Fractional Partial Differential Equation:
Introduction and Model

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April 12, 2018

- Division of Applied Mathematics, Brown University
- The OSD/ARO MURI Grant W911NF-15-1-0562 and the National Science Foundation under Grant DMS-1620194

- Diffusion describes the random movement of tracer particles from high concentration to low concentration.
- Two fundamental approaches were used to model diffusion.
- A deterministic/macroscopic description via a second-order diffusion PDE for the PDF of particle movements:
  - Fick sat up the diffusion equation (1855) when studying how nutrients travel through membranes in living organisms, by mimicking the heat conduction equation of Fourier (1822).
  - Einstein derived the diffusion equation from first principles as part of his work on Brownian motion (1905).
- A stochastic/microscopic description via random walk of particles
  - Brown observed and investigated irregular movement of small pollen grain under a microscope (1827).
  - Pearson modeled a diffusion process in terms of random walk, when he studied on how mosquitoes spread malaria (1905).
  - Bachelier used a Brownian motion to model asset prices (1900).

- The common assumptions of Einstein and Pearson
    - the existence of a mean free path,
    - the existence of a mean waiting time to perform a jump.
- Under these approximations
    - Pearson's approches of random walk yields Brownian motion, which then leads to stochastic differential equation and is suited for a microscopic description of diffusive transport.
    - Einstein's derivation yields a Fickian diffusion equation, which can be viewed as a Fokker-Planck equation of Brownian motion.

- Let X be a random variable, $F(x) = \mathbb{P}[X \le x]$ and $p(x) = F'(x)$ be its CDF and PDF, respectively, and $\mu_q = \mathbb{E}[X^q]$ be its qth moment.
- The common assumptions of Einstein and Pearson state the variance and mean waiting time of a randomly selected particle's motion are finite.
- If $\mu_1 = 0, \mu_2 = \sigma^2 < \infty$, the FT $\hat{p}$ has the expansion

$$\begin{aligned} \hat{p}(k) &= \mathbb{E}[e^{-ikX}] = \int_{\mathbb{R}} e^{-ikx} p(x)dx = 1 - ik\mu_1 k - \mu_2 k^2/2 + o(k^2) \\ &= 1 - \sigma^2 k^2/2 + o(k^2), \qquad k \to 0. \end{aligned} \tag{1}$$

- Let $X_1, X_2, \ldots$ be a sequence of iid random variables that represent the random jumps of a randomly selected particle with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = \sigma^2$.
- Lévy's continuity theorem $\Longrightarrow$ the particle's location $S_n := X_1 + \ldots + X_n$ satisfies

$$\begin{aligned} \mathbb{E}\big[e^{-ik(S_n/\sqrt{n})}\big] &= \prod_{j=1}^{n} \mathbb{E}\big[e^{-i(n^{-1/2}k)X_j}\big] = \Big[1 - \frac{\sigma^2 k^2}{2n} + o\Big(\frac{1}{n}\Big)\Big]^n \\ &\longrightarrow e^{-\frac{\sigma^2 k^2}{2}} = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} e^{-ikx} dx = \mathbb{E}\big[e^{-ikZ}\big], \\ &S_n/\sqrt{n} \Rightarrow Z \sim N(0, \sigma^2), \quad \text{as } n \to \infty. \end{aligned} \tag{2}$$

- For any fixed time $t > 0$ and $c >> 1$, the rescaled random walk

$$S_{\lfloor ct \rfloor} := X_1 + \ldots + X_{\lfloor ct \rfloor} \tag{3}$$

gives the particle location at time $t > 0$ after $\lfloor ct \rfloor$ jumps.

- As the jump size is reduced by $1/\sqrt{c}$, the normalized partition location $S_{\lfloor ct \rfloor}/\sqrt{c}$ satisfies

$$\begin{aligned} \mathbb{E}\big[e^{-ik(S_{\lfloor ct \rfloor}/\sqrt{c})}\big] &= \Big[1 - \frac{\sigma^2 k^2}{2c} + o\Big(\frac{1}{c}\Big)\Big]^{\lfloor ct \rfloor} \to e^{-\frac{t\sigma^2 k^2}{2}} \\ &= \mathbb{E}\big[e^{-ikZ_t}\big] =: \hat{p}(k,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{x^2}{2\sigma^2 t}}, \\ S_{\lfloor ct \rfloor}/\sqrt{c} &\Rightarrow Z_t \end{aligned} \tag{4}$$

by Lévy continuity theorem. Here $Z_t \sim N(0, \sigma^2 t)$ is a Brownian motion.

