Lecture II

In the last lecture we computed
\[
\frac{d}{dt} \int (\gamma + t \psi) \cdot \nu = - \int k \psi \, ds = - \int k ds \leq 0
\]

If \( \psi = k \) then this is decreasing (and it is the fastest decreasing direction)

Def: The curve shortening flow is the deformation of a curve in the normal direction with a speed equal to its curvature.

More precisely, we consider \( \gamma: I \times [0, \infty) \to \mathbb{R}^2 \)

\[
\frac{d\gamma}{dt} = k \nu = \kappa s
\]

Remark: For each \( t \) \( \gamma(t, t) \) is a curve.
Example 1:

Assume that $r(t) = (\cos \theta, \sin \theta)$ is a solution to curve shortening flow. Find $r(t)$.

$$\frac{d\theta}{dt} = r'(t) (\cos \theta, \sin \theta)$$

$$K = \frac{1}{r(t)}$$

$$\gamma = - (\cos \theta, \sin \theta)$$

$$\Rightarrow r'(t) (\cos \theta, \sin \theta) = - \frac{1}{r(t)} (\cos \theta, \sin \theta)$$

$$\Rightarrow (r^2)' = 2 \gamma r' = -2 \int_{0}^{t} \gamma d\sigma$$

$$\Rightarrow r^2(t) - r^2(0) = -2t$$

$$r(t) = \sqrt{r^2(0) - 2t}$$

$$\Rightarrow r'(t) = \sqrt{r^2(0) - 2t} (\cos \theta, \sin \theta)$$
What does the previous example represent?

\[ \sqrt{r(0)} - 2t \Rightarrow 2\pi r(t) \]

\[ t = \frac{r^2(0)}{2} \]
Another example:

We look for a solution of the form $(x, u(x) + \frac{x}{2}) = \gamma(x, t)$

\[
\frac{dn}{dt} = (0, 1) = k \cdot \nabla X / \cdot \nabla X
\]

\[
(-) \ (0, 1) \cdot \nabla X = k \cdot \frac{U_{xx}}{(1 + U_{xx})^{3/2}}
\]

\[
S(x) = \int [U_{x}(x, \tau)] d\lambda
\]

\[
= \int_{0}^{x} (1, U_{x}) \cdot d\lambda = \int_{0}^{x} \sqrt{1 + U_{x}^{2}(x, \tau)} d\lambda
\]

\[
\frac{ds}{ds} = \sqrt{1 + U_{x}^{2}} \cdot \frac{dx}{ds}
\]

\[
\frac{1}{U_{x}} \left( \frac{d}{dx} \right)_{0} \frac{dx}{ds}
\]

\[
\partial_{s} (x, t, 1) = \partial_{x} (x, t) \cdot \frac{dx}{ds}
\]

\[
\partial_{s} = \frac{(1, U_{x})}{\sqrt{1 + U_{x}^{2}}} = \zeta \quad \partial_{s} = \frac{(-U_{x}, 1)}{\sqrt{1 + U_{x}^{2}}}
\]

\[
\frac{\partial s}{\partial x} = \sqrt{1 + U_{x}^{2}} \cdot \frac{dx}{ds}
\]

\[
\partial_{ss} = \left[ \frac{(0, U_{xx})}{\sqrt{1 + U_{x}^{2}}} - \zeta \cdot \frac{U_{xx} U_{x}}{(1 + U_{x}^{2})} \right] \frac{dx}{ds}
\]

\[
k = \partial_{ss} \cdot \zeta = \frac{U_{xx}}{(1 + U_{x}^{2})^{3/2}}
\]
\[
\frac{1}{\sqrt{1 + u_x^2}} = \frac{U_{xx}}{(1 + u_x^2)^{3/2}}
\]

\[
\Rightarrow \frac{U_{xx}}{1 + u_x^2} = 1
\]

\[
\Rightarrow (\arctan u_x)_x = 1 \int dx
\]

\[
\arctan u_x = x
\]

\[
\Rightarrow \frac{du}{dx} = \tan x = \frac{\sin x}{\cos x}
\]

\[
u(x) = -\ln(\cos x)
\]

\[
\Rightarrow (x, -\ln(\cos x) + t)
\]

\[
x \in (-\frac{\pi}{2}, \frac{\pi}{2})
\]

\[
t = \theta
\]

Grime paper
The maximum principle

We change now the topic for a bit.

