Analysis and approximations of peridynamic models and nonlocal diffusion equations

Qiang Du

Penn State University

Joint work with
Richard Lehoucq (Sandia), Max Gunzburger (Florida State),
Kun Zhou, Li Tian, Tadele Mengesha, Thinh Le (Penn State),
Lili Ju (S. Carolina), Xuying Zhao (BCSRC)

Brown University 2012

Research supported in part by DOE, Sandia National Lab and Penn State
Outline

- Recap of model equations:
  peridynamics (Silling 2000), nonlocal diffusion

- Mathematics analysis:
  nonlocal calculus,
  volume constrained problems,
  local limit

- Numerical approximations:
  finite element methods,
  a priori/posteriori error analysis,
  adaptive FEM

- Comments/conclusions
Related works

- **Simulations/experiments with PD**
  - Silling/Askari (2005): meshfree methods
  - Parks/Lehoucq/Plimpton/Silling (2008): PD implemented via MD
  - Seleson/Parks/Gunzburger/Lehoucq (2009): PD as upscaling of MD
  - Chen/Gunzburger (2010): finite element for PD
  - Parks/Seleson (2011): role of influence function

- **Mathematical Analysis of PD**
  - Emmerich/Weckner 2006,2007: $L^p$ theory
  - Du/Zhou 2010: $L^2$ and $H^s$ theory, periodic boundary
  - Alali/Lipton 2010: $L^p$ theory/homogenization
  - Lipton/Mengesha 2011: homogeneization
  - Aksolyu/Mengesha 2011: $L^2$ theory
  - Du/Gunzburger/Lehoucq/Zhou 2011, 2012: $L^2/H^s$, PD state

- **Numerical analysis of PD/nonlocal diffusion**
  - Zhou/Du 2011: FEM convergence, conditioning, error estimates
  - Aksoyulu/Parks 2011: domain decomposition
  - Aksolyu/Mengesha 2010: conditioning, spectral analysis
  - Du/Gunzburger/Lehoucq/Zhou 2012: FEM convergence
  - Du/Ju/Tian/Zhou 2011: a posteriori error estimation
  - Du/Tian/Zhao 2012: convergence of Adaptive FEM
Recap of model equations

We only consider linear models: for displacement field $u$,

$$
\rho \ddot{u}(x, t) = \int_{H_x} C(x', x)(u(x', t) - u(x, t)) \, dx' + b(x, t)
$$

$\rho$: material density,

$b$: external body force,

$C(x', x)$: micromodulus function.

$\delta$: material horizon,

$H_x = B_\delta(x)$: family of $x$

(Silling 2000, 2007)

Mathematical studies of some nonlinear/nonlocal models can be found in
(Du/Kamm/Lehoucq/Parks 2011 SIAM Appl Math)
Three linear models

\[ \rho \ddot{u}(x, t) = \int_{H_x} C(x', x)(u(x', t) - u(x, t)) \, dx' + b(x, t) \]

- PD bond based model: force only depends the particles that form the bond: \( C(x, x') = K_1(x, x') \) where \( K_1(x, x') \) represents direct interaction between \( x \) and \( x' \)

- PD state based model: force is also determined through intermediate particles, \( C(x, x') = K_1(x, x') + C_0(x, x') \) where \( C_0(x, x') \) represents indirect interaction

- Scalar nonlocal PD/diffusion model: \( u(x) \) and \( C(x, x') \) are scalar fields, with \( C(x, x') \) possibly changing sign
Some questions to be addressed

- Well-posedness of variational problems
- Well-posedness of dynamic equations
- Regularity/stability of solutions
- Local limits of the nonlocal models (as $\delta \to 0$)
- Finite element approximations
- Convergence, accuracy, conditioning, adaptivity

- **Key**: a nonlocal calculus, providing a unified framework
Bond-based PD operator

A bond-based PD operator $\mathcal{L}_b$ defined as, $\forall \mathbf{x} \in \Omega_s \subset \mathbb{R}^n$ (solution domain)

$$\mathcal{L}_b \mathbf{u} \left( \mathbf{x} \right) = \int_{\Omega_S \cup \Omega_I} \mathcal{C}(\mathbf{x}, \mathbf{y}) \left( \mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}) \right) d\mathbf{y}$$

$$= \int_{\Omega_S \cup \Omega_I} \omega(|\mathbf{y} - \mathbf{x}|) \alpha(\mathbf{y}, \mathbf{x}) \otimes \alpha(\mathbf{y}, \mathbf{x}) \left( \mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}) \right) d\mathbf{y}.$$  

$\omega(\cdot)$: influence function; $\Omega_I$: interaction domain; $\alpha(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|}$.

