A Survey of Nonlocal Diffusion Equations and the Underlying Jump Processes

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This is joint work with:
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Outline

1. Background and Overview

2. Compound Poisson Processes

3. Survey of Other Topics
   - Infinite Activity Lévy Jump Processes
   - Exit-times and Escape Probabilities
   - Anisotropic Jump Processes
   - Non-Markovian Finite Activity Jump Processes

4. Conclusions
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4. Conclusions
Background

Consider the nonlocal diffusion equation,

\[ u_t(x, t) = \int_{\mathbb{R}^d} (u(y, t) - u(x, t)) \gamma(x, y) \, dy. \]

- \( u \) is a field, e.g., probability density
- “nonlocal”, in contrast to \( u_t(x, t) = \Delta u(x, t) \)
- \( \gamma \) is a nonnegative and symmetric propagator/dispersal kernel, i.e.,

\[ \gamma(x, y) = \gamma(y, x) \]

represents the mechanism “relating” \( x \) to \( y \)
- in many cases, \( \gamma \) has compactly support
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  \[ \gamma(x, y) = \gamma(y, x) \]
  represents the mechanism “relating” \( x \) to \( y \)
- in many cases, \( \gamma \) has compactly support
- \( u(x, t) \) is the probability density function for a jump process \( X_t \)
  note: analogous to classical diffusion and Brownian motion

\[ u_t(x, t) = \Delta u(x, t) \] is the Fokker-Planck equation for \( \sqrt{2} W_t \)
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● model for **anomalous diffusion**, i.e.,

the diffusing particle satisfies

\[ \langle X_t^2 \rangle \sim t^\beta \]

- **normal diffusion** if \( \beta = 1 \)
- **super-diffusion** if \( \beta > 1 \)
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- model for anomalous diffusion, i.e., the diffusing particle satisfies
  $$\langle X_t^2 \rangle \sim t^\beta$$
  - normal diffusion if $\beta = 1$
  - super-diffusion if $\beta > 1$

- an alternative to classical diffusion when Fick’s first law is not valid
  - contaminant flow in groundwater
  - sporadic movement of foraging spider monkeys
  - turbulence in fluids
  - dynamics of financial markets
  - long-range population/disease dispersion
Background

Consider the nonlocal diffusion equation

\[ u_t(x, t) = \int_{\mathbb{R}^d} (u(y, t) - u(x, t)) \gamma(x, y) \, dy. \]

- **master equation** for a jump process \( X_t \)
- **note**: analogous to classical diffusion and Brownian motion

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(a) Brownian Motion
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(a) Brownian Motion  (b) compound Poisson process  (c) \( \alpha \)-stable process
Background

Consider the nonlocal diffusion equation,

\[ u_t(x, t) = \int_{\mathbb{R}^d} (u(y, t) - u(x, t)) \gamma(x, y) \, dy. \]

- let \( \Omega \) denote a open, bounded domain
- constraints are posed on the volume \( \Gamma \subseteq \mathbb{R}^d \setminus \Omega \)
  - constraints are not posed solely on the surface \( \partial \Omega \)
  - intuitive, nonlocal dynamics \( \Rightarrow \) “nonlocal boundaries”
  - necessary, e.g., surface constraints may not be well-defined
Overview

Existing and recent work:

- theory, mathematical analysis, nonlocal calculus, variational formulations, and numerical methods for volume-constrained nonlocal operators
  - F. Andreu, J. M. Mazón, J. D. Rossi, and J. Toledo (2010)
  - X. Chen and M. Gunzburger (2011)

- probabilistic theory relating the unconstrained nonlocal operators to symmetric jump processes
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The contribution of this work:

- the volume-constrained nonlocal diffusion equation is the master equation for a symmetric jump process on a bounded domain
  - well-posed formulation of master equation for a general class of symmetric jump processes on bounded domains
  - boundary conditions for the jump processes prescribe volume constraints
  - numerical method for computing the density that avoids simulation
  - computation of statistics, e.g., exit-times and escape probabilities
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Compound Poisson Processes

Consider the nonlocal diffusion equation

$$u_t(x, t) = \int_{\mathbb{R}^d} (u(y, t) - u(x, t)) \gamma(x, y) \, dy.$$ 

Assumptions:

- \( \gamma \geq 0 \)
- \( \gamma \) is a symmetric, radial function, i.e., \( \gamma(x, y) = \gamma(y, x) = \gamma(|x - y|) \)
- \( \gamma \in L^1(\mathbb{R}^d) \)
- \( \gamma = \frac{1}{\lambda} \phi \), where \( \int_{\mathbb{R}^d} \phi(z) \, dz = 1 \)
Compound Poisson Processes

