}{A}\left(\int_{-\infty}^{+\infty} |\widehat{f}(\omega)|^2\, d\omega\right)^2 \end{aligned} \tag{4}$$

Theorem 1.3 (Time differentiation)**.** *Let $f \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$. Then $\forall n \in \mathbb{N}$*

$$F\left[\frac{d^n f}{dt^n};\, \omega\right] = (\mp i\omega)^n\, \widehat{f}(\omega) \tag{5}$$

2 Heisenberg's inequality

The uncertainty principle is partly a description of a characteristic feature of quantum mechanical system and partly a statement about the limitations of one's ability to perform measurements on a system without disturbing it. When translated into the language of quantum mechanics, it says that the values of a pair of canonically conjugate observables such as position and momentum cannot both be precisely determined in any quantum state. On the mathematical side, when one asks for a precise quantitative formulation of the uncertainty principle, the most common response is the following inequality, usually called *Heisenberg's inequality*:

Theorem 2.1 (Heisenberg's inequality). *If f, $tf(t)$ and $\omega\widehat{f}(\omega) \in L^2(\mathbb{R})$ then*

$$\left(\int_{-\infty}^{+\infty} t^2|f(t)|^2\,dt\right)\left(\int_{-\infty}^{+\infty} \omega^2|\widehat{f}(\omega)|^2\,d\omega\right) \geq \frac{A}{4B}\left(\int_{-\infty}^{+\infty} |f(t)|^2\,dt\right)^2 \tag{6}$$

$$= \frac{B}{4A}\left(\int_{-\infty}^{+\infty} |\widehat{f}(\omega)|^2\,d\omega\right)^2 \tag{7}$$

Proof. First we observe that, since f, $tf(t)$ and $\omega\widehat{f}(\omega) \in L^2(\mathbb{R})$, then (6) is well defined. Because $f \in L^2(\mathbb{R})$, from Proposition 1.1 also $\widehat{f} \in L^2(\mathbb{R})$ and this means that (7) is well defined too.

From (4) it follows immediately (7).

Let's now prove (6). Since $\widehat{(f')}(\omega) = \mp\omega\widehat{f}(\omega)$ from (5), the finiteness of $\int |\omega\widehat{f}|^2$ implies that f is absolutely continuous and $f' \in L^2(\mathbb{R})$. This allows us to define

$$I(\lambda) = \int_{-\infty}^{+\infty} |\lambda t f(t) + f'(t)|^2\,dt \geq 0 \qquad \lambda \in \mathbb{R}. \tag{8}$$

In fact, because $tf(t)$ and $f'(t) \in L^2(\mathbb{R})$, also $tf(t) + f'(t) \in L^2(\mathbb{R})$ and this means that $I(\lambda) < \infty$, so (8) is well defined. Since $|f(t)|^2 = f(t)\overline{f(t)}$ and

$$\begin{aligned}|\lambda t f(t) + f'(t)|^2 &= [\lambda t f(t) + f'(t)]\left[\lambda t\,\overline{f(t)} + \overline{f'(t)}\right]\\ &= \lambda^2 t^2\,|f(t)|^2 + \lambda t\left[f(t)\overline{f'(t)} + f'(t)\overline{f(t)}\right] + |f'(t)|^2\end{aligned}$$

then we obtain

$$\begin{aligned}I(\lambda) = \lambda^2\int_{-\infty}^{+\infty} t^2\,|f(t)|^2\,dt + \lambda\int_{-\infty}^{+\infty} t\left[f(t)\overline{f'(t)} + f'(t)\overline{f(t)}\right]dt\\ + \int_{-\infty}^{+\infty} |f'(t)|^2\,dt\end{aligned}$$

and integrating by parts

$$\begin{aligned}I(\lambda) = \lambda^2\int_{-\infty}^{+\infty} t^2\,|f(t)|^2\,dt + \lim_{a\to+\infty}\lambda\left|t\,|f(t)|^2\right|_{-a}^{a} - \lambda\int_{-\infty}^{+\infty} |f(t)|^2\,dt\\ + \int_{-\infty}^{+\infty} |f'(t)|^2\,dt.\end{aligned}$$

