Entropy-based artificial viscosity

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Outline Part 1

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Why L1 for PDEs?

- Solve 1D eikonal
  \[ |u'(x)| = 1, \quad u(0) = 0, \quad u(1) = 0 \]

- Exists infinitely many weak solutions
Why L1 for PDEs?

- Exists a unique (positive) viscosity solution, $u$

\[ |u'_\epsilon| - \epsilon u''_\epsilon = 1, \quad u_\epsilon(0) = 0, \quad u_\epsilon(1) = 0. \]

- $\|u - u_\epsilon\|_{H^1} \leq c\epsilon^{\frac{1}{2}}$

- Sloppy approximation.
Why \( L^1 \) for PDEs?

One can do better with \( L^1 \) (of course 😊)

- Define mesh \( \mathcal{T}_h = \bigcup_{i=0}^{N} [x_i, x_{i+1}], \ h = x_{i+1} - x_i \).
- Use continuous finite elements of degree 1.

\[
V = \{ v \in C^0[0,1]; \ v|_{[x_i, x_{i+1}]} \in \mathbb{P}_1, \ v(0) = v(1) = 0 \}.
\]
Consider $p > 1$ and set

$$J(v) = \int_0^1 |v' - 1| \, dx + h^{2-p} \sum_{i=1}^{N} (v'(x_i^+) - v'(x_i^-))^p$$

$L^1$-norm of residual

Entropy

Define $u_h \in V$

$$u_h = \arg \min_{v \in V} J(v)$$
Why L1 for PDEs?

- Implementation: use mid-point quadrature

\[ J_h(v) = \sum_{i=0}^{N} h \left| v'(x_{i+\frac{1}{2}}) - 1 \right| + \text{Entropy}. \]

- Define

\[ \tilde{u}_h = \arg \min_{v \in V} J_h(v) \]
Why L1 for PDEs?

Theorem (J.-L. G. & B. Popov (2008))

\[ u_h \to u \text{ and } \tilde{u}_h \to u \text{ strongly in } W^{1,1}(0, 1) \cap C^0[0, 1]. \]

- Fast solution in 1D (JLG&BP 2010) and in higher dimension (fast-marching/fast sweeping, Osher/Sethian) to compute \( \tilde{u}_h \).
- Similar results in 2D for convex Hamiltonians (JLG&BP 2008).
A new idea based on $L^1$ minimization

Some provable properties of minimizer $\tilde{u}_h$ (JLG&BP 2008, 2009, 2010). Minimizer $\tilde{u}_h$ is such that:

- **Residual is SPARSE:**

  $$|\tilde{u}'_h(x_{i+\frac{1}{2}})| - 1 = 0, \quad \forall i \text{ such that } \frac{1}{2} \not\in [x_i, x_{i+1}].$$

- **Entropy** makes it so that graph of $\tilde{u}'_h(x)$ is concave down in $[x_i, x_{i+1}] \ni \frac{1}{2}$. 

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High-Order Hydrodynamics
A new idea based on $L^1$ minimization

**Conclusion:**

- Residual is **SPARSE**: PDE solved almost everywhere. Entropy does not play role in those cells.
- Entropy plays a key role only in cell where PDE is not solved.
Can L1 help anyway?

New idea:

- Go back to the notion of viscosity solution
- Add smart viscosity to the PDE:
  \[
  |u'_{\epsilon}| - \partial_x (\epsilon(u_{\epsilon})\partial_x u_{\epsilon}) = 1
  \]

- Make \(\epsilon\) depend on the entropy production:
  1. Viscosity large (order h) where entropy production is large
  2. Viscosity vanish when no entropy production

- Entropy plays a key role in cell where PDE is not solved.
Transport, mixing
The PDE

- Solve the transport equation

\[ \partial_t u + \beta \cdot \nabla u = 0, \quad u|_{t=0} = u_0, \quad \text{+BCs} \]

- Use standard discretizations (ex: continuous finite elements)
- Deviate as little possible from Galerkin.
The idea