Reformulation: 

$$\mathcal{L}_b \mathbf{u} = -\mathcal{D} (\omega \mathcal{G}^* \mathbf{u})$$

for symmetric $\omega(\mathbf{x}, \mathbf{y}) = \omega(\mathbf{y}, \mathbf{x})$ with nonlocal divergence and its dual (adjoint) defined by, $\forall \mathbf{x} \in \Omega_s$,

$$(\mathcal{D} \Psi)(\mathbf{x}) = \int_{\mathbb{R}^n} \left( \Psi(\mathbf{x}, \mathbf{y}) + \Psi(\mathbf{y}, \mathbf{x}) \right) \cdot \alpha(\mathbf{x}, \mathbf{y}) d\mathbf{y}, \quad \forall \Psi,$$

$$(\mathcal{D}^* \mathbf{v})(\mathbf{x}, \mathbf{y}) = -\left( \mathbf{v}(\mathbf{y}) - \mathbf{v}(\mathbf{x}) \right) \otimes \alpha(\mathbf{x}, \mathbf{y}), \quad \forall \mathbf{v},$$

and $\mathcal{G}^* = Tr(\mathcal{D}^*)$, for anti-symmetric vector field $\alpha(\mathbf{x}, \mathbf{y}) = -\alpha(\mathbf{y}, \mathbf{x})$.
Nonlocal vector calculus

- \( \mathcal{D}, \mathcal{D}^*, \mathcal{G}^* \) along with other nonlocal gradient/curl are part of a nonlocal vector calculus recently developed in Du/Gunzburger/Lehoucq/Zhou 2011 Sandia Report (to appear in \( M^3 \)AS, 2013)

- Motivation of the new calculus: to represent nonlocal balance laws

- An axiomatic approach to define nonlocal quantities and relations

- Nonlocal operators like \( \mathcal{D}/\mathcal{D}^* \) map from 1-point to 2-point functions or vice-versa, in the spirit of maps between 0th and 1st order forms

- Past studies on nonlocal operators (nonlocal gradient, nonlocal mean, nonlocal graph Laplacian, ...) focus mostly on scalar fields, our vector and tensor versions systematically generalize existing works.

Eg. For scalar \( u \), consider the nonlocal diffusion operator \( \mathcal{L}_d \)

\[
\mathcal{L}_d u (x) = \int_{\Omega_S \cup \Omega_l} \omega(x, y)(u(y) - u(x)) \, dy.
\]

Reformulation: \[ \mathcal{L}_d u = -\mathcal{D}(\omega \mathcal{D}^* u) \]

Du/Gunzburger/Lehoucq/Zhou 2012, SIAM Rev
Nonlocal diffusion, variational problem

For two-point function $\Psi$, consider the nonlocal operator $\mathcal{N}_d$,

$$
\mathcal{N}_d \Psi (x) = - \int_{\mathbb{R}^n} (\Psi(x, y) + \Psi(y, x)) \cdot \alpha(x, y) \, dy, \quad \forall x \in \Omega_I.
$$

Nonlocal (generalized) Green’s identity

$$
\int_{\Omega_S} v D(\omega D^*(u)) \, dx - \int_{\Omega_S \cup \Omega_I} \int_{\Omega_S \cup \Omega_I} \omega D^*(v) \cdot D^*(u) \, dy \, dx = \int_{\Omega_I} v \mathcal{N}(\omega D^*(u)) \, dx.
$$

Volume Constrained Problems

<table>
<thead>
<tr>
<th>NL-Dirichlet</th>
<th>NL-Neumann</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\mathcal{L}_d u = f$ in $\Omega_S$</td>
<td>$-\mathcal{L}_d u = f$ in $\Omega_S$</td>
</tr>
<tr>
<td>$u = h$ in $\Omega_I$</td>
<td>$\mathcal{N}_d u = g$ in $\Omega_I$</td>
</tr>
</tbody>
</table>