Since $I(\lambda) < \infty$ and all the integrals in the above expression are convergent, then the second addend must be zero, for otherwise $|f(t)|^2$ would be comparable to t^{-1} for large t and f would not be in $L^2(\mathbb{R})$. Therefore

$$I(\lambda) = \lambda^2 \int_{-\infty}^{+\infty} t^2 |f(t)|^2 \, dt - \lambda \int_{-\infty}^{+\infty} |f(t)|^2 \, dt + \int_{-\infty}^{+\infty} |f'(t)|^2 \, dt.$$

Since $f' \in L^2(\mathbb{R})$, we can apply (3) to $f'(t)$ and, using (5) with $n = 1$, we find that

$$\begin{aligned}\int_{-\infty}^{+\infty} |f'(t)|^2 \, dt &= \frac{B}{A} \int_{-\infty}^{+\infty} \left|\widehat{(f')}(\omega)\right|^2 dt = \frac{B}{A} \int_{-\infty}^{+\infty} \left|\mp i\omega \widehat{f}(\omega)\right|^2 dt \\ &= \frac{B}{A} \int_{-\infty}^{+\infty} \omega^2 \left|\widehat{f}(\omega)\right|^2 d\omega,\end{aligned}$$

so we obtain

$$I(\lambda) = \lambda^2 \int_{-\infty}^{+\infty} t^2 |f(t)|^2 \, dt - \lambda \int_{-\infty}^{+\infty} |f(t)|^2 \, dt + \frac{B}{A} \int_{-\infty}^{+\infty} \omega^2 \left|\widehat{f}(\omega)\right|^2 d\omega$$

that is a quadratic equation in λ. Since $I(\lambda) \geq 0$ for any value of λ, its discriminant must be nonpositive:

$$\left(\int_{-\infty}^{+\infty} t^2 |f(t)|^2 \, dt\right)^2 - 4\frac{B}{A}\left(\int_{-\infty}^{+\infty} t^2 |f(t)|^2 \, dt\right)\left(\int_{-\infty}^{+\infty} \omega^2 \left|\widehat{f}(\omega)\right|^2 d\omega\right) \leq 0$$

and finally this leads to (6). □

Remark 2.1. We can rewrite the Heisenberg's inequality in terms of the L^2-norm:

$$\|tf(t)\|_2^2 \left\|\omega \widehat{f}(\omega)\right\|_2^2 \geq \frac{A}{4B} \|f\|_2^4 = \frac{B}{4A} \left\|\widehat{f}\right\|_2^4.$$

3 Examples

In this section we'll verify (6) and (7) for two special function: the Gaussian and $e^{-|t|}$.

3.1 Gaussian function

Let $f(t) = e^{-\alpha|t|^2}$ the Gaussian function; one can prove[1] that $f \in L^2(\mathbb{R})$ and

$$\widehat{f}(\omega) = A\sqrt{\frac{\pi}{\alpha}} e^{-\frac{\omega^2}{4\alpha}}.$$

[1] See Appendix A.1.

One can prove[2] that the integrals that appear in (6) are in this case

$$I_1 = \int_{-\infty}^{+\infty} |f(t)|^2 \, dt = \sqrt{\frac{\pi}{2\alpha}}$$
$$I_2 = \int_{-\infty}^{+\infty} \left|\widehat{f}(\omega)\right|^2 d\omega = A^2 \pi^{\frac{3}{2}} \sqrt{\frac{2}{\alpha}}$$
$$I_3 = \int_{-\infty}^{+\infty} t^2 \, |f(t)|^2 \, dt = \frac{\sqrt{\pi}}{2} (2\alpha)^{-\frac{3}{2}}$$
$$I_4 = \int_{-\infty}^{+\infty} \omega^2 \left|\widehat{f}(\omega)\right|^2 d\omega = A^2 \pi^{\frac{3}{2}} \sqrt{2\alpha}$$

