Matrix-based approaches as an emerging framework for numerical solution of initial and boundary value problems for ordinary and partial differential equations of arbitrary real order

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http://www.tuke.sk/podlubny/
A talk
in five short movements
Overture
Is it in the middle of nowhere?

Boston Marathon: 1897
Yonkers Marathon: 1907
Kosice Marathon: 1924
Top travel destinations for 2013

By Lara Brunt, Special to CNN, with CNN Travel staff

updated 4:22 PM EST, Wed January 2, 2013

Košice, Slovakia

Thanks in part to its medieval old town and vibrant mix of Renaissance, Baroque and art nouveau architecture, the compact yet captivating eastern Slovakian city of Košice has been chosen 2013’s European Capital of Culture (along with Marseille in France). This photo was taken at the time of the announcement.
Kosice and surroundings

Tatra Mountains

Caves
Main idea of fractional calculus: Interpolation of operators

\[ f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \frac{d^3 f}{dt^3}, \quad \ldots \]

\[ f, \quad \int f(t) dt, \quad \int dt \int f(t) dt, \quad \int dt \int dt \int f(t) dt, \quad \ldots \]

\[ \ldots, \quad \frac{d^{-2} f}{dt^{-2}}, \quad \frac{d^{-1} f}{dt^{-1}}, \quad f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \ldots \]
From integer to non-integer

\[ x^n = x \cdot x \cdot \ldots \cdot x \]

\[ x^n = e^{n \ln x} \]

\[ n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n - 1) \cdot n, \]

\[ \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0, \]

\[ \Gamma(n+1) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n = n! \]
Fractional Calculus: a response to S&T needs

1695

static models
dynamical models
geometry, algebra
differential and integral calculus

1960s
fractional order modeling
fractional calculus

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The current map of the fractional calculus
Movement I: Scherzo
3 definitions
Riemann–Liouville:
(gooses back to Letnikov, 1870)

\[ aD_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_a^t \frac{f(\tau) d\tau}{(t - \tau)^{\alpha-n+1}}, \quad (n - 1 \leq \alpha < n) \]
Caputo, 1967:

\[ C_a^\alpha D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (n-1 \leq \alpha < n) \]
Grünwald–Letnikov, 1860s
(goes back to Liouville, 1830s)

\[ D^\alpha f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{k=0}^{\left\lfloor \frac{t-a}{h} \right\rfloor} (-1)^k \binom{\alpha}{k} f(t - kh) \]
For good functions
RL, C, and GL definitions are equivalent

For \( aD_t^\alpha f(t) \) with \( n - 1 < \alpha \leq n \) “good” means

\[
f \in C^{(n)}[a, b], \quad f^{(k)}(a) = 0 \ (k = 0, \ldots, n - 1)
\]
3 flavors
\[ a D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_a^t \frac{f(\tau) d\tau}{(t - \tau)^{\alpha-n+1}} \]
$0D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left( -\frac{d}{dt} \right)^n \int_t^b \frac{f(\tau) d\tau}{(\tau-t)^{\alpha-n+1}}$

"past" of $f(t)$

"future" of $f(t)$
\[
\frac{d^\beta \phi(x)}{d|x|^\beta} = D_R^\beta \phi(x) = \frac{1}{2} \left( aD_x^\beta \phi(x) + xD_b^\beta \phi(x) \right)
\]
3 grades
Constant non-integer order (CO)

\[ aD_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_a^t \frac{f(\tau) d\tau}{(t - \tau)^{\alpha-n+1}}, \]

\[ (n - 1 \leq \alpha < n) \]
Variable order (VO)

\[
\mathcal{C}_0 D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(n - \alpha(t))} \int_0^t \frac{f^{(n)}(\tau) \, d\tau}{(t - \tau)^{\alpha(t) + 1 - n}},
\]

\[
(n - 1 \leq \alpha(t) < n)
\]
Distributed order (DO)

\[ aD_t^{\varphi(\alpha)} f(t) = \int_c^d \varphi(\alpha) aD_t^\alpha f(t) d\alpha \]

\[ \int_c^d \varphi(\alpha) d\alpha = 1 \]
Definitions

Grades

CO VO DO

Flavors

RL C GL L R S
Intelligent fitting of data with the help of solutions of differential equations

\[ y = kx + b \]
\[ y = a \sin(wx) + b \cos(wx) \]
\[ y = Ce^{kx} \]
\[ y = Ae^{kx} \sin(wx) + Be^{kx} \cos(wx) \]

\[ y'' = 0 \]
\[ y'' + w^2 y = 0 \]
\[ y' - ky = 0 \]
\[ a_2 y'' + a_1 y' + a_0 y = 0 \]

Instead of postulating the type of the fitting function, we can postulate the type of the differential equation; its coefficients must be determined.

\[ Ay'' + By' + Cy = 0 \]
The Mittag-Leffler function

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha > 0, \quad \beta > 0) \]
Fitting data using the Mittag-Leffler function

\[ y = y_0 e^{kt} \quad \text{and} \quad y'(t) - k y(t) = 0, \quad y(0) = y_0 \]

\[ y = y_0 t^{\beta-1} E_{\alpha, \beta}(a t^\alpha) \]

\[ y = y_0 E_{\alpha, 1}(a t^\alpha) \quad \text{and} \quad _0^C D_t^\alpha y(t) - k y(t) = 0, \quad y(0) = y_0 \]

Fitting the experimental data with the M-L function immediately gives the basic FDE describing the process.
Just supply your data...

Fitting data using the Mittag-Leffler function

by Igor Podlubny
11 Jul 2011 (Updated 02 Apr 2012)

Fitting data using the Mittag-Leffler function.

Watch this File
Just supply your data...

