Numerical method for stochastic delay differential equations

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1. Research objective

OUTLINE

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3. Mean square stability of the semi-implicit Euler method
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4. Mean square stability of the semi-implicit Milstein method
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5. T-stability of the semi-implicit Euler method for delay differential equations with multiplicative noise
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Research objective
General form of the stochastic differential delay equations (SDDEs)

The general form of the SDDEs is

\[
\begin{cases}
dX(t) = f(t, X(t), X(t - \tau))\,dt + g(t, X(t), X(t - \tau))\,dW(t), \\
X(t) = \psi(t), \quad t \in [-\tau, 0],
\end{cases}
\]

(1.1)

- where \( \tau \) is a positive fixed delay
- \( W(t) \) is a \( d \)-dimensional standard Wiener process
- \( \psi(t) \) is a \( C([-\tau, 0], R^m) \)-valued initial segment
- \( f : R^+ \times R^m \times R^m \to R^m \)
- \( g : R^+ \times R^m \times R^m \to R^{m \times d} \)
Objective of my research

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- It is necessary to develop numerical methods and study the properties of these methods.
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My research focused on the stability and convergence of several kinds of numerical methods for linear SDDEs.
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- My research focused on the stability and convergence of several kinds of numerical methods for linear SDDEs.

- In my research, Eq. (1.1) is interpreted in the Itô sense.
Convergence of the semi-implicit Euler method for a linear SDDE
a linear SDDE

\[
\begin{aligned}
\begin{cases}
    dX(t) &= [aX(t) + bX(t - \tau)]dt + [cX(t) + dX(t - \tau)]dW(t), \\
    X(t) &= \xi(t), \quad t \in [-\tau, 0],
\end{cases}
\end{aligned}
\]

(2.1)

- \(a, b, c, d \in \mathbb{R}\)
- \(\tau\) is a positive fixed delay
- \(W(t)\) is a 1-dimensional standard Wiener process
- \(\xi(t)\) is a \(C([-\tau, 0]; \mathbb{R})\)-valued initial segment
Some assumptions

- Let \((\Omega, \mathcal{F}, P)\) be a probability space with a filtration \((\mathcal{F}_t)_{t \geq 0}\), which satisfies the usual conditions.

- Let \(W(t), t \geq 0\) in Eq.(2.1) be \(\mathcal{F}_t\)-adapted and independent of \(\mathcal{F}_0\).

- \(|\cdot|\) is the Euclidean norm in \(\mathbb{R}\) and \(\|\xi\|\) is defined by \(\|\xi\| = \sup_{-\tau \leq t \leq 0} |\xi(t)|\).

- Assume \(\xi(t), t \in [-\tau, 0]\) to be \(\mathcal{F}_0\)-measurable and right continuous, and \(E\|\xi\|^2 < \infty\).

- Under the above assumptions, Eq.(2.1) has a unique strong solution \(X(t) : [-\tau, 0] \cup [0, +\infty) \to \mathbb{R}\).
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- Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$, which satisfies the usual conditions.
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- Let \(W(t), t \geq 0\) in Eq.(2.1) be \(\mathcal{F}_t\)-adapted and independent of \(\mathcal{F}_0\).
- \(|\cdot|\) is the Euclidean norm in \(R\) and \(\|\xi\|\) is defined by \(\|\xi\| = \sup_{-\tau \leq t \leq 0} |\xi(t)|\).
- Assume \(\xi(t), t \in [-\tau, 0]\) to be \(\mathcal{F}_0\)-measurable and right continuous, and \(E\|\xi\|^2 < \infty\).
- Under the above assumptions, Eq.(2.1) has a unique strong solution \(X(t) : [-\tau, 0] \cup [0, +\infty) \to R\).
- \(X(t)\) satisfies Eq.(2.1)
- \(X(t)\) is a measurable, sample-continuous and \(\mathcal{F}_t\)-adapted process.
Several important estimate inequalities

For any given $T > 0$, there exist positive numbers $C_1, C_2$ and $M$, such that the solution of Eq. (2.1) satisfies

$$E\left( \sup_{-\tau \leq t \leq T} |X(t)|^2 \right) \leq C_1 [1 + E\|\xi\|^2] \tag{2.2}$$

for all $t \in [-\tau, T]$,

$$E|X(t) - X(s)|^2 \leq C_2 (t - s) \tag{2.3}$$

for any $0 \leq s < t \leq T, t - s < 1$, and

$$E|aX(t) + bX(t - \tau)| \leq \sqrt{2M} (1 + E\|\xi\|^2) \tag{2.4}$$

for all $t \in [0, T]$. 
The semi-implicit Euler method for Eq. (2.1)

\[ X_{n+1} = X_n + \alpha (aX_{n+1} + bX_{n-m+1}) \]
\[ + (1 - \alpha) (aX_n + bX_{n-m}) h + [cX_n + dX_{n-m}] \Delta W_n, \]