- Notion of renormalized solution (DiPerna/Lions (1989))
  Good framework for non-smooth transport.
- $\forall E \in C^1(\mathbb{R}; \mathbb{R})$ is an entropy
- If solution is smooth $\Rightarrow E(u)$ solves PDE, $\forall E \in C^1(\mathbb{R}; \mathbb{R})$
  (multiply PDE by $E'(u)$ and apply chain rule)

\[
\partial_t E(u) + \beta \cdot \nabla E(u) = 0
\]
The idea

**Key idea 1:**

*Use entropy residual to construct viscosity*
The idea

viscosity $\sim$ entropy residual
The idea

\[ \text{viscosity} \sim \text{entropy residual} \]

- Viscosity \sim \text{residual} (Hughes-Mallet (1986) Johnson-Szepessy (1990))
- Entropy Residual \sim \text{a posteriori estimator} (Puppo (2003))
- Add entropy to formulation (For Hamilton-Jacobi equations Guermond-Popov (2007))
- Application to nonlinear conservation equations (Guermond-Pasquetti (2008))
The algorithm + time discretization

- Numerical analysis 101:
  \[ \text{Up-winding} = \text{centered approx} + \frac{1}{2} |\beta| h \text{ viscosity} \]

- Proof:
  \[
  \beta_i \frac{u_i - u_{i-1}}{h_i} = \beta_i \frac{u_{i+1} - u_{i-1}}{2h_i} - \frac{1}{2} \beta_i h_i \frac{u_{i+1} - 2u_i + u_{i-1}}{h_i^2}
  \]
Key idea 2:

Entropy viscosity should not exceed $\frac{1}{2} |\beta| h$
The algorithm

- Choose one entropy functional.
  EX1: $E(u) = |u - \bar{u}_0|$, 
  EX2: $E(u) = (u - \bar{u}_0)^2$, etc.

- Define entropy residual $D_h := \partial_t E(u_h) + \beta \cdot \nabla E(u_h)$,

- Define local mesh size of cell $K$: $h_K = \text{diam}(K)/p^2$

- Construct a wave speed associated with this residual on each mesh cell $K$:
  \[ v_K := h_K \| D_h \|_{\infty,K} / E(u_h) \]

- Define entropy viscosity on each mesh cell $K$:
  \[ \nu_K := \min(\frac{1}{2} \| \beta \|_{\infty,K} h_K, v_K h_K) \]
Summary

- Space approximation: Galerkin + entropy viscosity:
  \[
  \int_{\Omega} \left( \partial_t u_h + \beta \cdot \nabla u_h \right) v_h \text{d}x + \sum_K \int_K \nu_K \nabla u_h \nabla v_h \text{d}x = 0, \quad \forall v_h
  \]
  \[
  \text{Galerkin (centered approximation)} \quad \text{Entropy viscosity}
  \]

- Time approximation: Use an explicit time stepping: BDF2, RK3, RK4, etc.

- Idea: make the viscosity explicit ⇒ Stability under CFL condition.
Space + time discretization

- **EX**: 2nd-order centered finite differences 1D
- Compute the entropy residual $D_i$ on each cell $(x_i, x_{i+1})$

$$D_i := \max \left( \left| \frac{E(u^n_i) - E(u^{n-1}_i)}{\Delta t} \right| + \beta_{i+\frac{1}{2}} \frac{E(u^n_{i+1}) - E(u^n_i)}{h_i} \right),$$

$$\left| \frac{E(u^n_{i+1}) - E(u^{n-1}_{i+1})}{\Delta t} \right| + \beta_{i+\frac{1}{2}} \frac{E(u^n_{i+1}) - E(u^n_i)}{h_i} \right),$$

- Compute the entropy viscosity

$$\nu^n_i := \min \left( \frac{1}{2} |\beta_{i+\frac{1}{2}}| h_i, \frac{1}{2} \frac{D_i}{E(u^n_i) h_i^2} \right).$$
Space + time discretization

- Use RK to solve on next time interval $[t^n, t^n + \Delta t]$

$$u_i(t = t^n) = u^n_i$$

$$\partial_t u_i + \beta_{i+\frac{1}{2}} \frac{u_{i+1} - u_{i-1}}{2h_i} - \left( \nu^n_i \frac{u_{i+1} - u_i}{h_i} - \nu^n_{i-1} \frac{u_i - u_{i-1}}{h_{i-1}} \right) = 0$$

- The entropy viscosity can be computed on the fly for some RK techniques.
Space + time discretization: RK2 midpoint