Identification of Parameters of a Half-Order System

Petras, I.; Sierociuk, D.; Podlubny, I.
Signal Processing, IEEE Transactions on
Volume: 60, Issue: 10
Topic(s): Signal Processing & Analysis
Digital Object Identifier: 10.1109/TSP.2012.2205920
Publication Year: 2012, Page(s): 5561 - 5566

Experimental Evidence of Variable-Order Behavior of Ladders and Nested Ladders

Sierociuk, D.; Podlubny, I.; Petras, I.
Control Systems Technology, IEEE Transactions on
Volume: 21, Issue: 2
Topic(s): Signal Processing & Analysis
Digital Object Identifier: 10.1109/TCST.2012.2185932
Publication Year: 2013, Page(s): 459 - 466
Cited by 3
The Queen Function

“In fact, ... functions of Mittag-Leffler type enter as solutions of many problems dealt with fractional calculus so that they like to refer to the **Mittag-Leffler function** to as the **Queen function of Fractional Calculus**,

in contrast with its role of a **Cinderella** function played in the past.”
Mittag-Leffler function:
a complete replacement for the exponential function

Technical communiqué
Mittag–Leffler stability of fractional order nonlinear dynamic systems

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Fitting of experimental data using the Mittag-Leffler function

Igor Podlubny
Ivo Petráš
Tomáš Škovránek
Technical University of Kosice, Slovakia

ON MITTAG-LEFFLER FUNCTIONS AND RELATED DISTRIBUTIONS

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(Received August 29, 1988; revised February 27, 1989)

Abstract. The distribution $E_a(x) = 1 - E_a(-x^a)$, $0 < a ≤ 1; x ≥ 0$, where $E_a(x)$ is the Mittag-Leffler function is studied here with respect to its Laplace transform. Its infinite divisibility and geometric infinite divisibility are proved, along with many other properties. Its relation with stable distribution is established. The Mittag-Leffler process is defined and some of its properties are deduced.

Key words and phrases: Completely monotone function, Laplace transform, infinite divisibility, geometric infinite divisibility, stable process.
Movement 2: Allegro
Operational Matrices ...

... and Fourier series

... and Taylor series

... and Orthogonal Polynomials
- Legendre
- Jacobi
- Chebyshev
- Bernoulli
- Bernstein
- ...

... and Walsh functions

... and Wavelets

... or, Block-Pulse Matrices

... and, in general, any suitable basis functions

Matrix approach to discrete fractional calculus II: Partial fractional differential equations

Igor Podlubny* , Aleksei Chechkin†, Tomas Skovranek*, YangQuan Chen‡,
Blas M. Vinagre Jara§


Discretization of fractional-order operators and fractional differential equations on a non-equidistant mesh

Tomas Skovranek*, Viktor V. Verbičkij**, Yashodhan Tarte***, Igor Podlubny*
**Triangular strip matrices (TSM)**

Lower TSM:

\[
L_N = \begin{bmatrix}
\omega_0 & 0 & 0 & 0 & \cdots & 0 \\
\omega_1 & \omega_0 & 0 & 0 & \cdots & 0 \\
\omega_2 & \omega_1 & \omega_0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\omega_{N-1} & \vdots & \omega_2 & \omega_1 & \omega_0 & 0 \\
\omega_N & \omega_{N-1} & \vdots & \omega_2 & \omega_1 & \omega_0 \\
\end{bmatrix},
\]

Upper TSM:

\[
U_N = \begin{bmatrix}
\omega_0 & \omega_1 & \omega_2 & \cdots & \omega_{N-1} & \omega_N \\
0 & \omega_0 & \omega_1 & \cdots & \cdots & \omega_{N-1} \\
0 & 0 & \omega_0 & \cdots & \omega_2 & \cdots \\
0 & 0 & 0 & \cdots & \omega_1 & \omega_2 \\
\cdots & \cdots & \cdots & \cdots & \omega_0 & \omega_1 \\
0 & 0 & 0 & \cdots & 0 & \omega_0 \\
\end{bmatrix},
\]

If two TSMs are of the same type, then: \( CD = DC \).
Generating functions for TSMs

\[ q(z) = \sum_{k=0}^{\infty} \omega_k z^k \quad \rightarrow \quad \text{trunc}_N(q(z)) \overset{\text{def}}{=} \sum_{k=0}^{N} \omega_k z^k = q_N(z) \]

Function \( q(z) \) generates a sequence of lower TSMs:

\[ L_N, \quad N = 1, 2, \ldots \]

or upper TSMs

\[ U_N, \quad N = 1, 2, \ldots \]

Properties:

\[ \text{trunc}_N(\gamma \lambda(z)) = \gamma \text{trunc}_N(\lambda(z)) \]

\[ \text{trunc}_N(\lambda(z) + \mu(z)) = \text{trunc}_N(\lambda(z)) + \text{trunc}_N(\mu(z)) \]

\[ \text{trunc}_N(\lambda(z)\mu(z)) = \text{trunc}_N(\text{trunc}_N(\lambda(z)) \text{trunc}_N(\mu(z))) \]
Operations with TSMs

\[ A_N = \sum_{k=0}^{N} a_k (E_1^-)^k = \lambda_N(E_1^-), \quad B_N = \sum_{k=0}^{N} b_k (E_1^-)^k = \mu_N(E_1^-), \]

\[ \lambda_N(z) = \text{trunc}_N(\lambda(z)), \quad \mu_N = \text{trunc}_N(\mu(z)) \]

Addition and subtraction:

\[ A_N \pm B_N \leftrightarrow \text{trunc}_N(\lambda(z) \pm \mu(z)) \]

Multiplication by a constant:

\[ \gamma A_N \leftrightarrow \text{trunc}_N(\gamma \lambda(z)) \]

Product of TSMs:

\[ A_N B_N \leftrightarrow \text{trunc}_N(\lambda(z) \mu(z)) \]