Du/Gunzburger/Lehoucq/Zhou 2012, *SIAM Rev*
Assumptions

To study variational problems, we make assumptions that

- $\Omega_S$: solution domain, bounded with interior cone condition
- $\Omega_I$: set of $y \notin \Omega$ with $\omega(|xb?y|) \neq 0$ for some $x \in \Omega$
- $\Omega_C$: domain of constraints
- $\Omega = (\bar{\Omega}_S \cup \bar{\Omega}_I)^{\circ}$, a connected set

For influence function $\omega(x, y) = \omega(y, x)$, we typically have

- $\omega(x, y) \geq 0$, $\forall y \in B_\delta(x)$, and $\omega(x, y) = 0$ otherwise
- $\omega(x, y) \geq \omega_0 > 0$, $\forall y \in B_{\delta/2}(x)$
- $\int_{B_\delta(x)} |y - x|^2 \omega(x, y) \, dy < \infty$
  (a necessary/sufficient condition for finite elastic moduli/diffusion coefficient)
Nonlocal diffusion: variational formulation

For $-\mathcal{L}_d u = \mathcal{D}(\omega \mathcal{D}^* u) = f$, possibly with a volume constraint

Energy space:
$$V = \{ u \in L^2(\Omega), \sqrt{\omega} \mathcal{D}^* u \in L^2(\Omega^2) \}$$

Bilinear form:
$$B(u, v) = \int_{\Omega} \int_{\Omega} \omega \mathcal{D}^* u \mathcal{D}^* v \, dy \, dx,$$

Weak form: find $u \in V_c$ (constrained subspace of $V$)
$$B(u, v) = \int_{\Omega} fv \, dx, \quad \forall v \in V_c$$

Kernel space: $Z = \{ u \in V, B(u, u) = 0 \}$.

Under suitable conditions on $\omega$ and $V_c$, nonlocal Poincare’s inequalities can be established, thus, $B(\cdot, \cdot)$ gives a well-defined inner product on $V_c$ or $V/Z$, which leads to well-posedness of nonlocal constrained value problems.
Well-posedness

For $V_c$ that is compactly imbedded in $L^2(\Omega)$ and $V_c \cap Z = \{0\}$, the variational problem has a unique solution in $V_c$.

Key: nonlocal Poincare’s inequality.

Examples:

1) $V = L^2(\Omega)$ if $\omega \in L^2(\Omega^2)$.
   (the case of bounded $\mathcal{L}_d$ in $L^2$ with no elliptic smoothing).

2) $V = H^s(\Omega)$ for $s \in (0, 1)$, if there are constants $\gamma^*, \gamma^* > 0$, $\forall \mathbf{x} \in \mathbb{R}^n$,

\[
\omega(\mathbf{x}, \mathbf{y}) \leq \frac{\gamma^*}{|\mathbf{y} - \mathbf{x}|^{n+2s}}, \quad \forall \mathbf{y} \in B_\delta(\mathbf{x}),
\]

\[
\omega(\mathbf{x}, \mathbf{y}) \geq \frac{\gamma^*}{|\mathbf{y} - \mathbf{x}|^{n+2s}}, \quad \forall \mathbf{y} \in B_{\frac{\delta}{2}}(\mathbf{x}).
\]

(the case of unbounded $\mathcal{L}_d$ in $L^2$, elliptic smoothing with order $2s$).
State-based PD model

Strain energy of a linear PD solid (Silling 2007): for deformation state $Y$,

$$W(Y) = \frac{\kappa}{2} \vartheta^2 + \frac{\eta}{2} \int_{\mathbb{R}^3} \left( e\langle \xi \rangle - \frac{\vartheta |\xi|}{3} \right)^2 d\xi .$$

$e$: extension state; $\vartheta$: dilatation; $\varpi$: influence function; $\kappa$ and $\eta$: bulk/shear moduli.

Define, for any point function $U$ and a 2-point function $u$, weighted operators:

$$\mathcal{D}_\omega (U)(x) = \mathcal{D}(\omega U(x))(x) \text{ and } \mathcal{D}_*^\omega (u)(x) = \int_{\mathbb{R}^3} \mathcal{D}^* (u)(x,y) \omega \, dy , \quad \forall x \in \mathbb{R}^3.$$

The operator $\mathcal{D}_\omega$ resembles nonlocal divergence of point functions, indeed, it converges to the conventional divergence in the local limit. These weighted nonlocal operators are presented in Du/Gunzburger/Lehoucq/Zhou 2013 $M^3$AS.