- Z_t can be written as an Ito type stochastic differential equation, which gives a microscopic description of diffusion (Pearson's approach)

$$dZ_t = \mu dt + \sigma dB_t. \tag{5}$$

where $\mu = 0$ and $B_t \sim N(0, t)$ is the standard Brownian motion.

- Let $\hat{p}(k,t) := e^{-\frac{t\sigma^2 k^2}{2}}$ be the FT of of the PDF $p(x,t)$, which satisfies

$$\frac{\partial \hat{p}}{\partial t} = -\frac{\sigma^2}{2} k^2 \hat{p} = \frac{\sigma^2}{2} (ik)^2 \hat{p} = \frac{\sigma^2}{2} \widehat{\frac{\partial^2 p}{\partial x^2}} \quad \longrightarrow \quad \frac{\partial p}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 p}{\partial x^2}. \tag{6}$$

- This relates the dispersivity K to the particle jump variance σ^2.
- The PDF satisfies a Fickian diffusion equation (as the Fokker-Planck equation of the SDE), which decays exponentially.
- The equivalence between the PDE description and the stochastic formulation also has mathematical and numerical impact
  - One can solve a diffusion PDE (the Fokker-Planck equation) to find the PDF $p(x,t)$ of the underlying stochastic process.
  - One can also use a particle tracking method to numerically solve a diffusion PDE by simulating the underlying stochastic process.

- For $\mu \neq 0$, the stochastic process $\mu t + Z_t$ satisfies the Ito SDE (5). Moreover, it has FT

$$\mathbb{E}\Big[e^{-ik(\mu t+Z_t)}\Big] = e^{-ik\mu t-\frac{t\sigma^2 k^2}{2}} =: \hat{p}(k,t), \tag{7}$$

which solves

$$\frac{\partial \hat{p}}{\partial t} = \Big(-i\mu k - \frac{\sigma^2}{2}k^2\Big)\hat{p} = -\mu\widehat{\frac{\partial p}{\partial x}} + \frac{\sigma^2}{2}\widehat{\frac{\partial^2 p}{\partial x^2}}, \tag{8}$$

- which inverts to an advection-diffusion equation

$$\frac{\partial p}{\partial t} + \mu\frac{\partial p}{\partial x} - \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} = 0 \tag{9}$$

as the Fokker-Plank PDE of the SDE (5).

- Derivation of the classical conservation law
  - Let $c(x,t)$ be the concentration of a solute and $q(x,t)$ be the flux. In a small cube of side δx with the cross-sectional area $A = (\delta x)^2$, the mass change δM and δc over time δt are

$$\begin{aligned} \delta M(x,t) &= q(x - \delta x/2, t) A\delta t - q(x + \delta x/2, t) A\delta t = -\delta q(x,t) A\delta t, \\ \delta c(x,t) &= \frac{\delta M(x,t)}{A\delta x} = -\frac{\delta q(x,t)\delta t}{\delta x}. \end{aligned} \tag{10}$$

  - Taking the limit as $\delta x, \delta t \to 0^+$ yields a mass conservaltion law

$$\frac{\delta c(x,t)}{\delta t} = -\frac{\delta q(x,t)}{\delta x} \Longrightarrow \frac{\partial c}{\partial t} = -\frac{\partial q}{\partial x} \tag{11}$$

- The classical Fick's law

$$q(x,t) = -K\frac{\partial c(x,t)}{\partial x} \tag{12}$$

  - assumes the particles jump locally to the left and right neighboring cells with equal probability

$$q(x,t) \approx -K\frac{\delta c(x,t)}{\delta x} = -K\frac{c(x+\delta x/2,t) - c(x-\delta x/2,t)}{\delta x}. \tag{13}$$

  - The form (12) is clear for a constant K and is assumed for a variable K.