Matrix inversion:

\[ (A_N)^{-1} \leftrightarrow \text{trunc}_N(\lambda^{-1}(z)) \]
Left-sided R-L derivatives

\[ aD_{t_k}^\alpha f(t) \approx \frac{\nabla^\alpha f(t_k)}{h^\alpha} = h^{-\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f_{k-j}, \quad k = 0, 1, \ldots, N. \]

\[
\begin{bmatrix}
  h^{-\alpha} \nabla^\alpha f(t_0) \\
  h^{-\alpha} \nabla^\alpha f(t_1) \\
  h^{-\alpha} \nabla^\alpha f(t_2) \\
  \vdots \\
  h^{-\alpha} \nabla^\alpha f(t_{N-1}) \\
  h^{-\alpha} \nabla^\alpha f(t_N)
\end{bmatrix} = \frac{1}{h^\alpha}
\begin{bmatrix}
  \omega_0^{(\alpha)} & 0 & 0 & 0 & \cdots & 0 \\
  \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & 0 & \cdots & 0 \\
  \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  \omega_{N-1}^{(\alpha)} & \cdots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 \\
  \omega_N^{(\alpha)} & \omega_{N-1}^{(\alpha)} & \cdots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)}
\end{bmatrix}
\begin{bmatrix}
  f_0 \\
  f_1 \\
  f_2 \\
  \vdots \\
  f_{N-1} \\
  f_N
\end{bmatrix}
\]

\[ \omega_j^{(\alpha)} = (-1)^j \binom{\alpha}{j}, \quad j = 0, 1, \ldots, N. \]
\[ B_N^\alpha = \frac{1}{h^\alpha} \begin{bmatrix} \omega_0^{(\alpha)} & 0 & 0 & 0 & \cdots & 0 \\ \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & 0 & \cdots & 0 \\ \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_{N-1}^{(\alpha)} & \cdots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 \\ \omega_N^{(\alpha)} & \omega_{N-1}^{(\alpha)} & \cdots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} \end{bmatrix} \]

\[ \beta_\alpha(z) = h^{-\alpha}(1 - z)^\alpha. \]

\[ \begin{align*} B_N^\alpha B_N^\beta &= B_N^\beta B_N^\alpha = B_N^{\alpha+\beta}, \\
\alpha D_t^\alpha (\alpha D_t^\beta f(t)) &= \alpha D_t^\beta (\alpha D_t^\alpha f(t)) = \alpha D_t^{\alpha+\beta} f(t), \end{align*} \]

\[ f^{(k)}(a) = 0, \quad k = 1, 2, \ldots, r - 1, \]

\[ r = \max\{n, m\} \]
Left-sided R-L integration

\[ a D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha - 1} f(\tau) d\tau, \quad (a < t < b), \]

\[ I_N^\alpha = (B_N^\alpha)^{-1}. \]

\[ I_N^\alpha \leftrightarrow \varphi_N(z) = \text{trunc}_N \left( \beta_\alpha^{-1}(z) \right) = \text{trunc}_N \left( h^\alpha(1 - z)^{-\alpha} \right). \]

\[ I_N^\alpha = h^\alpha \begin{bmatrix}
\omega_0^{(-\alpha)} & 0 & 0 & 0 & \cdots & 0 \\
\omega_1^{(-\alpha)} & \omega_0^{(-\alpha)} & 0 & 0 & \cdots & 0 \\
\omega_2^{(-\alpha)} & \omega_1^{(-\alpha)} & \omega_0^{(-\alpha)} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\
\omega_{N-1}^{(-\alpha)} & \cdots & \omega_2^{(-\alpha)} & \omega_1^{(-\alpha)} & \omega_0^{(-\alpha)} & 0 \\
\omega_N^{(-\alpha)} & \omega_{N-1}^{(-\alpha)} & \cdots & \omega_2^{(-\alpha)} & \omega_1^{(-\alpha)} & \omega_0^{(-\alpha)}
\end{bmatrix} \]
Other generating functions for TSM

Christian Lubich’s formulas:

\[ \omega_1^{(\alpha)}(z) = (1 - z)^{\alpha} \]

\[ \omega_2^{(\alpha)}(z) = \left( \frac{3}{2} - 2z + \frac{1}{2}z^2 \right)^{\alpha}, \]

\[ \omega_3^{(\alpha)}(z) = \left( \frac{11}{6} - 3z + \frac{3}{2}z^2 - \frac{1}{3}z^3 \right)^{\alpha}, \]

\[ \omega_4^{(\alpha)}(z) = \left( \frac{25}{12} - 4z + \frac{3}{2}z^2 - \frac{4}{3}z^3 + \frac{1}{4}z^4 \right)^{\alpha}, \]

\[ \omega_5^{(\alpha)}(z) = \left( \frac{137}{60} - 5z + 5z^2 - \frac{10}{3}z^3 + \frac{5}{4}z^4 - \frac{1}{5}z^5 \right)^{\alpha}, \]

\[ \omega_6^{(\alpha)}(z) = \left( \frac{147}{60} - 6z + \frac{15}{2}z^2 - \frac{20}{3}z^3 + \frac{15}{4}z^4 - \frac{6}{5}z^5 + \frac{1}{6} \right)^{\alpha}. \]

Expand these functions in Taylor series and use the coefficients for generating the corresponding TSMs.
Movement 3: Rondo
Example (Bagley–Torvik equation)

\[ a y''(t) + b_0 D_{t}^{3/2} y(t) + c y(t) = f(t) \]

\[ y(0) = 0, \quad y'(0) = 0 \]

\[ \left( a B_n^{(2)} + b B_n^{(3/2)} + c B_n^{(0)} \right) Y_n = F_n \]
Example: Riesz kernel

\[ \frac{1}{\Gamma(1 - \alpha)} \int_{-1}^{1} \frac{y(\tau) \, d\tau}{|t - \tau|^\alpha} = 1, \quad (-1 < t < 1), \]

Exact solution:

\[ y(t) = \pi^{-1} \Gamma(1 - \alpha) \cos \left( \frac{\alpha \pi}{2} \right) (1 - t^2)^{(\alpha - 1)/2}. \]

Numerical solution:

\[ _{-1}D_t^{-(1-\alpha)} y(t) + _{t}D_t^{-(1-\alpha)} y(t) = 1, \]

\[ (B_N^{-(1-\alpha)} + F_N^{-(1-\alpha)}) Y_N = F_N \]

Historically the first example of numerical solution of equations with left-sided and right-sided fractional-order operators.
Example: Riesz kernel

\[ \alpha = 0.8 \]
Example (Caputo derivatives)

\[ y^{(\alpha)}(t) + y(t) = 1, \]
\[ y(0) = 0, \quad y'(0) = 0, \]

Exact solution:

\[ y(t) = t^\alpha E_{\alpha,\alpha+1}(-t^\alpha). \]