Assume that we have a solution to an ODE of the form

\[-af'' + bf' + c(x) = 0 \text{ for } x \in (0,1)\]

Assume \(a > 0\) and \(c \geq 0\).

**Claim:** \(f\) does not have a maximum in the interior

**Proof:** If \(f\) has a maximum at \(x_0\)

\[f'(x_0) = 0, \quad f''(x) \leq 0\]

\[0 < c(x) = af''(x_0) + bf'(x_0) \leq 0\]

This contradicts that \(c > 0\)
Remark: The same statement is true for the minimum if $c < 0$

Claim: If $f$ satisfies the same as before but $c \geq 0$, the same statements hold.

Idea of the proof:
Check the equation satisfied by $f_e = f(x) + e e^{lx}$

\[
\begin{align*}
f_e'(x) &= f' + \varepsilon L e^{lx} \\
 f_e''(x) &= f'' + \varepsilon L^2 e^{lx} \\
 -\alpha f_e'' + b f_e' &= -\alpha f'' + b f' + (\varepsilon L b - \varepsilon L^2 a) e^{lx} \\
 &= -c_e + (\varepsilon L b - \varepsilon L^2 a) e^{lx} \\
\end{align*}
\]

Eligiendo $L$ adecuado $c_e > 0$

$\Rightarrow f_e$ cumple las hipótesis del caso anterior $f_e(x) \leq \max \{ f_0(x), f_0' \}$

$\Rightarrow$ we conclude by taking $\varepsilon \to 0$
An application of the maximum principle

Assume that \( f_1 \) and \( f_2 \) satisfy

\[-a f_i'' + b f_i' + c = 0 \quad \text{and} \quad f_i(0) = f_i(1)\]

Then: \( f_i \equiv f_2 \)

**Proof:**

Check the equation satisfied

\[
g = f_1 - f_2
\]

\[
g' = f_1' - f_2'
\]

\[
g'' = f_1'' - f_2''
\]

\[
-a g'' + b g' = (-a f_1'' + b f_1') - (-a f_2'' + b f_2')
\]

\[
= -c - (-c) = 0
\]

\[
g(0) = f_1(0) - f_2(0) = 0
\]

\[
g(1) = f_1(1) - f_2(1) = 0
\]

\[
\max g = 0 = \min g
\]

\[
\Rightarrow g \leq 0 \Rightarrow f_1 = f_2
\]
We will be interested in equations of the form
\[-a f_{xx} + bf_x + c = f_t\]
Following the ideas before, we can define
\[\bar{f}(t) = \max_{x \in [0,1]} f(x,t)\]
If $\bar{f}$ is regular enough we would have $c \leq \bar{f}_t$ (or $\bar{f} = f(l_1,t)$ on $t=10+t$)
In particular, if $c \geq 0$, $\bar{f}$ is increasing.

Similarly we can define
\[\underline{f} = \min_{x \in [0,1]} f(x,t)\]
and
\[\underline{f}_t \leq c \text{ (or } \underline{f} = f(l_1,t) \text{ or } \underline{f}_t = f(l_1,t)\])
If $c \leq 0$, $\underline{f}$ is decreasing.
Then

\[
\max_{x,t} f(x,t) = \max_x \max_t f(x,t) = \max_{x,t} f(x,0), \quad \max_t f(0,t) \leq f(x,*) \leq \max_{x,t} f(x,t)
\]

Rem

This is usually called the parabolic boundary
Now we return to CSF:

Assume that \((x_1 u_1(x,t))\) and \((x_1 u_2(x,t))\) are solutions to CSF.

Assume in addition that

1) \(u_1(x,0) > u_2(x,0), \ x \in [-1, 1]\)
2) \(u_1(t,1,t) > u_2(t,1,t)\)

Then \(u_1(x,t) > u_2(x,t)\) for every \(x \in [-1, 1]\) and \(t \in [0,T]\)

**Exercise:** Show that

\[
(U_i)_t = \frac{(U_i)_x x}{1 + (U_i)_x^2}
\]

**Proof:**
Theorem: If $\gamma_1(x,t)$ and $\gamma_2(x,t)$ are two bounded closed curves such that $\gamma_1(x,0) \cap \gamma_2(x,0) = \emptyset$, then $\gamma_1(x,t) \cap \gamma_2(x,t) = \emptyset$ while the solutions are defined.

Proof: If they touch at some point, we can locally (after rotation and translation) write them as graphs and apply the previous result.
Corollary: If $\Omega(x,t)$ is a compact (bounded) curve, the solution can exist at most for a finite point.