- Inserting Fick's law into (11) yields the classical Fickian diffusion PDE

$$\frac{\partial c(x,t)}{\partial t} = \frac{\partial}{\partial x}\Big(K\frac{\partial c(x,t)}{\partial x}\Big) = K\frac{\partial^2 c(x,t)}{\partial x^2}. \tag{14}$$

The second equal sign holds for a constant K. (14) is self-adjoint.

- Consider the Ito SDE with variable drift μ and volatility σ

$$dZ_t = \mu(t, Z_t)dt + \sigma(t, Z_t)dB_t. \tag{15}$$

- For any smooth and rapidly decaying $f(x)$, Ito's Lemma states $Y_t = f(Z_t)$ satisfies

$$dY_t = f'(Z_t)dZ_t + \frac{1}{2}f''(Z_t)dZ_t^2 = \big(f'(Z_t)\mu + f''(Z_t)\sigma^2/2\big)dt + f'(Z_t)\sigma dB_t \tag{16}$$

- Integrating (16) on any time interval $[a, b]$ gives

$$Y_b - Y_a = f(Z_b) - f(Z_a) = \int_a^b \big(f'(Z_t)\mu + f''(Z_t)\sigma^2/2\big)dt + \int_a^b f'(Z_t)\sigma dB_t. \tag{17}$$

- Taking the expectation of (17) (recall $\mathbb{E}(B_t) = 0$) yields

$$\begin{aligned}\mathbb{E}[f(Z_b) - f(Z_a)] &= \int_{\mathbb{R}} f(x)\big[p(x,b) - p(a,t)\big]dt = \int_a^b \int_{\mathbb{R}} f(x)\frac{\partial p(x,t)}{\partial t}dxdt \\ &= \int_a^b \int_{\mathbb{R}} \big(f'(x)\mu(x,t) + f''(x)\sigma^2(x,t)/2\big)p(x,t)dxdt.\end{aligned} \tag{18}$$

- Integrating the terms on the right-hand side by parts and using the fact that f is arbitraty to get the Fokker-Planck PDE

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}\Big(\mu(x,t)p(x,t)\Big) - \frac{\partial^2}{\partial x^2}\Big(\frac{\sigma^2}{2}p(x,t)\Big) = 0. \tag{19}$$

- For variable drift and volatility, the governing PDE is in a conservative form. Retaining conservation is of crucial importance in many applications (e.g., subsurface porous medium flow and transport, especially when the problem has high uncertainty).
- The variable σ is in the different place from that in the Fickian diffusion PDE.

- "Prehistorical development"
  - Fractional calculus stemmed from a question by L'Hopital (1695) to Leibniz on the meaning of $\frac{d^n y}{dx^n}$ for $n = 1/2$. Leibniz's reply (Sept 30 1695): "... This is an apparent paradox from which, one day, useful consequences will be drawn. ..."
  - Euler observed that the differentiation formula

$$\frac{d^n x^\alpha}{dx^n} = \alpha(\alpha-1)\cdots(\alpha-n+1)x^{\alpha-n} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}x^{\alpha-n}$$

    has a meaning for non-integer n (1738).
  - Laplace proposed the idea of non-integer order differentiation by means of an integral (1812).
  - Fourier suggested some integral representation of fractional differentiation (1822).

- Fractional calculus really began with Abel and Liouville
  - Abel solved the integral equation (1823)

$$\int_a^x \phi(t)(x-t)^{-\mu} dt = f(x), \quad x > a,\ 0 < \mu < 1$$

  - Liouville made the major contribution to the theory (1832-1837)

$$D^\alpha f(x) = \sum_{k=0}^{\infty} c_k a_k^\alpha e^{a_k x}, \quad \text{for } f(x) = \sum_{k=0}^{\infty} c_k e^{a_k x}.$$

    He proposed $D^\alpha f(x) := \lim_{h\to 0} \frac{(\Delta_h^\alpha f)(x)}{h^\alpha}$ but didn't pursue it.
  - Riemann came up with today's fractional integration formula in 1847 (when still a student, but published in 1876 ten years after his death).
  - Grünwald (1867) and Letnikov (1868) introduced the definition

$$D^\alpha f(x) := \lim_{h\to 0} \frac{(\Delta_h^\alpha f)(x)}{h^\alpha}.$$

    Letnikov proved that this definition coincides with Riemann's.