Numerical solution:

\[ \{ B_{N-2}^\alpha + E_{N-2} \} \{ S_{0,1} Y_N \} = S_{0,1} F_N. \]

and from the initial conditions we have:

\[ y_0 = y_1 = 0 \]

Monday, June 3, 2013
Example (Caputo derivatives)
Nonlinear FDEs? Not a problem


\[ R(t)D_{*}^{1/2}R(t) = R(t)\ln R(t) + Eq(t), \quad R(0) = 0. \]  

Here \( D_{*}^{1/2} \) denotes the Caputo differential operator of order 1/2, defined by (see, e.g., [6])

\[ D_{*}^{1/2}y(t) = \frac{1}{\sqrt{\pi}} \int_{0}^{t} (t - s)^{-1/2}y'(s) \, ds. \]

Moreover the function \( q \) describes a point source energy that depends on the time, and therefore it is assumed to be nonnegative, continuous and integrable.

The nonlinear algebraic system is solved by Newton’s method.
Movement 4: Ritonello
Kronecker matrix product

\[ A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1m} \\
  a_{21} & a_{22} & \cdots & a_{2m} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nm}
\end{bmatrix}, \quad B = \begin{bmatrix}
  b_{11} & b_{12} & \cdots & b_{1q} \\
  b_{21} & b_{22} & \cdots & b_{2q} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{p1} & b_{p2} & \cdots & b_{pq}
\end{bmatrix} \]

Kronecker matrix product:

\[ A \otimes B = \begin{bmatrix}
  a_{11}B & a_{12}B & \cdots & a_{1m}B \\
  a_{21}B & a_{22}B & \cdots & a_{2m}B \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1}B & a_{n2}B & \cdots & a_{nm}B
\end{bmatrix} \]
Kronecker matrix product

Important properties:

- if $A$ and $B$ are band matrices, then $A \otimes B$ is also a band matrix,
- if $A$ and $B$ are lower (upper) triangular, then $A \otimes B$ is also lower (upper) triangular.
Kronecker matrix product

Kronecker products with identity matrices: example:

\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \]

\[ E_2 \otimes A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 & a_{21} & a_{22} & a_{23} \end{bmatrix} \]

\[ A \otimes E_3 = \begin{bmatrix} a_{11} & 0 & 0 & a_{12} & 0 & 0 & a_{13} & 0 & 0 \\ 0 & a_{11} & 0 & 0 & a_{12} & 0 & 0 & a_{13} & 0 \\ 0 & 0 & a_{11} & 0 & 0 & a_{12} & 0 & 0 & a_{13} \\ a_{21} & 0 & 0 & a_{22} & 0 & 0 & a_{23} & 0 & 0 \\ 0 & a_{21} & 0 & 0 & a_{22} & 0 & 0 & a_{23} & 0 \\ 0 & 0 & a_{21} & 0 & 0 & a_{22} & 0 & 0 & a_{23} \end{bmatrix} \]
Discretization schemes

Integer orders:

$$\tau$$

$$h$$

$$\frac{\partial U}{\partial t}$$

$$\frac{\partial^2 U}{\partial x^2}$$

Fractional orders:

$$\tau$$

$$h$$

$$\frac{\partial^2 U}{\partial x^2}$$

$$_{0}D_{t}^{\alpha}U$$

$$\frac{\partial^2 U}{\partial |x|^{\beta}}$$

$$_{0}D_{t}^{\alpha}U$$

Monday, June 3, 2013
Symmetric
Riesz fractional derivative

\[
\frac{d^\beta \phi(x)}{d|x|^\beta} = D_R^\beta \phi(x) = \frac{1}{2} \left( a D_x^\beta \phi(x) + x D_b^\beta \phi(x) \right)
\]

Riemann-Liouville

RIESZ POTENTIAL OPERATORS AND INVERSES VIA FRACTIONAL CENTRED DERIVATIVES

MANUEL DUARTE ORTIGUEIRA

Received 2 January 2006; Revised 4 May 2006; Accepted 7 May 2006

\[
R_{m}^{(\beta)} = h^{-\beta} \begin{bmatrix}
\omega_0^{(\beta)} & \omega_1^{(\beta)} & \omega_2^{(\beta)} & \omega_3^{(\beta)} & \cdots & \omega_m^{(\beta)} \\
\omega_1^{(\beta)} & \omega_0^{(\beta)} & \omega_2^{(\beta)} & \omega_3^{(\beta)} & \cdots & \omega_{m-1}^{(\beta)} \\
\omega_2^{(\beta)} & \omega_1^{(\beta)} & \omega_0^{(\beta)} & \omega_3^{(\beta)} & \cdots & \omega_{m-2}^{(\beta)} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\omega_m^{(\beta)} & \omega_{m-1}^{(\beta)} & \omega_{m-2}^{(\beta)} & \omega_{m-3}^{(\beta)} & \cdots & \omega_0^{(\beta)}
\end{bmatrix}
\]

\[
\omega_k^{(\beta)} = \frac{(-1)^k \Gamma(\beta + 1) \cos(\beta \pi/2)}{\Gamma(\beta/2 - k + 1) \Gamma(\beta/2 + k + 1)}
\]
Discretization grid

Nodes and their right-to-left, and bottom-to-top numbering.
Discretization using TSMs

\[ \tau \]

\[ \frac{\partial \beta U}{\partial |x|^\beta} \]

\[ 0D_t^\alpha U \]

\[ 0D_t^\alpha U - a^2 \frac{\partial \beta U}{\partial |x|^\beta} = F \]

\[ \left\{ B_n^\alpha \otimes E_m - a^2 E_n \otimes R_m^\beta \right\} u_{nm} = f_{nm} \]
Structure of the system matrix

\[ C_0 D_t^{\alpha} U - a^2 \frac{\partial^2 U}{\partial x^2} = F \]
Test example

\[ U_t = a^2 U_{xx} \]
\[ U(0, t) = 0, \quad U(L, t) = 0 \]
\[ U(x, 0) = \frac{4x(L - x)}{L^2} \]