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So, for now, we focus on the equation

\[ u_t(x, t) = \frac{1}{\lambda} \int_{\mathbb{R}^d} (u(y, t) - u(x, t)) \phi(x - y) \, dy. \]
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$$u_t(x, t) = \frac{1}{\lambda} \int_{\mathbb{R}^d} (u(y, t) - u(x, t)) \phi(x - y) \, dy.$$ 

- master equation for the compound Poisson process

$$Y_t = \sum_{k=1}^{N_t} R_k$$

- wait-times are exponentially distributed with mean $\lambda$
- $N_t$ is a Poisson process with intensity $1/\lambda$
- $R_k \overset{iid}{\sim} \phi$ and independent of $N_t$
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- “behaves like” classical diffusion as nonlocality vanishes, e.g., if \( \lambda = \varepsilon^2 \) and \( \phi(\xi) = 1 - \varepsilon^2 |\xi|^2 + o(\varepsilon^2) \), then as \( \varepsilon \to 0 \),

\[ \hat{u}_t(\xi, t) = \frac{1}{\varepsilon^2} (\phi(\xi) - 1) \hat{u}(\xi, t) \to -|\xi|^2 \hat{u}(\xi, t) \]
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note: we have weak convergence of the underlying processes, e.g.,

\[ Y_t \xrightarrow{d} \sqrt{2} W_t, \]

by the Levy Continuity Theorem
Consider the nonlocal diffusion equation

\[ u_t(x, t) = \frac{1}{\lambda} \int_{\mathbb{R}^d} (u(y, t) - u(x, t)) \phi(x - y) \, dy. \]

Solutions need not be differentiable (or continuous).

Note: operator maps \( L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d) \).

(a) \( \alpha = 2, \varepsilon = 0.0100 \)

(c) Classical diffusion
Compound Poisson Processes

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**Graphs:**

(a) \( \alpha = 2, \varepsilon = 0.0100 \)

(b) \( \alpha = 2, \varepsilon = 0.0025 \)

(c) classical diffusion
Volume Constraints

Consider the nonlocal diffusion equation

\[ u_t(x, t) = \frac{1}{\lambda} \int_{\mathbb{R}^d} (u(y, t) - u(x, t)) \phi(x - y) \, dy. \]

the absorbing volume-constrained problem reads

\[
\begin{cases}
  u_t(x, t) = \frac{1}{\lambda} \int_{\mathbb{R}^d} (u(y, t) - u(x, t)) \phi(x - y) \, dy, & x \in \Omega, \\
  u(x, t) = 0, & x \in \Gamma, \\
  u(x, 0) = u_0(x), & x \in \Omega.
\end{cases}
\]
Numerical Solutions and Simulations

We compare numerical solutions of the master equation to density estimates from simulations of a compound Poisson process.

- $u^h$ is the numerical solution to the nonlocal boundary value problem

$$u^h(x, t) = \sum_{i=1}^{n} c_i(t) 1_{\Omega_i}(x)$$

- $\mu_N^h$ is the density estimate from $N$ simulations

$$\mu_N^h(x, t) = \sum_{i=1}^{m} \left( \frac{\#(Y_t^{(i)} \in \hat{\Omega}_i)}{N\hat{h}} \right) 1_{\hat{\Omega}_i}(x)$$
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Take-home message:

- ability to compute the density of a jump process restricted to a bounded domain via a well-posed master equation suitably constrained
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\[ u_t(x, t) = \int_{\mathbb{R}^d} \left( u(y, t) - u(x, t) \right) \nu(x - y) \, dy, \]

where the so-called Lévy measure \( \nu \) satisfies

\[ \int_{\mathbb{R}^d \setminus B_\delta(0)} \nu(x) \, dx < \infty \quad \text{and} \quad \int_{B_\delta(0)} |x|^2 \nu(x) \, dx < \infty. \]
Infinite Activity Lévy Jump Processes

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\( \nu \in L^1(\mathbb{R}^d) \)

- finite activity, i.e., a finite number of jumps on every compact interval
- the operator maps \( L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d) \)
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- \( \nu \notin L^1(\mathbb{R}^d) \), e.g.,
  \[ \frac{c_1}{|x|^{d+\alpha}} \leq \nu(x) \leq \frac{c_2}{|x|^{d+\alpha}}, \quad \alpha \in (0, 2) \]
  - infinite activity, i.e., an infinite number of jumps on every compact interval
  - the operator maps \( H^{\alpha/2}(\mathbb{R}^d) \to H^{-\alpha/2}(\mathbb{R}^d) \)
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Activity of the jump process \( \Leftrightarrow \) “smoothing” of the solution operator
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we again compare numerical solutions of the master equation to density estimates from simulations of the jump process