(2.5)

- \( \alpha \) is a parameter with \( 0 \leq \alpha \leq 1 \)
The semi-implicit Euler method for Eq.(2.1)

\[ X_{n+1} = X_n + \left[ \alpha(aX_{n+1} + bX_{n-m+1}) \right. \]
\[ \left. + (1 - \alpha)(aX_n + bX_{n-m}) \right] h + [cX_n + dX_{n-m}] \Delta W_n, \]

(2.5)

- \(\alpha\) is a parameter with \(0 \leq \alpha \leq 1\)
- \(h > 0\) is a stepsize which satisfies \(\tau = mh\) for a positive integer \(m\), and \(t_n = nh\).
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- \( X_n \) is an approximation to \( X(t_n) \), if \( t_n \leq 0 \), we have \( X_n = \xi(t_n) \)
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- \( X_n \) is an approximation to \( X(t_n) \), if \( t_n \leq 0 \), we have \( X_n = \xi(t_n) \)
- increments \( \Delta W_n := W(t_{n+1}) - W(t_n) \), are independent \( N(0, h) \)-distributed Gaussian random variables
The semi-implicit Euler method for Eq.(2.1)

\[ X_{n+1} = X_n + \left[ \alpha (aX_{n+1} + bX_{n-m+1}) + (1 - \alpha)(aX_n + bX_{n-m}) \right] h + [cX_n + dX_{n-m}] \Delta W_n, \]

(2.5)

- \( \alpha \) is a parameter with \( 0 \leq \alpha \leq 1 \)
- \( h > 0 \) is a stepsize which satisfies \( \tau = mh \) for a positive integer \( m \), and \( t_n = nh \).
- \( X_n \) is an approximation to \( X(t_n) \), if \( t_n \leq 0 \), we have \( X_n = \xi(t_n) \)
- increments \( \Delta W_n := W(t_{n+1}) - W(t_n) \), are independent \( N(0, h) \)-distributed Gaussian random variables
- we assume that \( X_n \) is \( \mathcal{F}_{t_n} \)-measurable at the mesh-point \( t_n \).
The local truncation error is defined by

\[
\delta_{n+1} = X(t_{n+1}) - \left\{X(t_n) + \alpha[aX(t_{n+1}) + bX(t_{n-m+1})]h + (1 - \alpha)[aX(t_n) + bX(t_{n-m})]h + [cX(t_n) + dX(t_{n-m})]\Delta W_n \right\}
\]  
(2.6)

and the global error is defined by

\[
\epsilon_n = X(t_n) - X_n.
\]  
(2.7)
A Lemma for $\delta_n$

**Lemma**

The numerical solution produced by the semi-implicit Euler Scheme (2.5) to approximate the solution of Eq. (2.1) satisfies

$$\max_{0 \leq n \leq N} |E(\delta_n)| \leq C_3 h^2 \quad \text{as} \quad h \to 0,$$

(2.8)

and

$$\max_{0 \leq n \leq N} (E(\delta_n)^2)^{\frac{1}{2}} \leq C_4 h \quad \text{as} \quad h \to 0,$$

(2.9)

where $C_3, C_4$ are positive constants which are independent of $h$. 

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**WR Cao**

Numerical methods for SDDE
Assume that $ah\alpha < 1$. The numerical solution produced by the semi-implicit Euler method (2.5) is convergent to the exact solution of Eq. (2.1) on the mesh-point in the mean-square sense with order $1/2$, i.e. there exists a positive constant $C_0$ such that

$$\max_{1 \leq n \leq N} (E(\epsilon_n^2))^{\frac{1}{2}} \leq C_0 h^{\frac{1}{2}} \quad \text{as} \quad h \to 0. \quad (2.10)$$
Proof of the above conclusion

- Using the estimates (2.2)-(2.4) and Hölder inequality, Cauchy’s inequality, Doob’s inequality, the lemma can be proved.
- The lemma and the energy techniques were used to prove the theorem.
- Constant $C_0$ is independent on stepsize $h$ but dependent on $T$. 
We consider

\[
\begin{aligned}
\text{d}X(t) &= [aX(t) + bX(t-1)]\text{d}t + [cX(t) + dX(t-1)]\text{d}W(t), \\
X(t) &= t + 1, \quad t \in [-1, 0].
\end{aligned}
\]

The solution for \( t \in [0, 1] \) is given by

\[
X(t) = \Phi_{t, 0} \left( \xi(0) + \int_0^t \Phi_{s, 0}^{-1} (b - cd) s \, ds + \int_0^t d s \, \Phi_{s, 0}^{-1} \, dW_s \right),
\]

where

\[
\Phi_{t, 0} = \exp \left( \int_0^t (a - \frac{1}{2} c^2) ds + \int_0^t c \, dW_s \right).
\]
For time $t \in [1, 2]$, we obtain the explicit solution by using the explicit solution given above as a new initial function.