- Advance half time step to get $w^n$

$$w^n_i = u^n_i - \frac{1}{2} \Delta t \beta_{i+\frac{1}{2}} \frac{u^n_{i+1} - u^n_{i-1}}{2h_i}$$

- Compute entropy viscosity on the fly

$$D_i := \max \left( \left| \frac{E(w^n_i) - E(u^n_i)}{\Delta t/2} \right| + \beta_{i+\frac{1}{2}} \frac{E(w^n_{i+1}) - E(w^n_i)}{h_i} \right),$$

$$\left| \frac{E(w^n_{i+1}) - E(u^n_{i+1})}{\Delta t/2} + \beta_{i+\frac{1}{2}} \frac{E(w^n_{i+1}) - E(w^n_i)}{h_i} \right|$$

- Compute $u^{n+1}$

$$u^{n+1}_i = u^n_i - \Delta t \beta_{i+\frac{1}{2}} \frac{w^n_{i+1} - w^n_{i-1}}{2h_i}$$

$$+ \left( \nu^n_i \frac{w^n_{i+1} - w^n_i}{\Delta x} - \nu^n_{i-1} \frac{w^n_i - w^n_{i-1}}{\Delta x} \right)$$
Theory for linear steady equations

- Consider $\partial_t u + \beta \cdot \nabla u = f$, $u|_{\Gamma^-} = 0$. 

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High-Order Hydrodynamics
Theory for linear steady equations

Consider \( \partial_t u + \beta \cdot \nabla u = f, \quad u|_{\Gamma^-} = 0 \).

**Theorem**

Let \( u_h \) be finite element approximation with Euler time stepping and \( \nu_K := \min(\frac{1}{2} ||\beta||_\infty, K h_K, v_K) \) then \( u_h \) converges to \( u \).
Theory for linear steady equations

Consider $\partial_t u + \beta \cdot \nabla u = f$, $u|_{\Gamma^-} = 0$.

**Theorem**

Let $u_h$ be finite element approximation with Euler time stepping and $\nu_K := \min(\frac{1}{2}||\beta||_{\infty}, K h_K, v_K)$ then $u_h$ converges to $u$.

**Theorem**

Let $u_h$ be $P_1$ finite element approximation with RK2 time stepping and $\nu_K := \min(\frac{1}{2}||\beta||_{\infty}, K h_K, h^{-\frac{1}{2}} v_K)$ then $u_h$ converges to $u$.

**Conjecture**

The results should hold for nonlinear scalar conservation laws with convex, Lipschitz flux.
Theory for linear steady equations

Why convergence is so difficult to prove?

- Key a priori estimate

\[ \int_0^T \nu(u)|\nabla u|^2 dx \leq c \]

- Ok in \( \{ \nu(u)(x, t) = \frac{1}{2}\|\beta\|h \} \) (non-smooth region)
- The estimate is useless in smooth region.
- Explicit time stepping makes the viscosity depend on the past.
1D Numerical tests, BV solution

- linear transport

\[ \partial_t u + \partial_x u = 0, \quad u_0(x) = \begin{cases} 
  e^{-300(2x-0.3)^2} & \text{if } |2x-0.3| \leq 0.25, \\
  1 & \text{if } |2x-0.9| \leq 0.2, \\
  \left(1 - \left(\frac{2x-1.6}{0.2}\right)^2\right)^{\frac{1}{2}} & \text{if } |2x-1.6| \leq 0.2, \\
  0 & \text{otherwise.} 
\end{cases} \]

- Periodic boundary conditions.
1D Numerical tests, BV solution, Spectral elements

- Spectral elements in 1D on random meshes.
- Long time integration, 100 periods.

Long time integration, $t = 100$, for polynomial degrees $k = 2, \ldots, 8$, $\#d.o.f. = 200$. Galerkin (left); Constant viscosity (center); Entropy viscosity (right).
1D Numerical tests, BV solution, Finite differences

- Second-order finite differences in 1D on uniform and random meshes.
- Long time integration, 100 periods.