<table>
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<tr>
<th>n</th>
<th>m</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tr>
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<td>0.81333</td>
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<tr>
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<td>0.60037</td>
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<td>0.9600</td>
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<tr>
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<td>0.58790</td>
<td>0.78673</td>
<td>0.90667</td>
<td>0.9467</td>
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</tr>
<tr>
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<td>0</td>
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<td>0.34696</td>
<td>0.47731</td>
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<td>0.33572</td>
<td>0.46189</td>
<td>0.54280</td>
<td>1.57066</td>
<td></td>
</tr>
</tbody>
</table>

\[ 0.3484 \quad 0.6269 \quad 0.8267 \quad 0.9467 \quad 0.9867 \]
\[ 0.3380 \quad 0.6141 \quad 0.8135 \quad 0.9334 \quad 0.9733 \]
\[ 0.3287 \quad 0.6017 \quad 0.8003 \quad 0.9201 \quad 0.9600 \]
\[ 0.3203 \quad 0.5897 \quad 0.7873 \quad 0.9068 \quad 0.9467 \]
\[ 0.1843 \quad 0.3504 \quad 0.4817 \quad 0.5659 \quad 0.5948 \]
\[ 0.1813 \quad 0.3447 \quad 0.4740 \quad 0.5568 \quad 0.5853 \]
\[ 0.1784 \quad 0.3391 \quad 0.4664 \quad 0.5479 \quad 0.5760 \]
Example: Time-space fractional diffusion equation

\[ C_0 D_t^\alpha y - \frac{\partial^\beta y}{\partial |x|^\beta} = f(x, t), \quad \text{(with } f(x, t) \equiv 8) \]

\[ y(0, t) = 0, \quad y(1, t) = 0; \quad y(x, 0) = 0. \]

\[ \left\{ B_n^{(\alpha)} \otimes E_m - E_n \otimes R_m^{(\beta)} \right\} y_{nm} = f_{nm} \]
Example: Time-space fractional diffusion equation with delayed fractional derivative

\[ \frac{1}{2} \left\{ c D_t^\alpha y + c D_{t-\delta}^\gamma y \right\} - \frac{\partial^\beta y}{\partial |x|^\beta} = f(x, t) \quad (\text{with } f(x, t) \equiv 8) \]

\[ y(0, t) = 0, \quad y(1, t) = 0 \quad y(x, 0) = 0 \]

\[ \left\{ \frac{1}{2} (B_n^{(\alpha)} \otimes E_m + _k B_n^{(\gamma)} \otimes E_m) - E_n \otimes R_m^{(\beta)} \right\} y_{nm} = f_{nm} \]

Historically the first example of numerical solution of fractional differential equations with delayed fractional derivatives
Non-equidistant grids: they are everywhere

Venice, Italy, August 2010

\[
\frac{1}{\hat{h}_i} \left( \frac{w_{i+1} - w_i}{h_i} - \frac{w_i - w_{i-1}}{h_{i-1}} \right) \quad h_i = x_{i+1} - x_i, \\
\hat{h}_i = \hat{x}_i - \hat{x}_{i-1},
\]
Change the viewpoint:

Left-sided fractional derivatives: inverse of left-sided fractional integrals

\[ B^\alpha_N = (I^\alpha_N)^{-1} \]

Any approximation of fractional integration after inversion gives an approximation for fractional differentiation on the same grid!
The simplest approach: approximation by a piecewise constant function

\[ B_N^\alpha = (I_N^\alpha)^{-1} \]

Coefficients of \( I_N^\alpha \)

\[ I_{k,j} = \frac{(t_k - t_{j-1})^\alpha - (t_k - t_j)^\alpha}{\Gamma(\alpha + 1)}, \]

\[ j = 1, \ldots, k; \quad k = 1, \ldots, N. \]

For non-equidistant grids, the matrix is not a TSM.
Example: fractional integrals of $\sin(x)$
Example: fractional derivatives of $\sin(x)$
Example: two-term ordinary FDE

\[ y^{(\alpha)}(t) + y(t) = 1, \]
\[ y(0) = 0, \quad y'(0) = 0. \]

Exact solution:
\[ y(t) = t^\alpha E_{\alpha,\alpha+1}(-t^\alpha). \]

\[ \alpha = 1.8, \quad \text{number of (random) discretization nodes } N = 500 \]
Example: Bagley-Torvik equation

\[ Ay''(t) + By^{3/2}(t) + Cy(t) = F(t), \]

\[ F(t) = \begin{cases} 
8, & (0 \leq t \leq 1) \\
0, & (t > 1) 
\end{cases}, \quad y(0) = y'(0) = 0. \]

\[ A = 1, \quad B = 1, \quad C' = 1. \]
Variable-order fractional differentiation and integration (VO-FD, VO-FI)

\[ D_{c+}^{-q(t)} f(t) = \frac{1}{\Gamma[q(t)]]} \int_c^t (t - \sigma)^{q(t)-1} f(\sigma) d\sigma. \]

\[ D_{c+}^q f(t) = \frac{1}{\Gamma[1 - q(t)]]} \frac{d}{dt} \int_c^t \frac{f(\sigma)}{(t - \sigma)^{q(t)}} d\sigma. \]

Research Article
On the Selection and Meaning of Variable Order Operators for Dynamic Modeling

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Monday, June 3, 2013
Variable-order Fractional Differentiation (VOFD)

Left-sided

\[ C_0D_t^\alpha(t) f(t) = \frac{1}{\Gamma(n-\alpha(t))} \int_0^t \frac{f^{(n)}(\tau) \, d\tau}{(t-\tau)^{\alpha(t)+1-n}}, \quad (n-1 \leq \alpha(t) < n) \]

\[ GL_0D_t^\alpha(t) f(t) = \lim_{h \to 0} h^{-\alpha(t)} \sum_{k=0}^{n} (-1)^k \binom{\alpha(t)}{k} f(t-kh), \]

\[ (n-1 \leq \alpha(t) < n). \]