<table>
<thead>
<tr>
<th>Stepsize</th>
<th>1/8</th>
<th>1/16</th>
<th>1/32</th>
<th>1/64</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_I$</td>
<td>0.0288</td>
<td>0.0091</td>
<td>0.0044</td>
<td>0.0020</td>
</tr>
<tr>
<td>$\varepsilon_{II}$</td>
<td>0.0339</td>
<td>0.0154</td>
<td>0.0026</td>
<td>0.0013</td>
</tr>
</tbody>
</table>

- **Example I:** $a = -2, b = 0.1, c = 0.5, d = 0,$
- **Example II:** $a = -2, b = 0.1, c = 0, d = 0.5.$
- $T = 2, \alpha = 0.5$
- $\omega_{ij} : 1 \leq i \leq 20, 1 \leq j \leq 100$ are simulated.
- $\varepsilon(i) = 1/100 \sum_{j=1}^{100} |X(T, \omega_{ij}) - X_N(\omega_{ij})|^2$
- $\varepsilon = 1/20 \sum_{i=1}^{20} \varepsilon(i)$
Proof of the convergence theorem (1)

It is easy to see that (2.5) has a solution when \( ah\alpha < 1 \). From (2.5), (2.6) and (2.7), we have

\[
\epsilon_{n+1} = X(t_{n+1}) - \left\{ X_n + \alpha[aX_{n+1} + bX_{n-m+1}]h \right. \\
+ (1 - \alpha)[aX_n + bX_{n-m}]h + \left. [cX_n + dX_{n-m}]\Delta W_n \right\}
\]

and

\[
\epsilon_{n+1} = \epsilon_n + u_n + \delta_{n+1},
\]

where

\[
u_n : = ah\alpha(X(t_{n+1}) - X_{n+1}) + [ah(1 - \alpha) \\
+ c\Delta W_n](X(t_n) - X_n) + bh\alpha(X(t_{n-m+1}) - X_{n-m+1}) \\
+ [bh(1 - \alpha) + d\Delta W_n](X(t_{n-m}) - X_{n-m}).
\]  
\[\text{(2.12)} \]
Proof of the convergence theorem (2)

Clearly,

$$|E(u_n)| \leq C_u h \left( E|\epsilon_{n+1}| + E|\epsilon_n| + E|\epsilon_{n-m+1}| + E|\epsilon_{n-m}| \right), \quad (2.13)$$

where $C_u = \max\{|a|, |b|\}$, and

$$E(u_n^2) \leq C'_u h \left( E(\epsilon_{n+1}^2) + E(\epsilon_n^2) + E(\epsilon_{n-m+1}^2) + E(\epsilon_{n-m}^2) \right), \quad (2.14)$$

where $C'_u = \max\{a^2 + c^2 + |ab| + |cd|, b^2 + d^2 + |ab| + |cd|\}$. 

Proof of the convergence theorem (3)

Hence

\[
E(\epsilon_{n+1}^2 | \mathcal{F}_t) \leq E(\epsilon_n^2 | \mathcal{F}_t) + E(\delta_{n+1}^2 | \mathcal{F}_t) + E(u_n^2 | \mathcal{F}_t) \\
+ 2|E(\delta_{n+1} u_n | \mathcal{F}_t)| + 2|E(\delta_{n+1} \epsilon_n | \mathcal{F}_t)| + 2|E(\epsilon_n u_n | \mathcal{F}_t)|.
\]

(2.15)

Using the Hölder inequality, the properties of conditional expectation and inequalities (2.8), (2.9), (2.13) and (2.14), we have

\[
E(\delta_{n+1}^2 | \mathcal{F}_t) = E(E(\delta_{n+1}^2 | \mathcal{F}_{t_n}) | \mathcal{F}_t) \leq C_4^2 h^2,
\]

\[
E(u_n^2 | \mathcal{F}_t) \leq C'_u h \left[ E(\epsilon_{n+1}^2 | \mathcal{F}_t) + E(\epsilon_n^2 | \mathcal{F}_t) \\
+ E(\epsilon_{n-m+1}^2 | \mathcal{F}_t) + E(\epsilon_{n-m}^2 | \mathcal{F}_t) \right],
\]
Proof of the convergence theorem (4)

\[
2 \left| E(\delta_{n+1} u_n \mid \mathcal{F}_t_0) \right| \leq 2 \left[ E(\delta_{n+1}^2 \mid \mathcal{F}_t_0) \right]^{\frac{1}{2}} \left[ E(u_n^2 \mid \mathcal{F}_t_0) \right]^{\frac{1}{2}} \\
\leq E(\delta_{n+1}^2 \mid \mathcal{F}_t_0) + E(u_n^2 \mid \mathcal{F}_t_0) \\
\leq C_4^2 h^2 + C_u' h \left[ E(\epsilon_{n+1}^2 \mid \mathcal{F}_t_0) + E(\epsilon_n^2 \mid \mathcal{F}_t_0) \right. \\
\quad \left. + E(\epsilon_{n-m+1}^2 \mid \mathcal{F}_t_0) + E(\epsilon_{n-m}^2 \mid \mathcal{F}_t_0) \right],
\]