$\mathcal{D}_\omega$ and $\mathcal{D}_*^\omega$ can be used to reformulate a linearized PD state model.
Peridynamic Navier equation

Taking 2nd order approximation of $W(Y)$ w.r.t. the displacement field $u$:

$$\tilde{W}(u) = \frac{\kappa}{2} (\text{Tr}(D^*_\omega u))^2 + \frac{\eta}{2} \int_{\mathbb{R}^3} \omega(x, y) \left( \text{Tr}(D^*_u) - \frac{\text{Tr}(D^*_\omega u) |y - x|}{3} \right)^2 \, dy.$$  

Consider:  
\[ \min \int_{\Omega} \tilde{W}(u) \, dx - \int_{\Omega_S} u \cdot b \, dx, \quad \text{subject to } u = h_b, \quad \text{in } \Omega_I. \]

$\Rightarrow$ PD Navier equation

\[ \left\{ \begin{array}{l}
\eta D_{\omega} (D^*(u))^T + \sigma D_{\omega} (\text{Tr}(D^*_\omega u) I) = b, \quad \text{in } \Omega_s, \\
\quad u = h_d, \quad \text{in } \Omega_c.
\end{array} \right. \]

- Well-posedness in energy space $V$: nonlocal Korn's inequality

Example:  
$V = L^2(\Omega)$ if  
$$\int_{\Omega} \omega^2(x, y) \, dy \leq M, \quad \forall x \in \Omega.$$

- Local limit:  
$$\eta D_{\omega} (D^*(u))^T + \sigma D_{\omega} (\text{Tr}(D^*_\omega u) I) \to \mu \nabla \cdot \nabla + (\mu + \lambda) \nabla \nabla.$$  

(Du/Gunzburger/Lehoucq/Zhou 2011 Sandia Report)
Finite element approximations

Variational formulations of PD and nonlocal models allow systematic analysis of Galerkin type numerical approximations.

Some examples:

  If \( V \equiv H^s_p \) with \( s \in [0, 1) \), we have the error order estimate for the finite element approximation with piecewise polynomials of order \( m \),
  \[
  \| u - u_h \|_{H^s_p} \leq C(\delta) h^{m+1-s} \| b \|_{m+1-2s} \quad \text{if } b \in H^{m+1-2s}.
  \]
  And the stiffness matrix condition number estimate,
  \[
  \text{cond}(A^0) \leq C_1(\delta) h^{-2s}, \quad (\delta : \text{horizon}, \ h : \text{mesh size}).
  \]
  For some common kernels, with \( V \equiv L^2 \), \( \text{cond}(A^0) \leq c \min\{\delta^{-2}, h^{-2}\} \).

Adaptive FEM for nonlocal models

Nonlocal models potentially allow solutions that lack sufficient regularity, thus it is crucial to develop effective adaptive numerical solution algorithms:

Solve $\rightarrow$ Estimate $\rightarrow$ Mark $\rightarrow$ Refine

Adaptive algorithm generates a sequence of nested mesh $\mathcal{T}_0 \subset \mathcal{T}_1 \subset \mathcal{T}_2 \subset \ldots$

Robust error estimator: there are generic positive constants $\{C_i\}_{i=1}^2$:

\[ C_1 \eta^h \leq \| u - u_h \| \leq C_2 \eta^h. \]

A popular type of estimator: residual type a posteriori error estimator
A residual-type a posteriori error estimator

For finite element solution $u_h$ of the nonlocal diffusion problem,

\[
\begin{aligned}
- \mathcal{L}_d u &= b, \quad x \in \Omega_s, \\
 u &= 0 \quad x \in \Omega_I.
\end{aligned}
\]

Let $u_h$ be a Galerkin FE solution, error $e_h = u - u_h$, residual $R_h = b + \mathcal{L}_d u_h$.

**Lemma**

\[
\| e_h \|_V = \| R_h \|_{V^*}
\]

$V^*$: dual of $V_c$.

- True for 2nd order elliptic PDEs, but $H^{-1}$ norms not easily computable
- Meanwhile, for nonlocal $\mathcal{L}_d$ with an integrable kernel, $V \equiv L^2(\Omega) \equiv V^*$.