Right-sided

\[ C_bD_t^\alpha(t) f(t) = \frac{(-1)^n}{\Gamma(n-\alpha(t))} \int_t^b \frac{f^{(n)}(\tau) \, d\tau}{(\tau-t)^{\alpha(t)+1-n}}, \quad (n-1 \leq \alpha(t) < n). \]

\[ GL_bD_t^\alpha(t) f(t) = \lim_{h \to 0} (-1)^n h^{-\alpha(t)} \sum_{k=0}^{n} (-1)^k \binom{\alpha(t)}{k} f(t-kh). \]

\[ (n-1 \leq \alpha(t) < n) \]

Symmetric

\[ \frac{d^{\beta(x)} \phi(x)}{d|x|^{\beta(x)}} = D_R^{\beta(x)} \phi(x) = \frac{1}{2} \left( a D_x^{\beta(x)} \phi(x) + b D_b^{\beta(x)} \phi(x) \right), \]
Discretization of left-sided VOFD

\[
\begin{bmatrix}
    v^{(\alpha_n)}_n & v^{(\alpha_{n-1})}_{n-1} & \cdots & v^{(\alpha_1)}_1 & v^{(\alpha_0)}_0
\end{bmatrix}^T = B^{(\alpha(t))}_n
\begin{bmatrix}
v_n & v_{n-1} & \cdots & v_1 & v_0
\end{bmatrix}^T
\]

where

\[
B^{(\alpha(t))}_n =
\begin{bmatrix}
    \omega^{(\alpha_n)}_0 & \omega^{(\alpha_n)}_1 & \cdots & \cdots & \omega^{(\alpha_n)}_{n-1} & \omega^{(\alpha_n)}_n \\
    0 & \omega^{(\alpha_{n-1})}_0 & \omega^{(\alpha_{n-1})}_1 & \cdots & \cdots & \omega^{(\alpha_{n-1})}_{n-1} \\
    0 & 0 & \omega^{(\alpha_{n-2})}_0 & \omega^{(\alpha_{n-2})}_1 & \cdots & \cdots \\
    \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    0 & \cdots & 0 & 0 & \omega^{(\alpha_1)}_0 & \omega^{(\alpha_1)}_1 \\
    0 & 0 & \cdots & 0 & 0 & \omega^{(\alpha_0)}_0
\end{bmatrix}
\]

\[
\omega^{(\alpha_k)}_j = \tau^{-\alpha_k} (-1)^j \binom{\alpha_k}{j}, \quad j = 0, 1, \ldots, k; \quad k = 0, 1, \ldots, n.
\]

\[
\alpha_k = \alpha(t_k), \quad t_k = k\tau, \quad k = 0, 1, \ldots, n.
\]
Example 0: VO-FD of function \( y(t) = t \)

\[
\alpha = \begin{cases} 
0, & 0 \leq t < 2 \\
0.5, & 2 \leq t < 4 \\
1, & 4 \leq t \leq 5 
\end{cases}
\]

\[
\alpha = \begin{cases} 
0.25t, & 0 \leq t < 4 \\
1, & 4 \leq t \leq 5 
\end{cases}
\]

Matlab functions:
- voban
- vofan
- voran

```matlab
VO_alpha = '0 * ((t>0) & (t<2)) + 0.5 * ((t>=2) & (t<4)) + 1 * ((t>=4) & (t<=5))';
Dalphat = voban(VO_alpha, [0 5], h); % differentiation matrix
VOFDy = Dalphat * y'; % VO-derivative

VO_alpha = '0.25*t.*(t>=0)&(t<=4)) + 1*(t>4)'; % variable order
Dalphat = voban(VO_alpha, [0 5], h); % differentiation matrix
VOFDy = Dalphat * y'; % VO-derivative
```
Example 1: VO-fractional relaxation equation (1)

\[ 0 D_t^{\alpha(t)} x(t) + Bx(t) = f(t), \quad (0 < \alpha(t) \leq 1), \]
\[ x(0) = 1, \]

\[ \alpha(t) = e^{-At} \text{ with } A = 0.01 \]
\[ B = 0.1 \quad f(t) = 0. \]

"terminal" solutions for \( \alpha(t) = 1 \) and \( \alpha(t) = 0.9512 \)

\[ 0 D_t^{\alpha} x(t) + Bx(t) = 0, \quad (0 < \alpha \leq 1), \]
\[ x(0) = 1, \]
\[ x(t) = E_{\alpha,1}(-Bt^\alpha) \]

\[ \alpha = 1 \]
\[ x(t) = E_{1,1}(-Bt) = e^{-Bt} \]

\[ \alpha = e^{-0.05} = 0.9512 \]
\[ x(t) = E_{0.9512,1}(-Bt) \]

Figure 1: The shape of the variable order \( \alpha(t) = e^{-At} \) for \( A = 0.01 \)
Example 1: VO-fractional relaxation equation (3)

Figure 2: Solutions of problem (14) with \( f(t) = 0 \) and \( B = 0.1 \) for \( \alpha = 1 \) (red line), \( \alpha = \exp(-0.05) \) (green line), and \( \alpha(t) = \exp(-0.01 \cdot t) \) (black line). The discretization step is \( \Delta t = 0.01 \).
DO-fractional derivatives

Left-sided

\[ a D_t^{\varphi(\alpha)} f(t) = \int_c^d \varphi(\alpha) a D_t^\alpha f(t) d\alpha \]

Right-sided

\[ b D_t^{\varphi(\alpha)} f(t) = \int_c^d \varphi(\alpha) b D_t^\alpha f(t) d\alpha \]

Symmetric

\[ a R_b^{\varphi(\alpha)} f(t) = \int_c^d \varphi(\alpha) a R_b^\alpha f(t) d\alpha \]

Restriction:

\[ \int_c^d \varphi(\alpha) d\alpha = 1 \]
Interpretation of DO operators

\[ \varphi(\alpha) \]
Discretization of DO-FDs:

a piece of cake!
Discretization of DO-FDs:

a piece of cake!

\[ \int_c^d \varphi(\alpha) \, 0D^\alpha f(t) \, d\alpha = 0D^\varphi(\alpha) f(t) \]

\[ 0D_t^\alpha f(t) \approx B_{n,m}^{\varphi(\alpha)} f_n, \quad B_{n,m}^{\varphi(\alpha)} = \sum_{k=1}^m B_n^{\alpha_k} \varphi(\alpha_k) \Delta \alpha_k \]
Movement 5: “All Together Now!”
A toolbox for you!