\[
2 \left| E(\delta_{n+1} \epsilon_n \mid \mathcal{F}_t_0) \right| \leq 2 \left( E(E(\delta_{n+1} \mid \mathcal{F}_t_n))^2 \mid \mathcal{F}_t_0 \right)^{\frac{1}{2}} E(\epsilon_n^2 \mid \mathcal{F}_t_0)^{\frac{1}{2}} \\
\leq 2 \left[ E(C_3^2 h^4) \right]^{\frac{1}{2}} E(\epsilon_n^2 \mid \mathcal{F}_t_0)^{\frac{1}{2}} \\
\leq C_3^2 h^2 + h E(\epsilon_n^2 \mid \mathcal{F}_t_0),
\]
Proof of the convergence theorem (5)

\[ 2|E(\epsilon_n u_n | \mathcal{F}_{t_0})| \leq 2E\left(|E(u_n | \mathcal{F}_{t_n})||\epsilon_n | | \mathcal{F}_{t_0}\right) \]

\[ \leq 2E\left[C_u h (|\epsilon_n|^2 + |\epsilon_{n+1}| |\epsilon_n| \right] \]

\[ + |\epsilon_{n-m+1}| |\epsilon_n| + |\epsilon_{n-m}| |\epsilon_n| \right) | \mathcal{F}_{t_0} \right] \]

\[ \leq 5C_u h E(\epsilon_n^2 | \mathcal{F}_{t_0}) + C_u h \left[E(\epsilon_{n+1}^2 | \mathcal{F}_{t_0}) \right] \]

\[ + E(\epsilon_{n-m+1}^2 | \mathcal{F}_{t_0}) + E(\epsilon_{n-m}^2 | \mathcal{F}_{t_0}) \right].\]
Proof of the convergence theorem (6)

Adding the above inequalities, then (2.15) becomes

\[
(1 - C'_6 h) E(\epsilon^2_{n+1} | \mathcal{F}_t_0) \leq (1 + C'_5 h) E(\epsilon^2_n | \mathcal{F}_t_0) \\
+ C'_6 h E(\epsilon^2_{n-m+1} | \mathcal{F}_t_0) + C'_6 h E(\epsilon^2_{n-m} | \mathcal{F}_t_0) + C'_7 h^2,
\]

(2.16)

where \( C'_5 = 2C'_u + 5C_u + 1 \), \( C'_6 = 2C'_u + C_u \), \( C'_7 = 2C^2_4 + C^2_3 \). Let

\[
E_n = \max_{0 \leq i \leq n} \left\{ E(\epsilon^2_i | \mathcal{F}_t_0) \right\}.
\]

(2.17)
Proof of the convergence theorem (7)

Assume $1 - C_6' h \geq 1/2$. Due to $h \to 0$, the assumption is reasonable. We have from (2.16)

$$E_{n+1} \leq \frac{1 + C_5' h}{1 - C_6' h} E_n + \frac{C_6' h}{1 - C_6' h} (E_{n-m+1} + E_{n-m}) + \frac{C_7'}{1 - C_6' h} h^2$$

$$\leq \left(1 + \frac{C_5' + C_6'}{1 - C_6' h} h\right) E_n + 2C_6' h (E_{n-m+1} + E_{n-m}) + 2C_7' h^2. \quad (2.18)$$

Let $C_5 = 2(C_5' + C_6')$, $C_6 = 2C_6'$, $C_7 = 2C_7'$, then the inequality (2.18) becomes

$$E_{n+1} \leq (1 + C_5 h) E_n + C_6 h E_{n-m+1} + C_6 h E_{n-m} + C_7 h^2. \quad (2.19)$$
Proof of the convergence theorem(8)

Now we will proceed by using an induction argument over consecutive intervals of the length $\tau$ up to the end of the interval $[0, T]$.