Moreover, using $B = \frac{1}{2\pi A}$, we obtain

$$\frac{A}{4B}(I_1)^2 = \frac{A}{4B}\frac{\pi}{2\alpha} = \frac{A^2\pi^2}{4\alpha}$$

therefore

$$I_3 I_4 = \left[\frac{\sqrt{\pi}}{2}(2\alpha)^{-\frac{3}{2}}\right]\left[A^2\pi^{\frac{3}{2}}\sqrt{2\alpha}\right] = \frac{A^2\pi^2}{4\alpha} = \frac{A}{4B}(I_1)^2$$

so we have proved that (6) is an equality for the Gaussian. Furthermore, since

$$\frac{B}{4A}(I_2)^2 = \frac{B}{4A}A^4\pi^3\frac{2}{\alpha} = \frac{A^2\pi^2}{4\alpha} = \frac{A}{4B}(I_1)^2$$

we have shown that also (7) holds for the Gaussian.

3.2 Exponential function

Let $f(t) = e^{-|t|}$ an exponential function; one can prove[3] that $f \in L^2(\mathbb{R})$ and

$$\widehat{f}(\omega) = \frac{2A}{1+\omega^2}.$$

The integrals that appear in (6) are in this case[4]

$$I_1 = \int_{-\infty}^{+\infty} |f(t)|^2 \, dt = 1$$
$$I_2 = \int_{-\infty}^{+\infty} \left|\widehat{f}(\omega)\right|^2 d\omega = 2\pi A^2$$
$$I_3 = \int_{-\infty}^{+\infty} t^2 \, |f(t)|^2 \, dt = \frac{1}{2}$$
$$I_4 = \int_{-\infty}^{+\infty} \omega^2 \left|\widehat{f}(\omega)\right|^2 d\omega = 2\pi A^2.$$

[2]See Appendix A.1.
[3]See Appendix A.2.
[4]See Appendix A.2.

Then we find that

$$I_3 I_4 = \frac{1}{2} 2\pi A^2 = \pi A^2 \tag{9}$$

$$\frac{A}{4B}(I_1)^2 = \frac{A}{4B} = \frac{2\pi A^2}{4} = \frac{\pi A^2}{2} \tag{10}$$

$$\frac{B}{4A}(I_2)^2 = \frac{B}{4A}(2\pi A^2)^2 = \pi^2 A^3 B = \frac{\pi^2 A^3}{2\pi A} = \frac{\pi A^2}{2} \tag{11}$$

so we have proved that for the function $e^{-|t|}$ the left hand side in (6) is exactly twice the right hand side and this means that in this case (6) is a strict inequality. Finally, because we have shown

$$\frac{A}{4B}(I_1)^2 = \frac{\pi A^2}{2} = \frac{B}{4A}(I_2)^2$$

then (7) holds for the exponential.

A Appendix

In this appendix we give full proof of the results presented in the two previous examples.

A.1 Gaussian function

Let $f(t) = e^{-\alpha|t|^2}$ the Gaussian function, with $\alpha > 0$.

Lemma A.1. *The Gauss integral is*

$$\int_{-\infty}^{+\infty} e^{-\alpha|t|^2} dt = \sqrt{\frac{\pi}{\alpha}} \tag{12}$$

Proposition A.1. *The following properties are valid: $f \in L^2(\mathbb{R})$ and*

$$\widehat{f}(\omega) = A\sqrt{\frac{\pi}{\alpha}} e^{-\frac{\omega^2}{4\alpha}}. \tag{13}$$

Proof. First we have to show that $\int_{-\infty}^{+\infty} |f(t)|^2 dt < \infty$, i.e. $f \in L^2(\mathbb{R})$:

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \int_{-\infty}^{+\infty} |e^{-\alpha|t|^2}|^2 dt = \int_{-\infty}^{+\infty} e^{-2\alpha|t|^2} dt$$
$$= \sqrt{\frac{\pi}{2\alpha}} < \infty,$$

where in the last equality we have applied (12).