Instant solutions:
1. add water
2. microwave
3. stir

Contents
- 1. What is in the box?
- 2. Evaluation of integer order derivatives
- 3. Evaluation of left-sided Riemann-Liouville fractional derivatives of a constant
- 4. Evaluation of right-sided Riemann-Liouville fractional derivatives of a constant
- 5. Fractional integral equations: an equation with the Riesz kernel
- 6. Symmetric Riesz derivatives
- 7. Solution of ordinary fractional differential equations: the Bagley-Torvick equation
- 8. Solution of partial fractional differential equations: fractional diffusion equation
- 9. Partial fractional differential equations with delayed fractional derivatives
- 10. Conclusion
- 11. Acknowledgments
- References
What is in the cup?

- bcrecur.m
- eliminator.m
- shift.m

Constant orders
- ban.m
- fan.m
- ranort.m
- ransym.m

Variable orders
- voban.m
- vofan.m
- voranort.m
- voransym.m

Distributed orders
- doban.m
- dofan.m
- doranort.m
- doransym.m

ready for release
Example TI: CO-fractional relaxation equation

\[ cD_t^\alpha x(t) + Bx(t) = 0, \quad (0 < \alpha \leq 1), \]
\[ x(0) = 1, \]

\[ h = 0.01; \quad \% \text{step of discretization} \]
\[ t = 0:h:5; \quad \% \text{as in [DOPDS-paper, caption to Fig.6]} \]
\[ N = \text{length}(t) + 1; \quad \% \text{number of nodes} \]
\[ B = 0.1; \quad \% \text{coefficient of the equation} \]
\[ f = '0 + 0*t'; \quad \% \text{RHS, as in [DOPDS-paper, caption to Fig.6]} \]
\[ M = \text{zeros}(N,N); \quad \% \text{pre-allocate matrix } M \text{ for the system} \]
\[ \alpha = \exp(-0.01*5); \quad \% \text{beta} = 0.9512, \text{ order of equation} \]

\% First, we make the matrix for the entire equation -- this is really easy:
\[ M = \text{han}(\alpha, N-1, h) + B*\text{eye}(N-1,N-1); \]

\% Then we compute the right-hand side at discretization
\[ F = \text{eval}([f '-B'], t); \]

\% Utilize zero initial condition:
\[ M = \text{eliminator}(N-1,[1])*M*\text{eliminator}(N-1,[1]'); \]
\[ F = \text{eliminator}(N-1,[1])*F; \]

\% And solve the system \( MY = F \):
\[ Y = M\backslash F; \]

\% Pre-pend the zero initial value (that one due to zero\]
\[ Y0 = [0; Y]; \]

\% Plot the solution:
\[ \text{plot}(t, Y0+1, 'g') \]

Monday, June 3, 2013
Example T2: VO-fractional relaxation equation

$$0 D_t^{\alpha(t)} x(t) + B x(t) = f(t), \quad (0 < \alpha(t) \leq 1),$$

$$x(0) = 1,$$

$$\alpha(t) = e^{-At}, \quad \text{for } A = 0.01$$

\begin{verbatim}
\textbf{Example code:}

h = 0.01; \quad \% step of discretization
t = 0:h:5; \quad \% as in [DOFDS-paper, caption to Fig.6]
N = length(t) + 1; \quad \% number of nodes
B = 0.1; \quad \% coefficient of the equation
\quad \% as in [DOFDS-paper, caption to Fig.6]
f = '0 + 0*t'; \quad \% RHS, as in [DOFDS-paper, caption to Fig.6]
M = zeros(N,N); \quad \% pre-allocate matrix M for the system

\% First, we make the matrix for the entire equation -- this
M = voban('exp(-0.01*t)', [0 5], h) + B*eye(N-1,N-1);

\% Then we compute the right-hand side at discretization nodes:
F = eval ([f '-B'], t);

\% Utilize zero initial condition:
M = eliminator(N-1,[1])*M*eliminator(N-1, [1]')
F = eliminator(N-1,[1])*F;

\% And solve the system MY=F:
Y = M\F;

\% Pre-pend the zero initial value
\% (that one due to zero initial condition)
Y0 = [0; Y];

\% Plot the solution:
U = Y0 + 1;
plot(t, U, 'k')
\end{verbatim}

Monday, June 3, 2013
Example T3: DO-fractional relaxation equation

\[ 0 D_t^\varphi(\alpha)x(t) + Bx(t) = f(t), \]
\[ x(0) = 1 \]
\[ \varphi(\alpha) = 6 \alpha (1 - \alpha), \]
\[ 0 \leq \alpha \leq 1 \]

\[ h = 0.01; \quad \text{% step of discretization} \]
\[ t = 0:h:5; \quad \text{% as in [DOFDS-paper, caption to Fig.6]} \]
\[ N = \text{length}(t) + 1; \quad \text{% number of nodes} \]
\[ B = 0.1; \quad \text{% coefficient of the equation} \]
\[ f = '0 + 0*t'; \quad \text{% RHS, as in [DOFDS-paper, caption to Fig.6]} \]
\[ M = \text{zeros}(N,N); \quad \text{% pre-allocate matrix M for the system} \]

\% First, we make the matrix for the entire equation -- this is really easy:
\[ M = \text{doban}('6*alf.*(1-alf)', [0 1], 0.01, N-1, h) + B*\text{eye}(N-1,N-1); \]

\% Then we compute the right-hand side at discretization nodes:
\[ F = \text{eval} ([f ' -B'], t); \]

\% Utilize zero initial condition:
\[ M = \text{eliminator}(N-1,[1])*M*\text{eliminator}(N-1, [1]); \]
\[ F = \text{eliminator}(N-1,[1])*F; \]

\% And solve the system \( MY = F \):
\[ Y = M\backslash F; \]

\% Pre-pend the zero initial value
\% (that one due to zero initial condition)
\[ Y0 = [0; Y]; \]