- For any $n \in \mathbb{N}$, the iterated integrals can be expressed by

$$
\begin{aligned}
{}_aI_x^1 f(x) &:= \int_a^x f(y)dy, \qquad {}_aI_x^2 f(x) := \int_a^x \big({}_aI_z^1 f\big)(z)dz \\
&= \int_a^x \int_a^z f(y)dydz = \int_a^x \int_y^x f(y)dzdy = \int_a^x (x-y)f(y)dy, \cdots \\
{}_aI_x^n f(x) &:= \int_a^x \big({}_aI_z^{n-1} f\big)(z)dz = \int_a^x \int_a^z \frac{(z-y)^{n-2}}{(n-2)!} f(y)dydz \\
&= \int_a^x \int_y^x \frac{(z-y)^{n-2}}{(n-2)!} f(y)dzdy = \frac{1}{\Gamma(n)} \int_a^x (x-y)^{n-1} f(y)dy.
\end{aligned}
$$

Here the Gamma function $\Gamma(\beta) := \int_0^\infty e^{-t}t^{\beta-1}dt$ and $\Gamma(n) = (n-1)!$.

- For any $\beta \in \mathbb{R}^+$, define the left and right fractional integrals as

$$
\begin{aligned}
{}_aI_x^\beta f(x) &:= \frac{1}{\Gamma(\beta)} \int_a^x (x-y)^{\beta-1} f(y)dy, \\
{}_xI_b^\beta f(x) &:= \frac{1}{\Gamma(\beta)} \int_x^b (y-x)^{\beta-1} f(y)dy.
\end{aligned} \tag{20}
$$

- The Riemann-Liouville fractional derivatives of order $\alpha = n - \beta$, $0 < \beta < 1$

$$\begin{aligned} {}_a^{RL}D_x^\alpha f(x) &:= D^n {}_aI_x^\beta f(x) = \frac{1}{\Gamma(\beta)} \frac{d^n}{dx^n} \int_a^x (x-y)^{\beta-1} f(y)dy, \\ {}_x^{RL}D_b^\alpha f(x) &:= (-1)^n D^n {}_xI_b^\beta f(x) = \frac{(-1)^n}{\Gamma(\beta)} \frac{d^n}{dx^n} \int_x^b (y-x)^{\beta-1} f(y)dy. \end{aligned} \tag{21}$$

- The Caputo fractional derivatives of order $\alpha = n - \beta$, $0 < \beta < 1$

$$\begin{aligned} {}_a^{C}D_x^\alpha f(x) &:= {}_aI_x^\beta D^n f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-y)^{\beta-1} f^{(n)}(y)dy, \\ {}_x^{C}D_b^\alpha f(x) &:= (-1)^n {}_xI_b^\beta D^n f(x) = \frac{(-1)^n}{\Gamma(\beta)} \int_x^b (y-x)^{\beta-1} f^{(n)}(y)dy. \end{aligned} \tag{22}$$

- Fractional derivatives defined via Fourier transform

$${}_{-\infty}D_x^\alpha f(x) := \mathcal{F}^{-1}\big[(ik)^\alpha \hat{f}(k)\big], \quad {}_xD_\infty^\alpha f(x) := \mathcal{F}^{-1}\big[(-ik)^\alpha \hat{f}(k)\big]. \tag{23}$$