Since $f \in L^2(\mathbb{R})$, we can verify (13). Let $\omega \in \mathbb{R}$, then

$$\begin{aligned}
\widehat{f}(\omega) &= A \int_{-\infty}^{+\infty} e^{\pm i\omega t} e^{-\alpha |t|^2}\, dt = A \int_{-\infty}^{+\infty} e^{-\alpha\left(t^2 \mp \frac{i\omega}{\alpha} t\right)}\, dt \\
&= A \int_{-\infty}^{+\infty} e^{-\alpha\left[t^2 \mp \frac{i\omega}{\alpha} t + \left(\frac{i\omega}{2\alpha}\right)^2\right] + \alpha\left(\frac{i\omega}{2\alpha}\right)^2}\, dt \\
&= A \int_{-\infty}^{+\infty} e^{-\alpha\left(t \mp \frac{i\omega}{2\alpha}\right)^2 - \frac{\omega^2}{4\alpha}}\, dt \\
&= A e^{-\frac{\omega^2}{4\alpha}} \int_{-\infty}^{+\infty} e^{-\alpha t^2}\, dt \\
&= A \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\omega^2}{4\alpha}}
\end{aligned}$$

□

Proposition A.2.

$$I_1 = \int_{-\infty}^{+\infty} |f(t)|^2\, dt = \sqrt{\frac{\pi}{2\alpha}} \tag{14}$$

$$I_2 = \int_{-\infty}^{+\infty} \left|\widehat{f}(\omega)\right|^2 d\omega = A^2 \pi^{\frac{3}{2}} \sqrt{\frac{2}{\alpha}} \tag{15}$$

$$I_3 = \int_{-\infty}^{+\infty} t^2 |f(t)|^2\, dt = \frac{\sqrt{\pi}}{2} (2\alpha)^{-\frac{3}{2}} \tag{16}$$

$$I_4 = \int_{-\infty}^{+\infty} \omega^2 \left|\widehat{f}(\omega)\right|^2 d\omega = A^2 \pi^{\frac{3}{2}} \sqrt{2\alpha} \tag{17}$$

Proof. We have already proved (14) in Proposition A.1.

Let's show (15):

$$\begin{aligned}
I_2 &= \int_{-\infty}^{+\infty} \left|\widehat{f}(\omega)\right|^2 d\omega \\
&= \int_{-\infty}^{+\infty} \left| A \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\omega^2}{4\alpha}} \right|^2 d\omega \\
&= A^2 \frac{\pi}{\alpha} \int_{-\infty}^{+\infty} e^{-\frac{\omega^2}{2\alpha}}\, d\omega \\
&= A^2 \frac{\pi}{\alpha} \sqrt{2\pi\alpha} \\
&= A^2 \pi^{\frac{3}{2}} \sqrt{\frac{2}{\alpha}}.
\end{aligned}$$

Let's show (16):

$$\begin{aligned}
I_3 &= \int_{-\infty}^{+\infty} t^2 |f(t)|^2 dt \\
&= \int_{-\infty}^{+\infty} t^2 e^{-2\alpha t^2} dt \\
&= -\frac{1}{4\alpha} \int_{-\infty}^{+\infty} t \left(-4\alpha t e^{-2\alpha t^2}\right) dt \\
&= -\frac{1}{4\alpha} \left[\left| t e^{-2\alpha t^2} \right|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} e^{-2\alpha t^2} dt \right] \\
&= \left(-\frac{1}{4\alpha}\right) \left(0 - \sqrt{\frac{\pi}{2\alpha}}\right) \\
&= \frac{\sqrt{\pi}}{2} (2\alpha)^{-\frac{3}{2}} .
\end{aligned}$$

Finally we prove (17):

$$\begin{aligned}
I_4 &= \int_{-\infty}^{+\infty} \omega^2 \left| \widehat{f}(\omega) \right|^2 d\omega \\
&= \int_{-\infty}^{+\infty} \omega^2 \left| A \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\omega^2}{4\alpha}} \right|^2 d\omega \\
&= A^2 \frac{\pi}{\alpha} \int_{-\infty}^{+\infty} \omega^2 e^{-\frac{\omega^2}{2\alpha}} d\omega \\
&= A^2 \frac{\pi}{\alpha} \frac{\sqrt{\pi}}{2} (2\alpha)^{\frac{3}{2}} \\
&= A^2 \pi^{\frac{3}{2}} \sqrt{2\alpha}.
\end{aligned}$$

□

A.2 Exponential function

Let $f(t) = e^{-|t|}$ an exponential function.