Long time integration, $t = 100$, for 2nd order finite differences
#d.o.f.=200. Uniform mesh (left); Random mesh (right).
Numerical tests, smooth solution

- $\Omega = \{ (x, y) \in \mathbb{R}^2, \sqrt{x^2 + y^2} \leq 1 \} := B(0, 1)$,
- Speed: rotation about origin, angular speed $2\pi$
- $u(x, y) = \frac{1}{2} \left( 1 - \tanh \left( \frac{(x-r_0\cos(2\pi t))^2 + (y-r_0\sin(2\pi t))^2}{a^2} - 1 \right) + 1 \right)$,
- $a = 0.3$, $r_0 = 0.4$
2D numerical tests, smooth solution, $P_1$ FE

- $P_1$ finite elements

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Table: $P_1$ approximation.
Linear transport problem with smooth initial condition. Errors in $L^1$ (at left) and $L^2$ (at right) norms vs $h$ for $N = 2, 4, 6, 8, 12$. 

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2D Numerical tests, BV solution

- \( \Omega = \{(x, y) \in \mathbb{R}^2, \sqrt{x^2 + y^2} \leq 1\} := B(0, 1) \),
- Speed: rotation about origin, angular speed \( 2\pi \)
- \( u(x, y) = \chi_{B(0,a)}(\sqrt{(x - r_0 \cos(2\pi t))^2 + (y - r_0 \sin(2\pi t))^2}) \),
- \( a = 0.3, r_0 = 0.4 \)
2D Numerical tests, BV solution, $P_2$ FE

- $P_2$ finite elements

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Table: $P_2$ approximation.
2D Nonlinear scalar conservation laws

- Solve

\[ \partial_t u + \partial_x f(u) + \partial_y g(u) = 0 \quad u|_{t=0} = u_0, \quad +\text{BCs.} \]

- The unique entropy solution satisfies

\[ \partial_t E(u) + \partial_x F(u) + \partial_y G(u) \leq 0 \]

for all entropy pair \( E(u), F(u) = \int E'(u)f'(u)du \), \( G(u) = \int E'(u)g'(u)du \)
2D scalar nonlinear conservation laws

- Choose one entropy $E(u)$
- Define entropy residual, $D_h(u) := \partial_t E(u) + \partial_x F(u) + \partial_y G(u)$
- Define local mesh size of cell $K$: $h_K = \text{diam}(K)/p^2$
- Construct a speed associated with residual on each cell $K$:

$$v_K := h_K \| D_h \|_{\infty,K}/E(u_h)$$

- Compute maximum local wave speed:

$$\beta_K = \| \sqrt{f'(u)^2 + g'(u)^2} \|_{\infty,K}$$

- Define entropy viscosity on each mesh cell $K$:

$$\nu_K := h_K \min\left(\frac{1}{2} \beta_K, v_K\right)$$
Summary

- **Space approximation**: Galerkin + entropy viscosity:

\[
\int_{\Omega} \left( \frac{\partial_t u_h}{\partial t} + \frac{\partial_x f(u_h)}{\partial x} + \frac{\partial_y g(u_h)}{\partial y} \right) v_h \text{d}x + \sum_K \int_K \nu_K \nabla u_h \nabla v_h \text{d}x = 0, \quad \forall v_h
\]

Galerkin (centered approximation)  \quad \text{Entropy viscosity}

- **Time approximation**: explicit RK
The algorithm + time discretization

**EX:** 2nd-order centered finite differences 1D

- Compute local speed on each cell \((x_i, x_{i+1})\)

\[
\beta_{i+\frac{1}{2}} := \frac{1}{2} (f'(u_i) + f'(u_{i+1}))
\]

- Compute the entropy residual \(D_i\) on each cell \((x_i, x_{i+1})\)

\[
D_i := \max \left( \left| \frac{E(u^n_i) - E(u^{n-1}_i)}{\Delta t} + \beta_{i+\frac{1}{2}} \frac{E(u^n_{i+1}) - E(u^{n-1}_i)}{h_i} \right|, 
\left| \frac{E(u^n_{i+1}) - E(u^{n-1}_{i+1})}{\Delta t} + \beta_{i+\frac{1}{2}} \frac{E(u^n_{i+1}) - E(u^{n-1}_i)}{h_i} \right| \right)
\]
The algorithm + time discretization

- Compute the entropy viscosity

\[ \nu_i^n := \min \left( \frac{1}{2} |\beta_i + \frac{1}{2}| h_i, \frac{1}{2} \frac{D_i}{E(u^n)} h_i^2 \right) \]