**Theorem** [Du/Ju/Tian/Zhou 2011]

- For $\eta_\delta^h = M(\delta) \| R_h \|_{L^2}$ ($M(\delta)$: a constant determined by influence function and horizon $\delta$), is a **reliable and efficient** a posteriori error estimator: for some positive constants $C_1$ and $C_2$, independent of $\delta$ and $h$, $C_1 \eta_\delta^h \leq \| e_h \|_V \leq C_2 \eta_\delta^h$.

- For any given mesh, as $\delta \to 0$, the nonlocal residual $R_h$ converges weakly to its **local counterpart** (element-wise residual and flux jump across element.)
A posteriori error estimator

For $L_d$ with a singular influence function such that $V \equiv H^s$ with $s \in (0, 1/2)$, we have

**Theorem [Du/Tian/Zhao 2012]** $\eta^h_\delta = h^s M(\delta)\|R_h\|_{L^2}$ is a reliable estimator: for a constant $C_3$ independent of $\delta$ and $h$, $\|e_h\|_V \leq C_3 \eta^h_\delta$.

**Adaptive algorithm**: pick marking parameter $\theta \in (0, 1]$, initial mesh $T_0$, $k = 0$.

1. **Solve** for discrete solution $u_k$ over the mesh $T_k$;
2. **Estimate** the error $\eta_k|_T$ for each element $T \in T_k$;
3. **Mark** a set $M_k$ of $T_k$ with minimal cardinality such that
   $\sum_{T \in M_k} \eta^2_k|_T \geq \theta \sum_{T \in T_k} \eta^2_k|_T$;
4. **Refine** $M_k$ by a shape regular and nested local refinement to get $T_{k+1}$;
5. Set $k := k + 1$ and go to step (1) until an error tolerance is met.

**Theorem [Du/Tian/Zhao 2012]** Let $\{T_k, u_k, \eta_k\}$ be the sequence of meshes, discrete solutions, and error estimators produced by the above algorithm, then there exists constants $\gamma > 0$ and $0 < \rho < 1$, such that

$$\|u - u_{k+1}\|_V^2 + \gamma \eta^2_{k+1} \leq \rho(\|u - u_k\|_V^2 + \gamma \eta^2_k).$$

In short, the AFEM reduces error at each level of refinement and is convergent.
Numerical Experiments

Example: a 1-d nonlocal diffusion equation with an exact solution having a discontinuity inside mesh element, using discontinuous linear on uniformly refined meshes

Figure: Exact solution and numerical solution.
Numerical Experiments

\[ \delta = 2 \]

<table>
<thead>
<tr>
<th>N</th>
<th>( | \varepsilon^h |_{L^2} )</th>
<th>CR</th>
<th>( | \varepsilon^h |_V )</th>
<th>CR</th>
<th>( \eta^h_{\delta} )</th>
<th>CR</th>
<th>( \eta^h_{\delta} / | \varepsilon^h |_{L^2} )</th>
<th>( \eta^h_{\delta} / | \varepsilon^h |_V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>1.2848e-02</td>
<td>-</td>
<td>9.0822e-03</td>
<td>-</td>
<td>9.0800e-03</td>
<td>-</td>
<td>0.707</td>
<td>1.000</td>
</tr>
<tr>
<td>50</td>
<td>3.9030e-05</td>
<td>8.36</td>
<td>1.5046e-05</td>
<td>9.24</td>
<td>1.5011e-05</td>
<td>9.24</td>
<td>0.385</td>
<td>0.998</td>
</tr>
<tr>
<td>100</td>
<td>9.8456e-06</td>
<td>1.99</td>
<td>3.7442e-06</td>
<td>2.01</td>
<td>3.7398e-06</td>
<td>2.00</td>
<td>0.380</td>
<td>0.999</td>
</tr>
<tr>
<td>200</td>
<td>2.4863e-06</td>
<td>1.99</td>
<td>9.3605e-07</td>
<td>2.00</td>
<td>9.3496e-07</td>
<td>2.00</td>
<td>0.376</td>
<td>0.999</td>
</tr>
<tr>
<td>400</td>
<td>6.2784e-07</td>
<td>1.99</td>
<td>2.3401e-07</td>
<td>2.00</td>
<td>2.3374e-07</td>
<td>2.00</td>
<td>0.372</td>
<td>0.999</td>
</tr>
</tbody>
</table>

\[ \delta = 0.2 \]