- Integer-order derivatives can be expressed as limit of difference quotients

$$
\begin{aligned}
f'(x) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(I-B_\varepsilon)f(x) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big[f(x)-f(x-\varepsilon)\big] \\
f^{(n)}(x) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}(I-B_\varepsilon)^n f(x) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\sum_{k=0}^{n} g_k^{(n)} f(x-k\varepsilon).
\end{aligned}
\tag{24}
$$

  with $g_k^{(n)} := (-1)^k\binom{n}{k}$ being the binormial coefficients.
  - The n in ε^n and $\binom{n}{k}$ in (24) counts for the order of the derivative.
  - The n in $\sum_{k=0}^{n}$ in (24) counts for the number of summands.
- If we replace n in the former by α and n in the latter by the number of summands to the left boundary $x=a$, we obtain the definition of the Grünwald-Letnikov fractional derivatives of order α

$$
\begin{aligned}
{}^{GL}_{a}D^{\alpha}_{x} f(x) &:= \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^\alpha}\sum_{k=0}^{\lfloor (x-a)/\varepsilon\rfloor} g_k^{(\alpha)} f(x-k\varepsilon), \\
{}^{GL}_{x}D^{\alpha}_{b} f(x) &:= \lim_{\varepsilon\to 0^+}\frac{(-1)^{\lceil\alpha\rceil}}{\varepsilon^\alpha}\sum_{k=0}^{\lfloor (b-x)/\varepsilon\rfloor} g_k^{(\alpha)} f(x+k\varepsilon).
\end{aligned}
\tag{25}
$$

- Under appropriate smoothness assumptions, the Riemann-Liouville fractional derivatives and Grünwald-Letnikov fractional derivatives coincide

$$ {}_{a}^{GL}D_x^\alpha f(x) = {}_{a}^{RL}D_x^\alpha f(x), \quad {}_{x}^{GL}D_b^\alpha f(x) = {}_{x}^{RL}D_b^\alpha f(x). \tag{26} $$

- The Riemann-Liouville fractional derivatives and the Caputo fractional derivatives differ by singular boundary terms. For example, for $0 < \alpha < 1$,

$$ {}_{0}^{C}D_x^\alpha f(x) = {}_{0}^{RL}D_x^\alpha f(x) - \frac{f(0)x^{-\alpha}}{\Gamma(1-\alpha)}. \tag{27} $$

- All the three fractional derivatives (with $a = -\infty$ and $b = \infty$) coincide for rapidly decaying f on $\mathbb{R}$ and equal to those defined by Fourier transforms (Multidimensional cases much subtle).

- Example: Let $f(x) = 1$ for $x > 0$ Then for $0 < \alpha < 1$, ${}_0^C D_x^\alpha f(x) = 0$ but

$$\begin{aligned} {}_0^{RL} D_x^\alpha f(x) &:= D_0 I_x^{1-\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-y)^{-\alpha} dy \\ &= \frac{d}{dx} \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \neq 0. \end{aligned} \tag{28}$$

  This is consistent with (27).

- Let $\tilde{f}(s) \equiv \mathcal{L}[f](s) := \int_0^\infty e^{-st} f(t) dt$ be the LT of f. It is shown that

$$\mathcal{L}\big[{}_0^C D_t^\alpha f(t)\big] = s^\alpha \tilde{f}(s) - s^{\alpha-1} f(0), \quad \mathcal{L}\big[{}_0^{RL} D_t^\alpha f(t)\big] = s^\alpha \tilde{f}(s), \quad 0 < \alpha < 1. \tag{29}$$

  $\mathcal{L}\big[{}_0^C D_t^\alpha f(t)\big]$ resembles that of f', and has been used in time FPDE.

- It was found that the dispersive transport of electrons in operation of photocopiers and laser printers could not be modeled properly by the classical Fickian diffusion PDE (Scher & Montroll 1975).
  - Charges moving in media get trapped by local imperfections and then get released due to thermal fluctuations.
- In groundwater contaminant transport, remediation is not so effective as predicted by the integer-order advection-diffusion PDEs
  - The contaminant in groundwater gets trapped to low peameability zone and gets released when the contaminant is cleaned.
- Einstein and Pearson's assumptions are violated in these processes
  - These assumptions hold for homogeneous medium,
  - but fail for heterogeneous medium.