- Use RK to solve on next time interval \([t^n, t^n + \Delta t]\)

\[ u_i(t = t^n) = u_i^n \]

\[ \partial_t u_i + \frac{f(u_{i+1}) - f(u_{i-1})}{2h_i} - \left( \nu_i^n \frac{u_{i+1} - u_i}{h_i} - \nu_{i-1}^n \frac{u_i - u_{i-1}}{h_{i-1}} \right) = 0 \]
EX: 1D burgers + 2nd-order Finite Differences

- Second-order Finite Differences + RK2/RK3/RK4

\[ u_h \]

Burgers, \( t = 0.25, \ N = 50, 100, \text{ and } 200 \text{ grid points} \).
EX: 1D burgers + Fourier

- Solution method: Fourier + RK4 + entropy viscosity

Burgers at $t = 0.25$ with $N = 50, 100, \text{ and } 200$. 

$u_N$

$\nu_N(u_N)$
EX: 1D Nonconvex flux + Fourier (WENO5 + SuperBee (or minmod 2) fails)

Consider \( \partial_t + \partial_x f(u) = 0 \), \( u(x, 0) = u_0(x) \)

\[
f(u) = \begin{cases} 
\frac{1}{4}u(1-u) & \text{if } u < \frac{1}{2}, \\
\frac{1}{2}u(u-1) + \frac{3}{16} & \text{if } u \geq \frac{1}{2},
\end{cases}
\]

\[
u_0(x) = \begin{cases} 
0, & x \in (0, 0.25], \\
1, & x \in (0.25, 1]
\end{cases}
\]
EX: 1D Nonconvex flux + Fourier (WENO5 + SuperBee (or minmod 2) fails)

Consider $\partial_t + \partial_x f(u) = 0$, $u(x, 0) = u_0(x)$

$$f(u) = \begin{cases} \frac{1}{4}u(1-u) & \text{if } u < \frac{1}{2}, \\ \frac{1}{2}u(u-1) + \frac{3}{16} & \text{if } u \geq \frac{1}{2}, \end{cases}$$

$$u_0(x) = \begin{cases} 0, & x \in (0, 0.25], \\ 1, & x \in (0.25, 1]. \end{cases}$$

Non-convex flux problem $u_N$ at $t = 1$ with $N = 200, 400, 800, \text{ and } 1600$. 
Convergence tests, 2D Burgers

- Solve 2D Burgers

\[ \partial_t u + \partial_x \left( \frac{1}{2} u^2 \right) + \partial_y \left( \frac{1}{2} u^2 \right) = 0 \]

- Subject to the following initial condition

\[ u(x, y, 0) = u^0(x, y) = \begin{cases} 
  -0.2 & \text{if } x < 0.5 \text{ and } y > 0.5 \\
  -1 & \text{if } x > 0.5 \text{ and } y > 0.5 \\
  0.5 & \text{if } x < 0.5 \text{ and } y < 0.5 \\
  0.8 & \text{if } x > 0.5 \text{ and } y < 0.5 
\end{cases} \]

- Compute solution in \((0, 1)^2\) at \(t = \frac{1}{2}\).
Convergence tests, 2D Burgers

Initial data

$P_1$ FE, 3 $10^4$ nodes
Convergence tests, 2D Burgers

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**Table:** Burgers, $P_1$ approximation.
Convergence tests, 2D Burgers

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Table: Burgers, $\mathcal{P}_2$ approximation.
Buckley Leverett, \( \mathbb{P}_2 \) FE

- Solve \( \partial_t u + \partial_x f(u) + \partial_y g(u) = 0 \).

\[
f(u) = \frac{u^2}{u^2 + (1-u)^2}, \quad g(u) = f(u)(1 - 5(1 - u)^2)
\]

Non-convex fluxes (composite waves)

\[
u(x, y, 0) = \begin{cases} 
1, & \sqrt{x^2 + y^2} \leq 0.5 \\
0, & \text{else}
\end{cases}
\]
Buckley Leverett, $P_2$ FE
KPP (WENO + superbee limiter fails), $\mathbb{P}_2$ FE

- Solve $\partial_t u + \partial_x f(u) + \partial_y g(u) = 0$.

  $$f(u) = \sin(u), \quad g(u) = \cos(u)$$

Non-convex fluxes (composite waves)

$$u(x, y, 0) = \begin{cases} \frac{7}{2} \pi, & \sqrt{x^2 + y^2} \leq 1 \\ \frac{1}{4} \pi, & \text{else} \end{cases}$$
KPP (WENO + superbee limiter fails)

\( P_2 \) approx

\( Q_4 \) entrop visco.