Proposition A.3. *The function $f \in L^2(\mathbb{R})$ and for $\omega \in \mathbb{R}$*

$$\widehat{f}(\omega) = \frac{2A}{1+\omega^2} \tag{18}$$

Proof. First we have to show that $\int_{-\infty}^{+\infty} |f(t)|^2 dt < \infty$, i.e. $f \in L^2(\mathbb{R})$:

$$\begin{aligned}
\int_{-\infty}^{+\infty} |f(t)|^2 dt = \int_{-\infty}^{+\infty} |e^{-|t|}|^2 dt &= \int_{-\infty}^{+\infty} e^{-2|t|} dt \\
&= 2 \int_{0}^{+\infty} e^{-2t} dt = 2 \left| \frac{e^{-2t}}{-2} \right|_{0}^{+\infty} \\
&= 2[0 + \frac{1}{2}] = 1 < \infty.
\end{aligned}$$

Since $f \in L^2(\mathbb{R})$, we can verify (18). Let $\omega \in \mathbb{R}$, then

$$\begin{aligned}
\widehat{f}(\omega) &= A \int_{-\infty}^{+\infty} e^{\pm i\omega t} e^{-|t|}\, dt \\
&= A \int_{-\infty}^{0} e^{\pm i\omega t + t}\, dt + A \int_{0}^{+\infty} e^{\pm i\omega t - t}\, dt \\
&= A \int_{-\infty}^{0} e^{t(\pm i\omega + 1)}\, dt + A \int_{0}^{+\infty} e^{-t(\mp i\omega + 1)}\, dt \\
&= A \left| \frac{e^{t(\pm i\omega+1)}}{\pm i\omega + 1} \right|_{-\infty}^{0} + A \left| \frac{e^{-t(\mp i\omega+1)}}{\pm i\omega - 1} \right|_{0}^{+\infty} \\
&= A \frac{1}{\pm i\omega + 1} - A \frac{1}{\pm i\omega - 1} \\
&= A \frac{\pm i\omega - 1 - (\pm i\omega + 1)}{-\omega^2 - 1} \\
&= A \frac{-2}{-\omega^2 - 1} \\
&= \frac{2A}{1 + \omega^2}
\end{aligned}$$

□

Proposition A.4.

$$I_1 = \int_{-\infty}^{+\infty} |f(t)|^2\, dt = 1 \tag{19}$$

$$I_2 = \int_{-\infty}^{+\infty} \left|\widehat{f}(\omega)\right|^2 d\omega = 2\pi A^2 \tag{20}$$

$$I_3 = \int_{-\infty}^{+\infty} t^2 |f(t)|^2\, dt = \frac{1}{2} \tag{21}$$

$$I_4 = \int_{-\infty}^{+\infty} \omega^2 \left|\widehat{f}(\omega)\right|^2 d\omega = 2\pi A^2. \tag{22}$$

Proof. We have already proved (19) in Proposition A.3.

Let's show first (21).

$$\begin{aligned}
I_3 &= \int_{-\infty}^{+\infty} t^2 |f(t)|^2 dt = \int_{-\infty}^{+\infty} t^2 e^{-2|t|} dt \\
&= 2 \int_{0}^{+\infty} t^2 e^{-2t} dt = \left|-t^2 e^{-2t}\right|_0^{+\infty} + \int_0^{+\infty} 2t e^{-2t} dt \\
&= 0 + \left|-t e^{-2t}\right|_0^{+\infty} + \int_0^{+\infty} e^{-2t} dt \\
&= 0 + \left| -\frac{e^{-2t}}{2} \right|_0^{+\infty} \\
&= \frac{1}{2}
\end{aligned}$$

We prove (20) using a consequence of the residue theorem.