**Case 1** \(0 \leq t_n < \tau, t_{n+1} \leq \tau\).

Since \(\epsilon_{n-m} = \epsilon_{n-m+1} = 0\) in this case, we have from (2.19)

\[
E_{n+1} \leq (1 + C_5 h)E_n + C_7 h^2
\]

\[
\leq C_7 h^2 \sum_{i=0}^{n} (1 + C_5 h)^i
\]

\[
= C_7 h^2 \frac{(1 + C_5 h)^{n+1} - 1}{1 + C_5 h - 1} \leq C_8 h,
\]

where \(C_8 = C_7(e^{C_5 T} - 1)/C_5\).
Proof of the convergence theorem(9)

Case 2 \( t_n = \tau, \tau < t_{n+1} \leq 2\tau. \)

In this case, \( \epsilon_{n-m} = 0 \), by (2.19), we obtain

\[
E_{n+1} \leq (1 + C_5 h) E_n + C_6 h E_{n-m+1} + C_7 h^2.
\]

Since \( 0 \leq t_{n-m+1} \leq \tau \), we have the following estimation about \( E_{n-m+1} \) from Case 1:

\[
E_{n-m+1} \leq C_8 h.
\]

Hence, by (2.19), it is obvious that

\[
E_{n+1} \leq (1 + C_5 h) E_n + C_6 C_8 h^2 + C_7 h^2 \\
\leq (1 + C_5 h) E_n + C_9 h (e^{C_5 T} - 1)
\]

for \( C_9 = (C_6 C_8 + C_7)/C_5 \).
Combining Case 1 and Case 2, we obtain

\[ E_{n-m} = 0, \ E_{n-m+1} \leq C_8 h, \ E_{n+1} \leq C_{10} h \]

for \( t_n \in [0, \tau] \), where \( C_{10} = C_9 (e^{C_5 T} - 1) \).

**Case 3** \( t_n \in [k\tau, (k+1)\tau], \ k \leq s - 1 \).

We make the assumption

\[ E_{n-m} \leq C_{11} h, \ E_{n-m+1} \leq C_{11} h \quad (2.21) \]

for a positive constant \( C_{11} \), then we have from (2.19) and (2.21)
Proof of the convergence theorem (11)

\[ E_{n+1} \leq (1 + C_5 h) E_n + C_6 h (E_{n-m+1} + E_{n-m}) + C_7 h^2 \]

\[ \leq (1 + C_5 h) E_n + 2C_6 C_{11} h^2 + C_7 h^2 \]

\[ \leq C_{12} h (e^{C_5 T} - 1) \]

by the same arguments as above, where \( C_{12} = (2C_6 C_{11} + C_7) / C_5 \).

This implies

\[ (E_{n+1})^{\frac{1}{2}} \leq C_0 h^{\frac{1}{2}}, \]

i.e.

\[ \max_{1 \leq n \leq N} \left[ E(\epsilon_n^2 | \mathcal{F}_{t_0}) \right]^{\frac{1}{2}} \leq C_0 h^{\frac{1}{2}}, \]

where \( C_0 = \sqrt{C_{12} (e^{C_5 T} - 1)} \). The theorem is proved.
Mean square stability of the semi-implicit Euler method
Mean square stability of the analytical solutions

Lemma

If

\[ a < -|b| - \frac{1}{2} (|c| + |d|)^2, \]

(3.1)

then the solution of Eq. (2.1)

\[
\begin{cases}
    dX(t) = [aX(t) + bX(t - \tau)]dt + [cX(t) + dX(t - \tau)]dW(t), \\
    X(t) = \xi(t), t \in [-\tau, 0],
\end{cases}
\]

is mean square stable, that is

\[
\lim_{t \to \infty} E|X(t)|^2 = 0.
\]

(3.2)
Definition

Under the condition (3.1), a numerical method is said to be mean square stable (MS-stable), if there exists a $h_0(a, b, c, d) > 0$, such that any application of the method to the problem (2.1) generates numerical approximations $X_n$, which satisfy

$$\lim_{n \to \infty} E|X_n|^2 = 0$$

for all $h \in (0, h_0(a, b, c, d))$ with $h = \tau/m$. A numerical method is said to be general mean square stable (GMS-stable), if any application of the method to the problem (2.1) generates numerical approximations $X_n$, which satisfy

$$\lim_{n \to \infty} E|X_n|^2 = 0$$

for every stepsize $h = \tau/m$. 
Assume the condition (3.1) is satisfied and let

\[ K = \frac{|a| + |b|}{2|a|} + \frac{2a + 2|b| + (|c| + |d|)^2}{2|a|(|a| + |b|)}. \]  

(3.3)

(1) If \( K < 0 \), then for every \( \alpha \in [0, 1] \), the semi-implicit Euler method is GMS-stable.

(2) If \( K \geq 0 \), then for \( \alpha \in (K, 1] \), the semi-implicit Euler method is GMS-stable; for \( \alpha \in [0, K] \), it is MS-stable.
Theorem

\[ h_0(a, b, c, d) = \min \{ h', h'' \} \]

where \( h' = \max \{ h_1, h_2 \} \), \( h'' = \max \{ \frac{1}{|a|}, h_2 \} \), and

\[ h_1 = \min \left\{ \frac{1}{|a|}, \frac{-(2a + 2|b| + (|c| + |d|)^2)}{(a + |b|)^2} \right\}, \]

\[ h_2 = \frac{-(2a + 2|b| + (|c| + |d|)^2)}{(|a| + |b|)^2}. \]
Proof of the theorem (1)