- The current modeling of transport process in heterogeneous media is
  - to use integer-order PDEs (valid for homogeneous medium),
  - to tweak free parameters that multiply pre-set integer-order PDEs.
- Field tests show that
  - contaminant plumes often exhibit a power-law decaying tail in heterogeneous media,
  - integer-order PDE model, characterized by an exponentially decaying tail, struggles a variable coefficient fit (of the data at each location),
  - FPDE model, characterized by a power-law decaying tail, can fit all the data with a constant coefficient.
- Many anomalous diffusion processes were found in various disciplines
  - signaling of biological cells, anomalous electrodiffusion in nerve cells
  - foraging behavior of animals, electrochemistry, physics, finance
  - fluid and continuum mechanics, viscoelastic and viscoplastic flow

- A fractional Fick's law assumes that the underlying particles have global jumps, i.e., $\delta c(x,t)$ is increased by an amount of $g_k^{(\alpha-1)} c(x - k\delta x, t)$, i.e.,

$$q(x,t) = -K \frac{\partial^{\alpha-1} c(x,t)}{\partial x^{\alpha-1}}, \quad 1 < \alpha < 2. \tag{30}$$

Here the fractional derivatives are Grünwald-Letnikov type. Since $g_k^{(\alpha-1)}$ decay like $O(k^{-\alpha})$, the particle jumps have a heavy tail.

- Inserting (30) into (11) yields a space FPDE

$$\frac{\partial c}{\partial t} = -\frac{\partial q}{\partial x} = \frac{\partial}{\partial x}\Big(K \frac{\partial^{\alpha-1} c(x,t)}{\partial x^{\alpha-1}}\Big) = K \frac{\partial^{\alpha} c(x,t)}{\partial x^{\alpha}}. \tag{31}$$

The second equal sign holds for a constant K.

- In anomalous diffusion a particle's motion may have very different waiting times/jump sizes. Classical random walk does not apply.
  - Let $X_1, X_2, \ldots$ be a sequence of iid random jumps of a particle with $\mathbb{E}[X_i] = 0$ and $\mathbb{P}[X > x] = Cx^{-\alpha}$ where $C > 0$ and $1 < \alpha < 2$.
  - $\mathbb{E}[X^p] = \alpha/(\alpha - p)$ for $0 < p < \alpha$ or ∞ for $p \geq \alpha$. Central limit theorem (Fickian diffusion or SDE by Brownian motion) fails to apply.
  - The FT of a different scaling of $S_{\lfloor ct \rfloor}$ yields

$$\mathbb{E}\left[e^{-ikc^{-1/\alpha} S_{\lfloor ct \rfloor}}\right] = \left[1 + \frac{(ik)^\alpha}{c} + O\left(c^{-2/\alpha}\right)\right]^{\lfloor ct \rfloor} \to e^{t(ik)^\alpha}. \tag{32}$$

  - Lévy's continuity theorem concludes that a properly scaled $S_{\lfloor ct \rfloor}$ converges to an α stable Lévy process Z_t

$$\begin{aligned} & e^{t(ik)^\alpha} = \mathbb{E}\left[e^{-ikZ_t}\right] = \hat{p}(k,t) = \int_{\mathbb{R}} e^{-ikx} p(x,t)\,dx, \\ & i.e., \quad c^{-1/\alpha} S_{\lfloor ct \rfloor} \Rightarrow Z_t. \end{aligned} \tag{33}$$

  - Unlike Gaussian case, there is no analytical expression for $p(x,t)$ now.

- The Pearson's viewpoint gives rise to an SDE driven by an Lévy process

$$dX_t = \mu dt + \sigma dL_t. \tag{34}$$

- Einstein's approach: Note that $\hat{p}(k,t) = e^{t(ik)^\alpha}$ solves

$$\frac{d\hat{p}}{dt} = (ik)^\alpha \hat{p} = \widehat{\frac{\partial^\alpha p}{\partial x^\alpha}} \tag{35}$$

- which inverts to

$$\frac{\partial p}{\partial t} = \frac{\partial^\alpha p}{\partial x^\alpha}. \tag{36}$$

  - The PDF of finding a particle somewhere in space satisfies a (space-fractional) PDE, which decays algebraically $O(x^{-(\alpha+1)})$.
  - This justifies why FPDEs model transport processes exhibiting anomalous diffusion, long-range time memory or space interactions more accurately than classical integer-order PDEs.

Thank You

for Your Attention!