Jean-Luc Guermond

High-Order Hydrodynamics
Leonhard Euler

COMPRESSIBLE EULER EQUATIONS
LAGRANGIAN HYDRODYNAMICS

Euler equations
The algorithm
1D-2D Tests + Fourier
2D tests, $\mathbb{P}_1$ finite elements

NONLINEAR SCALAR CONSERVATION EQUATIONS
Euler flows

- Solve compressible Euler equations

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0 \\
\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p \mathbb{I}) &= 0 \\
\partial_t (E) + \nabla \cdot (\mathbf{u}(E + p)) &= 0 \\
\rho e &= E - \frac{1}{2} \rho \mathbf{u}^2, \quad T = (\gamma - 1)e \quad T = \frac{p}{\rho}
\end{align*}
\]

Initial data + BCs

- Use continuous finite elements of degree \( p \).
- Deviate as little possible from Galerkin.
The algorithm

- Compute the entropy $S_h = \frac{\rho_h}{\gamma - 1} \log\left(\frac{\rho_h}{\rho_h^\gamma}\right)$
- Define entropy residual, $D_h := \partial_t S_h + \nabla \cdot (u_h S_h)$
- Define local mesh size of cell $K$: $h_K = \text{diam}(K) / p^2$
- Construct a speed associated with residual on each cell $K$:

$$v_K := h_K \| D_h \|_{\infty, K}$$

- Compute maximum local wave speed:

$$\beta_K = \| u \| + \left(\gamma T\right)^{\frac{1}{2}} \| \|_{\infty, K}$$

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High-Order Hydrodynamics
The algorithm

- Use Navier-Stokes regularization: define $\mu_K$ and $\kappa_K$.
- Entropy viscosity and thermal conductivity on each mesh cell $K$:
  \[ \mu_K := h_K \min\left(\frac{1}{2} \beta_K \| \rho_h \|_{\infty, K}, v_K \right), \quad \kappa_K = P \mu_K \]
- In practice use $P = \frac{1}{10}$.
- Solution method: Galerkin + entropy viscosity + thermal conductivity
1D Euler flows + Fourier

- Solution method: Fourier + RK4 + entropy viscosity
1D Euler flows + Fourier

- Solution method: Fourier + RK4 + entropy viscosity

Figure: Lax shock tube, $t = 1.3, 50, 100, 200$ points. Shu-Osher shock tube, $t = 1.8, 400, 800$ points. Right: Woodward-Collela blast wave, $t = 0.038, 200, 400, 800, 1600$ points.
2D Euler flows + Fourier

- Domain $\Omega = (-1, 1)^2$
- Riemann problem with the initial condition:
  
  $0 < x < 0.5$ and $0 < y < 0.5$, \quad $p = 1, \rho = 0.8, u = (0, 0)$,
  
  $0 < x < 0.5$ and $0.5 < y < 1$, \quad $p = 1, \rho = 1, u = (0.7276, 0)$,
  
  $0.5 < x < 1$ and $0 < y < 0.5$, \quad $p = 1, \rho = 1, u = (0, 0.7276)$,
  
  $0 < x < 0.5$ and $0.5 < y < 1$, \quad $p = 0.4, \rho = 0.5313, u = (0, 0)$.