\[
(1 - a h \alpha)^2 X_{n+1}^2 = [1 + ah(1 - \alpha) + c \Delta W_n]^2 X_n^2 + b^2 h^2 \alpha^2 X_{n-m+1}^2
\]
+ \([bh(1 - \alpha) + d \Delta W_n]^2 X_{n-m}^2\]
+ \([b h(1 - \alpha) + c \Delta W_n] bh \alpha X_n X_{n-m+1}\]
+ \(2[bh(1 - \alpha) + c \Delta W_n] bh \alpha X_{n-m+1} X_{n-m}\]
+ \([1 + ah(1 - \alpha) + c \Delta W_n] [bh(1 - \alpha) + d \Delta W_n] X_n X_{n-m}\].

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Proof of the theorem (2)

Note that

$$E(\Delta W_n) = 0, \ E[(\Delta W_n)^2] = h$$

and $X_n, X_{n-m+1}, X_{n-m}$ are $\mathcal{F}_{t_n}$-measurable, hence

$$E(\Delta W_n X_i X_j) = E[X_i X_j E(\Delta W_n|\mathcal{F}_{t_n})] = 0,$$
$$E[(\Delta W_n)^2 X_i^2] = E[X_i^2 E((\Delta W_n)^2|\mathcal{F}_{t_n})] = hE(X_i)^2,$$

$i, j \in \{n, n-m+1, n-m\}$.

Let $Y_n = E|X_n|^2$, we have
Proof of the theorem (3)

\[(1 - ah\alpha)^2 Y_{n+1} \leq P(a, b, c, d, h, \alpha) Y_n + Q(a, b, h, \alpha) Y_{n-m+1} + R(a, b, c, d, h, \alpha) Y_{n-m},\]

where

\[P(a, b, c, d, h, \alpha) = \left[1 + ah(1 - \alpha)\right]^2 + |1 + ah(1 - \alpha)| \left(|bh\alpha| + |bh(1 - \alpha)|\right) + |cd|h + c^2h,\]

\[Q(a, b, h, \alpha) = b^2h^2\alpha^2 + |bh\alpha| \left(|1 + ah(1 - \alpha)| + |bh(1 - \alpha)|\right),\]

\[R(a, b, c, d, h, \alpha) = b^2h^2(1 - \alpha)^2 + d^2h + |cd|h + |bh(1 - \alpha)| \left(|1 + ah(1 - \alpha)| + |bh\alpha|\right).\]
Proof of the theorem (4)

Note that condition (3.1) implies $1 - ah\alpha \neq 0$, then

$$Y_{n+1} \leq \frac{1}{(1 - ah\alpha)^2} \left( P(a, b, c, d, h, \alpha) + Q(a, b, h, \alpha) + R(a, b, c, d, h, \alpha) \right) \max\{Y_n, Y_{n-m+1}, Y_{n-m}\}.$$ 

By recursive calculation we conclude that $Y_n \to 0$ ($n \to \infty$) if

$$[P(a, b, c, d, h, \alpha) + Q(a, b, h, \alpha) + R(a, b, c, d, h, \alpha)] < (1 - ah\alpha)^2$$

(3.4)
In Figs 1-3, we consider the equation (2.1) with coefficients $a = -10$, $b = 7$, $c = 1$ and $d = 0.5$. In this case $K = 0.8390$. By Theorem 5, the semi-implicit Euler method is GMS-stable if $0.8390 < \alpha \leq 1$ and MS-stable if $0 \leq \alpha \leq 0.8390$ and $h_0(a, b, c, d) = 1/10$. 

**Figure**: Simulations with fixed parameter $\alpha = 0.9$. Left: $h = 1/4$, right: $h = 1/8$. 

![Numerical examples Fig 1](image-url)
Figure: Simulations with fixed stepsize $h = 1/4$. Upper left: $\alpha = 0$, upper right: $\alpha = 0.2$, lower left: $\alpha = 0.85$, lower right: $\alpha = 1$. 
Figure: Simulations with fixed parameter $\alpha = 0.1$. Upper: $h = 1/5$, middle: $h = 1/10$, lower: $h = 1/20$. 
Figure: Simulations with fixed stepsize $h = 1/2$. Left: $\alpha = 0.01$, right: $\alpha = 0$.

