Correlation structure of fractional Pearson diffusions

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Fractional differential equations are an important and useful tool in many areas of science and engineering
[1, 2, 3, 4, 5]. In a heterogeneous environment, the coefficients of the diffusion equation will naturally vary in space. Pearson diffusions form a tractable class of variable coefficient diffusion models with polynomial coefficients. The process $X_1(t)$ is called a Pearson diffusion if it solves the stochastic differential equation

$$dX_1(t) = \mu(X_1(t))dt + \sigma(X_1(t))dW(t)$$

with $\mu(x) = a_0 + a_1x$ and $D(x) = \frac{\sigma^2(x)}{\alpha} = d_0 + d_1x + d_2x^2$.

These processes include the Ornstein-Uhlenbeck process and the Cox-Ingersoll-Ross (CIR) process, which are used in finance. Their steady state distributions belong to the class of Pearson distributions. In a fractional Pearson diffusion, the time variable is replaced by an inverse $\alpha$-stable subordinator independent of the process $X_1$ [6]. The resulting stochastic process is non-Markovian, but its density $p_\alpha(x,t)$ of the one dimensional distribution of $X_\alpha(t)$ is governed by the fractional Pearson diffusion equation, obtained by replacing the first time derivative in the Pearson diffusion equation with a Caputo fractional derivative of the same order $0 < \alpha < 1$ [7]:

$$\frac{\partial^\alpha p_\alpha(x,t)}{\partial t^\alpha} = -\frac{\partial}{\partial x} [\mu(x)p_\alpha(x,t)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [\sigma^2(x)p_\alpha(x,t)] .$$

The purpose of this paper is to study the correlation structure of fractional Pearson diffusions in steady state [8]. We show that if $X_1(t)$ is a Pearson diffusion in steady state, so that its correlation function is given by $\text{corr}[X_1(s), X_1(t)] = \exp(-\theta|t-s|)$, $\theta > 0$, $t, s > 0$, then the correlation function of the corresponding fractional Pearson diffusion $X_\alpha(t) = X_1(E_t)$, where $E_t$ is the standard inverse $\alpha$-stable subordinator independent of $X_1$, is given by

$$\text{corr}[X_\alpha(t), X_\alpha(s)] = E_\alpha(-\theta t^\alpha) + \frac{\theta \alpha t^\alpha}{\Gamma(1+\alpha)} \int_0^{\frac{s}{t}} \frac{E_\alpha(-\theta t^\alpha(1-z)^\alpha)}{z^{1-\alpha}} dz$$

(1)

for $t \geq s > 0$, where $E_\alpha(z)$ is the Mittag-Leffler function.

It follows from the expression for the correlation that fractional Pearson diffusions exhibit long-range dependence in the following sense: if $s > 0$ is fixed and $t \to \infty$, then the correlation of $X_\alpha(t)$ and $X_\alpha(s)$ falls off like a power law $t^{-\alpha}$, with exponent equal to the order of the fractional derivative in time.

References