<table>
<thead>
<tr>
<th>N</th>
<th>( | \varepsilon^h |_{L^2} )</th>
<th>CR</th>
<th>( | \varepsilon^h |_V )</th>
<th>CR</th>
<th>( \eta^h_{\delta} )</th>
<th>CR</th>
<th>( \eta^h_{\delta} / | \varepsilon^h |_{L^2} )</th>
<th>( \eta^h_{\delta} / | \varepsilon^h |_V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>1.2830e-02</td>
<td>-</td>
<td>9.0735e-02</td>
<td>-</td>
<td>9.0765e-02</td>
<td>-</td>
<td>7.075</td>
<td>1.000</td>
</tr>
<tr>
<td>100</td>
<td>1.0611e-05</td>
<td>1.98</td>
<td>3.7301e-05</td>
<td>2.00</td>
<td>3.7272e-05</td>
<td>2.00</td>
<td>3.513</td>
<td>0.999</td>
</tr>
<tr>
<td>200</td>
<td>2.6690e-06</td>
<td>1.99</td>
<td>9.3211e-06</td>
<td>2.00</td>
<td>9.3175e-06</td>
<td>2.00</td>
<td>3.491</td>
<td>1.000</td>
</tr>
<tr>
<td>400</td>
<td>6.7150e-07</td>
<td>1.99</td>
<td>2.3298e-06</td>
<td>2.00</td>
<td>2.3293e-06</td>
<td>2.00</td>
<td>3.469</td>
<td>1.000</td>
</tr>
</tbody>
</table>

\[ \delta = 0.02 \]

<table>
<thead>
<tr>
<th>N</th>
<th>( | \varepsilon^h |_{L^2} )</th>
<th>CR</th>
<th>( | \varepsilon^h |_V )</th>
<th>CR</th>
<th>( \eta^h_{\delta} )</th>
<th>CR</th>
<th>( \eta^h_{\delta} / | \varepsilon^h |_{L^2} )</th>
<th>( \eta^h_{\delta} / | \varepsilon^h |_V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>4.2893e-05</td>
<td>8.22</td>
<td>1.4908e-03</td>
<td>9.25</td>
<td>1.4906e-03</td>
<td>9.25</td>
<td>34.752</td>
<td>1.000</td>
</tr>
<tr>
<td>100</td>
<td>1.0831e-05</td>
<td>1.99</td>
<td>3.7269e-04</td>
<td>2.00</td>
<td>3.7265e-04</td>
<td>2.00</td>
<td>34.404</td>
<td>1.000</td>
</tr>
<tr>
<td>200</td>
<td>2.7206e-06</td>
<td>1.99</td>
<td>9.3173e-05</td>
<td>2.00</td>
<td>9.3168e-05</td>
<td>2.00</td>
<td>34.245</td>
<td>1.000</td>
</tr>
<tr>
<td>400</td>
<td>6.8441e-07</td>
<td>1.99</td>
<td>2.3293e-05</td>
<td>2.00</td>
<td>2.3292e-05</td>
<td>2.00</td>
<td>34.032</td>
<td>1.000</td>
</tr>
</tbody>
</table>

**Table:** Convergence results for different mesh sizes and \( \delta \)'s (CR: convergence rate)
Numerical Experiments

Example: a 2D nonlocal diffusion model with a solution having discontinuities along a line inside elements, discontinuous linear with uniform refinement.

Figure: Exact error (left) and estimated error (right) ($\delta = 0.1$, $20 \times 20$ mesh).
**Numerical Experiments**

\[ \delta = 0.2 \]