In Fig 4 we consider the equation (2.1) with $a = -0.8$, $b = 0.2$, $c = 0.2$ and $d = 0.2$. We use a large stepsize $h = 1/2$ and choose $\alpha = 0.01$ (left figure), $\alpha = 0$ (right figure). In this case, $K = -0.025 < 0$. It is shown that the method is mean square stable for any $\alpha$ and $h > 0$. 
Mean square stability of the semi-implicit Milstein method
the semi-implicit Milstein method

\[ X_{n+1} = X_n + \left[ \alpha(aX_{n+1} + bX_{n-m+1}) \right. \\
\left. + (1 - \alpha)(aX_n + bX_{n-m}) \right] h + (cX_n + dX_{n-m}) \Delta W_n \\
\left. + \frac{1}{2} (c + d)(cX_n + dX_{n-m})(\Delta W_n^2 - h) \right] 
\]  
\[ (4.1) \]

The convergence result of the semi-implicit Milstein method is

\[ \max_{1 \leq n \leq N} \left( E(X(t_n) - X_n)^2 \right)^{1/2} \leq C h \quad \text{as} \quad h \to 0. \]  
\[ (4.2) \]
Assume the condition (3.1) is satisfied and let

\[ M = \frac{H}{2|a|(|a| + |b|)} \]

\[ K = \frac{H + 2a + 2|b| + (|c| + |d|)^2}{2a(|a| + |b|)} \]

where, \( H = \frac{1}{2} (c + d)^2 (|c| + |d|)^2 + (|a| + |b|)^2 \). Then

- If \( \alpha \geq M \), the semi-implicit Milstein method is GMS-stable.
- If \( \alpha < M \) and \( \alpha > K \), the semi-implicit Milstein method is GMS-stable; if \( \alpha \leq K \), then the semi-implicit Milstein method is MS-stable.

Specially, if \( K < 0 \), then for all \( 0 \leq \alpha \leq 1 \), the semi-implicit Milstein method is GMS-stable.
T-stability of the semi-implicit Euler method for delay differential equations with multiplicative noise
a linear delay differential equations with multiplicative noise

\[
\begin{cases}
    dX(t) = [aX(t) + bX(t - \tau)]dt + cX(t)dW(t),
    t \geq 0, \\
    X(t) = \xi(t), t \in [-\tau, 0],
\end{cases}
\]

- \(a, b, c \in \mathbb{R}\)
- \(\tau\) is a positive fixed delay
- \(W(t)\) is a 1-dimensional standard Wiener process
- \(\xi(t)\) is a \(C([-\tau, 0]; \mathbb{R})\)-valued initial segment
Stochastically asymptotically stable in the large of the analytical solution

Lemma

If

\[ a < -|b| - \frac{1}{2}c^2, \]  

then the solution of Eq.(5.1) is stochastically asymptotically stable in the large, that is

\[ P( \lim_{t \to \infty} X(t, \xi) = 0 ) = 1 \]

for all \( \xi \).
Definition of T-stability

Assume that the condition (5.2) is fulfilled. A numerical scheme equipped with a specified driving process is said to be T-stable if $|X_n| \to 0 (n \to \infty)$ almost surely holds for the driving process, where $X_n$ is the numerical solution produced by the numerical scheme applied to the test equation (5.1).
The semi-implicit Euler method equipped with two-point random variables

The semi-implicit Euler method for Eq. (5.1) has a form as follows:

\[ X_{n+1} = X_n + \left[ \alpha(aX_{n+1} + bX_{n-m+1}) + (1 - \alpha)(aX_n + bX_{n-m}) \right] h + cX_n \Delta W_n, \quad (5.3) \]

where \( \alpha \) is a parameter with \( 0 \leq \alpha \leq 1 \). \( h > 0 \) is a stepsize which satisfies \( \tau = mh \) for a positive integer \( m \), and \( t_n = nh \). \( X_n \) is an approximation to \( X(t_n) \), if \( t_n \leq 0 \), we have \( X_n = \xi(t_n) \).

Moreover, the increments \( \Delta W_n := U_n \sqrt{h} \), \( P(U_n = \pm 1) = 1/2 \), where \( P \) indicates probability.
If condition (5.2) holds, we have $a < 0$, then $1 - ah\alpha > 0$. By (5.3) we have

$$|X_{n+1}| = \frac{1}{(1 - ah\alpha)} |(1 + ah(1 - \alpha) + c\Delta W_n)X_n + bh\alpha X_{n-m+1} + bh(1 - \alpha)X_{n-m}| \leq \frac{1}{(1 - ah\alpha)} \left[ |1 + ah(1 - \alpha) + c\Delta W_n| + |b|h\alpha + |b|h(1 - \alpha) \right] \max\{|X_n|, |X_{n-m+1}|, |X_{n-m}|\}.$$

Obviously, $|X_n| \to 0$ a.s. ($n \to \infty$) if $R(h; \alpha; a, b, c) < 1$ with probability 1, where

$$R(h; \alpha; a, b, c) = \frac{1}{1 - ah\alpha} \left[ |1 + ah(1 - \alpha) + c\Delta W_n| + |b|h \right].$$
Stability function (2)