- Solution at time $t = 0.2$. 
2D Euler flows + Fourier (Riemann test case 12)

Euler benchmark, Fourier approximation: Density (at left), $0.528 < \rho_N < 1.707$ and viscosity (at right), $0 < \mu_N < 3.410^{-3}$, at $t = 0.2$, $N = 400$. 
Riemann problem test case no 12, $\mathbb{P}_1$ FE

movie, Riemann no 12
Mach 3 Wind Tunnel with a Step, $P_1$ finite elements

- Mach 3 Wind Tunnel with a Step (Standard Benchmark since Woodward and Colella (1984))
- Inflow boundary, density 1.4, pressure 1, and $x$-velocity 3, (Mach $=3$)
Mach 3 Wind Tunnel with a Step, $P_1$ finite elements

- Mach 3 Wind Tunnel with a Step (Standard Benchmark since Woodward and Colella (1984))
- Inflow boundary, density 1.4, pressure 1, and $x$-velocity 3, (Mach = 3)

$P_1$ FE, $1.3 \times 10^5$ nodes

Log(density)

Movie, density field (entropy visc.) Movie, density field (viscous)

Flash Code, adaptive $PPM$, $\sim 4.9 \times 10^6$ nodes
Viscous flux of entropy Viscosity.
Mach 10 Double Mach reflection, $\mathbb{P}_1$ finite elements

- Right-moving Mach 10 shock makes $60^\circ$ angle with $x$-axis (Standard Benchmark, Woodward and Colella (1984))
- Shock interacts with flat plate $x \in \left(\frac{1}{6}, +\infty\right)$.
- The un-shocked fluid $\rho = 1.4$, $p = 1$, and $u = 0$
Mach 10 Double Mach reflection, $\mathcal{P}_1$ finite elements

- Right-moving Mach 10 shock makes $60^\circ$ angle with $x$-axis (Standard Benchmark, Woodward and Colella (1984))
- Shock interacts with flat plate $x \in (\frac{1}{6}, +\infty)$.
- The un-shocked fluid $\rho = 1.4$, $p = 1$, and $u = 0$

$\mathcal{P}_1$ FE, $4.5 \times 10^5$ nodes, $t = 0.2$

Movie, density field
Mach 10 Double Mach reflection

Entropy  Viscosity
NONLINEAR SCALAR CONSERVATION EQUATIONS

Leonhard Euler
Solve compressible Euler equations in Lagrangian form

\[ \rho \partial_t \mathbf{u} + \nabla p = 0 \]
\[ \rho \partial_t e + p \nabla \cdot \mathbf{u} = 0 \]
\[ J \rho = \rho_0 \]
\[ \partial_t \mathbf{x} = \mathbf{u}(\mathbf{x}, t) \]
\[ T = (\gamma - 1)e \quad T = \frac{p}{\rho} \]

Initial data + BCs

Work with \( \rho \) and nonconservative variables \( \mathbf{u}, e \).
EULER IN LAGRANGIAN COORDINATES

- Weak forms

\[
\int_{\Omega_0} \rho_0 \partial_t u(\phi_t(x_0)) \psi(\phi_t(x_0)) \, dx_0 = - \int_{\Omega_t} \psi(x) \nabla p(x, t) \, dx \\
- \int_{\Omega_t} \nu(x, t) \nabla \psi(x) \nabla u(x, t) \, dx \\
\int_{\Omega_0} \rho_0 \partial_t e(\phi_t(x_0)) \psi(\phi_t(x_0)) \, dx_0 = - \int_{\Omega_t} \psi(x) p(x, t) \nabla \cdot u(x, t) \, dx \\
- \int_{\Omega_t} \frac{1}{2} \nu(x, t) \nabla \psi(x) \nabla |u(x, t)|^2 \, dx - \int_{\Omega_t} \kappa(x, t) \nabla \psi(x) \nabla T(x, t) \, dx \\
\int_{\Omega_t} \rho(x) \psi(x) \, dx = \int_{\Omega_0} \rho_0(x_0) \psi(\phi_t(x_0)) \, dx_0 \quad \partial_t x = u(x, t) \\
T = (\gamma - 1)e = \frac{p}{\rho}
\]
Specific entropy $s = \frac{1}{\gamma - 1} \log(p/\rho^\gamma)$

Entropy residual

$$D := \max(|\rho \partial_t s|, |s(\partial_t \rho + \rho \nabla \cdot u)|)$$

Algorithm similar to Eulerian formulation
COMPRESSIBLE EULER EQUATIONS
LAGRANGIAN HYDRODYNAMICS

Euler equations
Weak formulation
Numerical tests

SOD TUBE

Jean-Luc Guermond
High-Order Hydrodynamics
1D NOH PROBLEM
TWO WAVE PROBLEM