Proposition A.5. *Assume f is an analytic function on $\mathbb{C}$ with a finite number of poles. Considering only the poles z_1, z_2, ..., z_n that are in the upper half plane, then*

$$\int_{-\infty}^{+\infty} f(x)dx = 2\pi i \sum_{j=1}^{n} Res[f; z_j] \tag{23}$$

where $Res[f; z_j]$ is the residue of f in z_j. Recall that, if z_j is a pole of order m for the function f, then

$$Res[f; z_j] = \lim_{x \to z_j} \frac{1}{(m-1)!} \frac{d^{m-1}}{dx^{m-1}} [(x - z_j)^m f(x)] \tag{24}$$

Defining

$$g(\omega) = \left|\widehat{f}(\omega)\right|^2 = \frac{4A^2}{(1+\omega^2)^2} = \frac{4A^2}{(\omega+i)^2(\omega-i)^2},$$

we observe that there are two poles $z_1 = i$ and $z_2 = -i$ of order 2. For Theorem A.5 we can consider only $z_1 \in \mathbb{C}^+$. Let's compute $Res[g; i]$, using (24):

$$\begin{aligned}
Res[g; i] &= \lim_{\omega \to i} \frac{d}{d\omega} [(\omega - i)^2 \frac{4A^2}{(\omega+i)^2(\omega-i)^2}] \\
&= 4A^2 \lim_{\omega \to i} \frac{d}{d\omega} [\frac{1}{(\omega+i)^2}] \\
&= 4A^2 \lim_{\omega \to i} \left[-2(\omega+i)^{-3}\right] \\
&= 4A^2[-2(2i)^{-3}] = -A^2 i^{-3} = -iA^2
\end{aligned}$$

so we have proved that $Res[g; i] = -iA^2$. Using this result and (23) we obtain

$$\begin{aligned}
I_2 &= \int_{-\infty}^{+\infty} \left|\widehat{f}(\omega)\right|^2 d\omega \\
&= \int_{-\infty}^{+\infty} \left|\frac{2A}{1+\omega^2}\right|^2 d\omega \\
&= \int_{-\infty}^{+\infty} \frac{4A^2}{(1+\omega^2)^2} d\omega \\
&= 2\pi i \, Res[g; i] = -2\pi A^2 i^2 = 2\pi A^2
\end{aligned}$$

Finally we prove (22) imitating the proof of (20). Defining

$$h(\omega) = \omega^2 \left|\widehat{f}(\omega)\right|^2 = \frac{4A^2\omega^2}{(\omega+i)^2(\omega-i)^2},$$

we observe that there are two poles $z_1 = i$ and $z_2 = -i$ of order 2. For Theorem

A.5 we can consider only $z_1 \in \mathbb{C}^+$. Let's compute $Res[h;i]$:

$$\begin{aligned}
Res[h;i] &= \lim_{\omega\to i} \frac{d}{d\omega}\left[(\omega-i)^2 \frac{4A^2\omega^2}{(\omega+i)^2(\omega-i)^2}\right] \\
&= 4A^2 \lim_{\omega\to i} \frac{d}{d\omega}\left[\frac{\omega^2}{(\omega+i)^2}\right] \\
&= 4A^2 \lim_{\omega\to i} \frac{2\omega(\omega+i)^2-\omega^2 2(\omega+i)}{(\omega+i)^4} \\
&= 4A^2 \lim_{\omega\to i} \frac{2\omega(\omega+i)\left[\omega+i-\omega\right]}{(\omega+i)^4} \\
&= 4A^2 \lim_{\omega\to i} \frac{2i\omega}{(\omega+i)^3} \\
&= 4A^2 \frac{2i^2}{(2i)^3} = -\frac{4iA^2}{4} = -iA^2
\end{aligned}$$

so we have proved that $Res[h;i] = -iA^2$. Using this result and (23) we obtain

$$\begin{aligned}
I_4 &= \int_{-\infty}^{+\infty} \left|\widehat{f}(\omega)\right|^2 d\omega \\
&= \int_{-\infty}^{+\infty} \omega^2 \left|\frac{2A}{1+\omega^2}\right|^2 d\omega \\
&= \int_{-\infty}^{+\infty} \omega^2 \frac{4A^2}{(1+\omega^2)^2} d\omega \\
&= 2\pi i\, Res[h;i] = -2\pi A^2 i^2 = 2\pi A^2
\end{aligned}$$

and this conclude our proof. □

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