Considering on the selection of the driving process and the two-point distribution, it is enough to take two steps for averaging. That is

\[ R^{(2)}(h; \alpha; a, b, c) = \frac{1}{(1 - ah\alpha)^2} \left( |1 + ah(1 - \alpha) + c\sqrt{h}| + |b|h \right) \cdot \left( |1 + ah(1 - \alpha) - c\sqrt{h}| + |b|h \right) \]

and \( |X_n| \to 0 \) a.s. as \( n \to \infty \) if

\[ R^{(2)}(h; \alpha; a, b, c) < 1. \quad (5.4) \]
Let condition (5.2) be satisfied. For every \( \alpha \in [0, 1] \), there is a constant \( h_0(\alpha, a, b, c) \). The semi-implicit Euler method equipped with two-point random variables for the driving process is \( T \)-stable, if \( h \in (0, h_0(\alpha, a, b, c)) \).
In the following tests, we show the influence of parameter $\alpha$ and stepsize $h$ on T-stability of the semi-implicit Euler method.

In Figs 1 and 2, we consider the equation (5.5) with coefficients $a = -5$, $b = 4$, $c = 1$. 

\[
\begin{align*}
\left\{ \begin{array}{l}
dX(t) &= [aX(t) + bX(t - 1)]dt + cX(t)dW(t), \quad t \geq 0, \\
X(t) &= t + 1, \quad t \in [-1, 0]. 
\end{array} \right. 
\end{align*}
\] (5.5)
Figure: Simulations with fixed parameter $\alpha = 0.1$. Upper: $h = 1/2$, middle: $h = 1/8$, lower: $h = 1/32$.

$$h_0(\alpha, a, b, c) = 0.0417$$

It is shown that the numerical method is unstable if $h$ is large enough. The fact that the method is stable when $h = 1/8$ demonstrates that the range of $h$ in Theorem 10 is not optimal.
Figure: Simulations with fixed stepsize $h = 1/4$. Upper: $\alpha = 0$, middle: $\alpha = 0.5$, lower: $\alpha = 0.95$.

- $\alpha = 0$, then $h_0(\alpha, a, b, c) = 0.0370$
- $\alpha = 0.5$, then $h_0(\alpha, a, b, c) = 0.4799$
- $\alpha = 0.95$, then $h_0(\alpha, a, b, c) = 23.3137$

We usually take $h < 1$ in numerical computation, so the condition $h < 23.3137$ can be explained as that the method is T-stable for arbitrary $h$. 

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We usually take $h < 1$ in numerical computation, so the condition $h < 23.3137$ can be explained as that the method is T-stable for arbitrary $h$.
Figure: Simulations with fixed parameter $\alpha = 1$ and $h = 1$. Left: $a = -5$, $b = 4$, $c = 1$, right: $a = -4.5$, $b = 2$, $c = 2$.

In Fig 7, we show the T-stability of the semi-implicit Euler method with $\alpha = 1$. In fact, based on the theorem, the numerical method is T-stable for arbitrary $h$. It is shown that the numerical method is T-stable when $\alpha = 1$ even $h$ is very large.
Additionally, we do a numerical test (see Fig 4) to show the T-stability of the semi-implicit Euler method for a two Wiener processes linear problem

\[
\begin{align*}
    dX(t) &= [aX(t) + bX(t - \tau)]dt \\
    &\quad + cX(t)dW_1(t) + c_1X(t - \tau)dW_2(t), \quad t \geq 0, \quad (5.6)
\end{align*}
\]

where \( W_1(t) \) and \( W_2(t) \) are independent 1-dimensional standard Wiener processes. We take \( a = -9, b = 7, c = c_1 = 0.5, \tau = 1 \) and \( X(t) = t + 1, \quad t \in [-\tau, 0] \).
The semi-implicit Euler method for Eq. (5.6) is

\[ X_{n+1} = X_n + \left[ \alpha (aX_{n+1} + bX_{n-m+1}) \right. \\
+ (1 - \alpha) (aX_n + bX_{n-m}) \left. \right] h + cX_n \Delta W^1_n + c_1 X_{n-m} \Delta W^2_n, \]

(5.7)

where the increments

\[ \Delta W^1_n := U^1_n \sqrt{h}, \; \Delta W^2_n := U^2_n \sqrt{h}, \]

\[ P(U^1_n = \pm 1) = P(U^2_n = \pm 1) = 1/2. \]
The T-stability of the semi-implicit Euler method for Eq. (5.6) is similar to it for Eq. (5.1).

**Figure:** Simulations with the semi-implicit Euler method for Eq.(5.6). Upper left: $\alpha = 0.1$, $h = 0.25$, Upper right: $\alpha = 0.1$, $h = 0.125$, lower left: $\alpha = 0.9$, $h = 0.25$, lower right: $\alpha = 1$, $h = 0.5$. 

[Image of the plots showing the simulations for different parameters]
Reference

Reference

Future research

- Numerical scheme for stochastic partial differential equations
- Numerical methods for multi-delay and multi-white noises differential equations
- High order numerical method for stochastic differential equations
- Numerical methods for stochastic differential equations with other stochastic process.
Thank you!

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