Now we return to CSF.

Assume that \((x_i u_1(x,t))\) and \((x_i u_2(x,t))\) are solutions to CSF.

Assume in addition that

1) \(u_1(x, 0) > u_2(x, 0), \ x \in [-1, 1]\)
2) \(u_1(x,t_1,t) > u_2(x,t_1,t)\)

Then \(u_1(x,t) > u_2(x,t)\) for every \(x \in [-1, 1]\) and \(t \in [0, T]\)

**Exercise**: Show that

\[
(U_i)_t = \frac{(U_i)_x x}{1 + (U_i)_x^2}
\]

**Proof**: 

\(u_1(x, 0)\)

\(u_2(x, 0)\)
\[ (u_1)_t = \frac{(u_1)_x x}{9 + (u_1)_x^2} \]
\[ (u_2)_t = \frac{(u_2)_x x}{1 + (u_2)_x^2} \]

We prove the result by contradiction: we assume there is a first time \( t > 0 \) where

\[ u_1(x_0, t_0) = u_2(x_0, t_0) \]

\( x_0 \in (-1, 1) \)

\( 1 \) \( u_1(x_0, t) > u_2(x_0, t), \quad \forall t > t_0 \)

\( 2 \) \( u_1(x_1, t_0) \geq u_2(x, t) \)

Consider the function

\[ v(x, t) = u_1(x, t) - u_2(x, t) \geq 0 \]

for \( x \in [-1, 1] \) and \( t \in [0, t_0] \).
For $t_0$ fixed, $x_0$ is a minimum of $u(x,t_0)$:

$u_x(x_0,t_0) = 0 \Rightarrow (u_1)_x(x_0,t_0) = (u_2)_x(x_0,t_0)$

$u_{xx} \geq 0 \Rightarrow (u_1)_{xx}(x_0,t_0) \geq (u_2)_{xx}(x_0,t_0)$

$(u^2)_t(x_0,t_0) \leq 0 =: (u_1)_t(x_0,t_0) \geq (u_2)_t(x_0,t_0)$

From the evolution equations:

$[u_1(x_0,t_0) - u_2(x_0,t_0)]_t = \frac{(u_1)_{xx} - (u_2)_{xx}}{1 + (u_1)_x^2} + 3\frac{(u_1)_x}{1 + (u_1)_x^2}$

This doesn't allow us to conclude, but we can perturb $u$:

$\tilde{u}_e = u_1 - u_2 + \varepsilon \tilde{u} > 0$

Taking $\varepsilon \to 0 \Rightarrow u(\varepsilon, x, t) \to 0$
Theorem: If $\gamma_1(x,t)$ and $\gamma_2(x,t)$ are two bounded closed curves such that $\gamma_1(x,0) \cap \gamma_2(x,0) = \emptyset$, then $\gamma_1(x,t) \cap \gamma_2(x,t) = \emptyset$ while the solutions are defined.

Proof: If they touch at some point, we can locally (after rotation and translation) write them as graphs and apply the previous result.
Corollary: If \( \Phi(x,t) \) is a compact (bounded) curve, the solution can exist at most for a finite point.
Lecture III

Theorem:
Every compact, embedded, $C^2$ curve converges to a point in finite time.

This result was first proved in the mid 80's by a combination of results by Gage & Hamilton and Grayson.

A few alternative proofs have been published later. Here we will analyze one provided by G. Huisken.

There is another proof by B. Andrews.

- T. Bryan

We first check the following simpler result:
**Theorem**

Let $\mathcal{C} : I \times [0,T] \to \mathbb{R}^2$ be an open curve that evolves under curve shortening flow (in its interior). Let

$$d(x,y,t) = \| \mathcal{C}(x,t) - \mathcal{C}(y,t) \|$$

$$l(x,y,t) = \int_x^y \| \mathcal{C}'(\lambda,t) \| \, d\lambda$$

Assume that $\frac{d}{l}$ attains an infimum in the interior at time $t_0$ then

$$\frac{d}{dt} \frac{d}{l} (x,y,t_0) \geq 0$$

with equality when $\mathcal{C}$ is a straight line.

**Remark:** In general, $\frac{d}{l} \leq 1$, since $l$ is the shortest distance. The "isoperimetric quantity" $\frac{d}{l}$ gives a quantitative measure of how $\mathcal{C}$ differs from a straight line.
Some Remarks

For an open curve we need to specify the behavior at the boundary.

Two standard choices to have a well defined problem

1) To prescribe the points

2) To prescribe an angle with fixed lines

Note that if we fix the end points, then

\[
\frac{d}{dt} \left( \frac{d}{e} (P, Q, t) \right) = -\frac{d}{E^2} \frac{d}{dt} E = \frac{d}{E^2} \int_0^a k^2 ds \geq 0
\]

With equality only for a straight line
Ideas of the proofs

Fix $t_0$ and pick $x_0, y_0$ such that $d \left[ (x_0, y_0, t_0) \right]$ is minimum.