<table>
<thead>
<tr>
<th>N</th>
<th>( | e_h^n |_2 )</th>
<th>CR</th>
<th>( | e_h^n |_V )</th>
<th>CR</th>
<th>( \eta_h^\delta )</th>
<th>CR</th>
<th>( \eta_h^\delta / | e_h^n |_2 )</th>
<th>( \eta_h^\delta / | e_h^n |_V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>7.0078e-02</td>
<td>-</td>
<td>6.9594e-01</td>
<td>-</td>
<td>7.8443e-01</td>
<td>-</td>
<td>11.19</td>
<td>1.13</td>
</tr>
<tr>
<td>20</td>
<td>4.7783e-02</td>
<td>0.55</td>
<td>4.7410e-01</td>
<td>0.55</td>
<td>5.3456e-01</td>
<td>0.55</td>
<td>11.19</td>
<td>1.13</td>
</tr>
<tr>
<td>40</td>
<td>3.2846e-02</td>
<td>0.54</td>
<td>3.2822e-01</td>
<td>0.53</td>
<td>3.7036e-01</td>
<td>0.53</td>
<td>11.28</td>
<td>1.13</td>
</tr>
<tr>
<td>80</td>
<td>2.2966e-02</td>
<td>0.52</td>
<td>2.2950e-01</td>
<td>0.52</td>
<td>2.5896e-01</td>
<td>0.52</td>
<td>11.28</td>
<td>1.13</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>N</th>
<th>( | e_h^n |_2 )</th>
<th>CR</th>
<th>( | e_h^n |_V )</th>
<th>CR</th>
<th>( \eta_h^\delta )</th>
<th>CR</th>
<th>( \eta_h^\delta / | e_h^n |_2 )</th>
<th>( \eta_h^\delta / | e_h^n |_V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>7.5791e-02</td>
<td>-</td>
<td>1.3561e+00</td>
<td>-</td>
<td>1.5005e+00</td>
<td>-</td>
<td>19.80</td>
<td>1.11</td>
</tr>
<tr>
<td>20</td>
<td>5.2745e-02</td>
<td>0.52</td>
<td>9.4233e-01</td>
<td>0.53</td>
<td>1.0575e+00</td>
<td>0.50</td>
<td>20.05</td>
<td>1.12</td>
</tr>
<tr>
<td>40</td>
<td>3.3323e-02</td>
<td>0.66</td>
<td>6.5578e-01</td>
<td>0.52</td>
<td>7.3942e-01</td>
<td>0.52</td>
<td>22.19</td>
<td>1.13</td>
</tr>
<tr>
<td>80</td>
<td>2.3269e-02</td>
<td>0.52</td>
<td>4.5888e-01</td>
<td>0.52</td>
<td>5.1766e-01</td>
<td>0.51</td>
<td>22.25</td>
<td>1.13</td>
</tr>
</tbody>
</table>

**Table:** Convergence results for different mesh sizes and \( \delta \)'s (CR: convergence rate)
**Numerical Experiments**

Example: a 1-d nonlocal diffusion model with a discontinuous exact solution using discontinuous piecewise linear on uniformly/adaptively refined grids.

\[ \omega(x, y) = \delta^{-3/2}|x - y|^{-3/2} \implies V \equiv H^s \text{ with } s = 1/4. \]

Observed optimal rate of convergence:

\[ N^{-1/2+s} = N^{-1/4} \text{ uniform refinement vs. } N^{-2+s} = N^{-7/4} \text{ adaptive refinement.} \]

![Diagram showing convergence rates with uniform and adaptive refinements.](image)

**Figure:** Convergence rate with uniform (left) and adaptive (right) refinements.
Numerical Experiments

Example: a 2-d nonlocal diffusion model with a discontinuous exact solution using discontinuous piecewise constant on uniformly/adaptively refined grids.

\[ \omega(x, y) = \delta^{-3/2}|x - y|^{-5/2} \Rightarrow V \equiv H^s \text{ with } s = 1/4. \]

Figure: Adaptively refined mesh and numerical solution.

Observed optimal rate of convergence:

\[ N^{-1/2+s} = N^{-1/4} \text{ uniform refinement vs. } N^{-1+s} = N^{-3/4} \text{ adaptive refinement.} \]
Summary

> Mathematical properties of the bond-based and state-based PD models, as well as nonlocal diffusion equations, are analyzed via a nonlocal vector calculus framework.

> Error order, condition number, a posteriori error estimators and convergent adaptive algorithms are developed for some linear peridynamic and nonlocal problems.

> Connections/differences between nonlocal/local models are explored.
Comment: motivation for new math concepts

- Widely studied linear advection

  \[ u_t + u_x = 0 \]

  well-defined characteristics, finite speed of propagation,…

- Less studied linear nonlocal advection

  \[ u_t + \int \rho(x, y)u(y)dy = 0 \]

  (for a nonlinear model, see Du-Kamm-Lehoucq-Parks 2011 SIAP)

- Yet, there is a popular discrete analog (or, numerical scheme):

  \[ u_t(x, t) + \frac{(u(x + h, t) - u(x, h))}{h} = 0 \]

  For \( h \) small, nonzero, speed of propagation = \( \infty \) (classical sense)!
  But a dominant finite traveling speed exists \( \Rightarrow \) new (broader) definition.
References


Thank you!