- simulations of the process are performed via the Poisson approximation
- note the smoothing of the solution operator

\[ u^h(x, t) \text{ and } \mu_N^{5h} \text{ for } \tau = 0.001 \]
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Exit-times and Escape Probabilities

Consider the volume-constrained nonlocal diffusion equation

\[
\begin{cases}
    u_t(x, t) = \int_{\mathbb{R}^d} (u(y, t) - u(x, t)) \gamma(x, y) \, dy, & x \in \Omega, \\
    u(x, t) = 0, & x \in \Gamma, \\
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\end{cases}
\]

\[F_T(t) = \text{Pr}(T \leq t) = 1 - \text{Pr}(T > t) = 1 - \int_{\Omega} u(x, t) \, dx\]

let \(T\) denote the exit-time random variable for the process, then
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- We have the decomposition into escape probabilities

\[
\int_{\Omega} u(x, t) \, dx = 1 - \sum_k \sum_j M_{\Gamma_j}^{k}(t)
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\]

The formulation of such volume-constrained problems allows for “non-standard” domains, e.g., unconnected domains

- as an illustration, consider \( \Omega = (0, 0.5) \cup (0.6, 1) \) with \( u_0(x) = 2 \cdot 1_{(0,0.5)}(x) \)
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Anisotropic Jump Processes

Consider the nonlocal diffusion equation

\[ u_t(x, t) = \int_{\mathbb{R}^d} (u(y, t) - u(x, t)) \gamma(x, y) \, dy, \]

where \( \gamma \) is not a radial kernel, e.g.,

\[ \gamma(x, y) = 50 \exp \left( -5 \left( x - \frac{3}{4} \right)^2 - 5 \left( y - \frac{3}{4} \right)^2 \right) \mathbf{1}_{(-1/2, 1/2)}(x - y). \]
Anisotropic Jump Processes

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Non-Markovian Finite Activity Jump Processes

The master equation for more general finite activity jump processes is

$$u_t(x, t) = \int_0^t \Lambda(t - t') \int_{\mathbb{R}^d} (u(y, t') - u(x, t')) \gamma(x, y) \, dy \, dt'.$$

$\Lambda$ is a so-called memory kernel capable of incorporating temporal effects of the material, non-Markovian effects, etc.
Non-Markovian Finite Activity Jump Processes

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- \( \Lambda \) is a so-called memory kernel capable of incorporating temporal effects of the material, non-Markovian effects, etc.
- we obtain the nonlocal diffusion equation,

\[ u_t(x, t) = \frac{1}{\lambda} \int_{\mathbb{R}^d} (u(y, t) - u(x, t)) \phi(x - y) \, dy, \]

by taking \( \Lambda(t - t') = \frac{1}{\lambda} \delta(t - t') \) and \( \gamma(x, y) = \phi(x - y) \in L^1(\mathbb{R}^d) \).
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- $\Lambda$ is a so-called memory kernel capable of incorporating temporal effects of the material, non-Markovian effects, etc.
- We obtain the nonlocal diffusion equation,

$$u_t(x, t) = \frac{1}{\lambda} \int_{\mathbb{R}^d} (u(y, t) - u(x, t)) \phi(x - y) \, dy,$$

by taking $\Lambda(t - t') = \frac{1}{\lambda} \delta(t - t')$ and $\gamma(x, y) = \phi(x - y) \in L^1(\mathbb{R}^d)$
- We obtain the nonlocal Cattaneo-Vernotte equation,

$$u_t(x, t) + \frac{\tau}{2} u_{tt}(x, t) = \frac{1}{\beta} \int_{\mathbb{R}^d} (u(y, t) - u(x, t)) \phi(x - y) \, dy,$$

by taking $\Lambda(t - t') = \frac{1}{\beta \tau} \frac{2}{\tau} \exp \left( -\frac{t-t'}{\tau/2} \right)$ and $\gamma(x, y) = \phi(x - y) \in L^1(\mathbb{R}^d)$
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   - Infinite Activity Lévy Jump Processes
   - Exit-times and Escape Probabilities
   - Anisotropic Jump Processes
   - Non-Markovian Finite Activity Jump Processes
4. Conclusions
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• give a probabilistic interpretation of the nonlocal diffusion equation,

\[ u_t(x, t) = \int_{\mathbb{R}^d} (u(y, t) - u(x, t)) \gamma(x, y) \, dy, \]

with volume constraints as the master equation for a symmetric jump process on a bounded domain

• well-posed formulation of master equation for a general class of symmetric jump processes on bounded domains
• boundary conditions for the jump processes prescribe volume constraints
• numerical method for computing the density that avoids simulation
• computation of statistics, e.g., exit-times and escape probabilities
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Thanks for your attention.
References