We can assume without loss of generality that $\mathcal{D}$ is parametrized by arc-length.

Consider

\[ f_1(\lambda) = \frac{d \left( (x_0 + \lambda, y_0, t_0) \right)}{\ell \left( (x_0 + \lambda, y_0, t_0) \right)} = \frac{d \left( (x_0 + \lambda, y_0, t_0) \right)}{\ell (x_0 + 1, y_0, t_0)} \]

\[ f_1(\lambda) \text{ tiene un mínimo en } \lambda = 0 \Rightarrow (f_1)'(0) = 0 \]
\[
\begin{align*}
\frac{d}{dx} \left[ \left( \nabla (x_0 + t, t_0) - \nabla(y_0, t_0) \right)^2 \right] \\
= \frac{d}{dx} \left[ \nabla(x_0 + t, t_0) - \nabla(y_0, t_0), \nabla(x_0 + t, t_0) - \nabla(y_0, t_0) \right] \\
= 2 \left( \frac{d}{dx} \nabla(x_0 + t, t_0) \right) \cdot \nabla(x_0, t_0) - \nabla(y_0, t_0) \\
= (d^2)_x = 2 \text{d}x \text{d}y \\
\end{align*}
\]

where

\[
\begin{align*}
d_x &= \left< \nabla(x_0, t_0), \mathbf{w} \right> \\
\mathbf{w} &= \nabla(x_0, t_0) - \nabla(y_0, t_0) \\
|\mathbf{w}| &= 1
\end{align*}
\]
On the other hand

\[ l = \int_{x_0}^{x_{0} + l} m(x, s, t) \, dt = y_0 - x_0 - \lambda \]

in arc length parameter

\[ (l')_x = -1 \]

\[ (f'(1))_x (0) = \frac{d}{d x} (l)_x \]

\[ = \frac{d}{d e} \frac{d}{d e} (l')_x \]

\[ = \frac{d}{d e} \frac{d}{d e} (l - \frac{d}{d e} (l')_x) \]

\[ = \langle x(x_0), w \rangle + \frac{d}{d e} \frac{d}{d e} = 0 \]
We can do the same computation for \( f_2(A) = \frac{d}{\ell} \left( x_0, y_0 + 1, t_0 \right) \).

We obtain

\[
\left( f_2 \right)_2 = -\frac{\langle w, \tau(x_0, t_0) \rangle}{\ell} - \frac{d}{\ell^2} = 0
\]

\[
\langle w, \tau(x_0) \rangle - \langle w, \tau(y_0) \rangle = 0
\]

\[
\langle w, \tau(x_0) - \tau(y_0) \rangle = 0
\]

\( |w| = 1, w \neq 0 \)

There are two cases

1. \( \tau(x_0, t_0) = \tau(y_0, t_0) \)

2. \( \tau(x_0, t_0) - \tau(y_0, t_0) \neq 0 \) and \( \tau(x_0, t_0) - \tau(y_0, t_0) \perp w \)
Case 1: Consider

\[ g_1(A) = \frac{d(x_0+1, y_0,A)}{c} \]

\( g_1 \) has a minimum at \( A = 0 \)

\[ \Rightarrow (g_1)_A = 0 \quad \text{and} \quad (g_1)_{AA} \geq 0 \]

\[ (g_1)_{xx}(0) = \langle w, k(x_0,t) - k(x_0,t) \langle y_0, k \rangle \rangle \]

\[ \Rightarrow 0 \leq \langle w, \frac{d}{dt} \left( \frac{e}{\pi (x_0,t)} - \pi (y_0,t) \right) \rangle \]

from CSF

\[ \left( \frac{d}{c} \right)_t = \frac{d_t}{c} - \frac{d_s}{c} \quad \text{and} \quad \left( \frac{d}{c} \right)_x = \frac{d_t}{c} + \frac{d_s}{c} \]

\[ \Rightarrow \frac{y_0}{c} = \frac{\langle w, k(x_0,t) - k(x_0,t) \rangle}{c} \]
Case 2:

\[ g_2(A) = \frac{d(x_0 + A, y_0 - 1, t_0)}{k} \]

and the same result follows
case 2: $f$ case 1
For our main theorem a similar idea can be used but we need to consider a different "isoperimetric profile"

Instead of $\frac{d}{2}$ we use $\frac{d}{\eta(L)}$ where

$$\Psi(L) = \frac{L(t)}{\pi} \sin \left[ \frac{\pi \left( L(t) \right)}{L(\Pi)} \right]$$

where $L(t)$ is the total length at time $t$.

The computation is similar and we leave it to the interested reader.