\% Plot the solution:
\[ U = Y0 + 1; \]
\[ \text{plot}(t, U, 'k') \]
Example T4: CO-order Bagley-Torvik equation

\[ Ay''(t) + By^{3/2}(t) + Cy(t) = F(t), \quad F(t) = \begin{cases} 8, & (0 \leq t \leq 1) \\ 0, & (t > 1) \end{cases} \quad y(0) = y'(0) = 0 \]

(1) Prepare constants and nodes (this is the longest part of the script):
\[
\begin{align*}
\text{alpha} & = 1.5; \\
A & = 1; \ B = 1; \ C = 1; \ % \ coefficients \ of \ the \ Bagley-Torvik \ equation \\
h & = 0.075; \ % \ step \ of \ discretization \\
T & = 0:h:30; \ % \ nodes \\
N & = 30/h + 1; \ % \ number \ of \ nodes \\
M & = \text{zeros}(N,N); \ % \ pre-allocate \ matrix \ M \ for \ the \ system
\end{align*}
\]

(2) Make the matrix for the entire equation -- this is really easy:
\[ M = A*\text{bann}(2,N,h) + B*\text{bann}(\text{alpha},N,h) + C*\text{eye}(N,N); \]

(3) Make right-hand side:
\[ F = 8*(T<=1)'; \]

(4) Utilize zero initial conditions:
\[ M = \text{eliminator}(N,[1 2])*M*\text{eliminator}(N,[1 2])'; \\
F = \text{eliminator}(N,[1 2])*F; \]

(5) Solve the system \( MY=F \):
\[ Y = M\backslash F; \]

(6) Pre-pend the zero values (those due to zero initial conditions):
\[ Y0 = [0; \ 0; \ Y]; \]

Plot the solution:
\[ \text{plot}(T,Y0) \]
Example T5: DO-order Bagley-Torvik equation

% (1) Prepare constants and nodes (this is just the setup)
alpha = 1.5;
A = 1; B = 1; C = 1; % coefficients of equation
h = 0.075; % step of discretization
T = 0:h:30; % nodes
N = 30/h + 1; % number of nodes
M = zeros(N,N); % pre-allocate matrix M for the system

% (2) Make the matrix for the entire equation -- this is really easy:
M = A*ban(2,N,h) + B*doban('5*alf.*(1-alf)', [0 1], 0.01, N, h) + C*eye(N,N);

% (3) Make right-hand side:
F = 8*(T<=1)';

% (4) Utilize zero initial conditions:
M = eliminator(N,[1 2])*M*eliminator(N, [1 2])';
F = eliminator(N,[1 2])*F;

% (5) Solve the system M*Y=F:
Y = M\F;

% (6) Pre-pend the zero values (those due to zero initial conditions)
Y0 = [0; 0; Y];

% Plot the solution:
plot(T,Y0)
Example T7: CO-order
time- and space-fractional diffusion equation

\[ C \int_0^t \frac{\partial^\beta y}{\partial |x|^\beta} = f(x,t) \]

\[ y(0,t) = 0, \quad y(1,t) = 0; \]
\[ y(x,0) = 0. \]

\[
\begin{align*}
\alpha &= 0.7; \\beta &= 1.8; \\
a2 &= 1; & \text{coefficient from the diffusion equation} \\
L &= 1; & \text{length of spatial interval} \\
m &= 21; & \text{Number of spatial steps of discretization} \\
n &= 148; & \text{Number of steps in time} \\
h &= L / (m-1); & \text{spatial step} \\
\tau &= h^2 / (6*a2); & \text{time step} \\
B1 &= \text{ban}(alpha, n-1, \tau)'; & \text{alpha-th order derivative with respect to time} \\
TD &= \text{kron}(B1, \text{eye}(m)); & \text{time derivative matrix} \\
B2 &= \text{ransym}(beta, m, h); & \text{beta-th order derivative with respect to X} \\
SD &= \text{kron}([\text{eye}(n-1), B2]); & \text{spatial derivative matrix} \\
\text{SystemMatrix} &= TD - a2*SD; & \text{matrix corresponding to discretization in space and time} \\
S &= \text{eliminator}(m, [1 \ m]); \\
SK &= \text{kron}(\text{eye}(n-1), S); \\
\text{SystemMatrix}_\text{without_columns_1_m} &= \text{SystemMatrix} * SK'; \\
S &= \text{eliminator}(m, [1 \ m]); \\
SK &= \text{kron}(\text{eye}(n-1), S); \\
\text{SystemMatrix}_\text{without_rows_columns_1_m} &= SK * \text{SystemMatrix}; \\
F &= 8*\text{ones(size(SystemMatrix}_\text{without_rows_columns_1_m},1)); \\
Y &= \text{SystemMatrix}_\text{without_rows_columns_1_m}\F; \\
YS &= \text{reshape}(Y, m-2, n-1); \\
YS &= \text{flipud}(YS); \\
U &= YS;
\end{align*}
\]
Example T8: DO-diffusion-wave equation

\[ C_D \varphi(\alpha) y - \frac{\partial^\beta y}{\partial |x|^{\beta}} = f(x, t) \]

\[ y(0, t) = 0, \quad y(1, t) = 0; \quad y(x, 0) = 0. \]

\[ \varphi(\alpha) = 1 \]

DO \downarrow \varphi(\alpha) = \delta(\alpha - \lambda) \downarrow 0D^\lambda_t \]

MATLAB: '0 + 100*(alf>0.99)'
Example T8 (cont’d): DO-diffusion-wave equation

\[ \varphi(\alpha) = 2(1 - \alpha) \]
\[ \varphi(\alpha) = 2\alpha \]
Variable step length?

As seen in MATLAB: ode23.m and ode45.m solvers
Method of “large steps”
Method of "large steps"

\[ 0D_t^\alpha y(t) = f(y(t), t), \quad (t > 0), \]
\[ y(0) = 0, \]

Suppose we obtained its solution in the interval \((0,a)\) (and the final value \(y_a\) at \(t=a\)), then we can use this for transforming the above problem to

\[ aD_t^\alpha y(t) = f(y(t), t) - 0R_a^\alpha y(t), \quad (t > a), \]
\[ y(a) = y_a, \]

where

\[ 0R_a^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^a (t-\tau)^{\alpha-1} y(\tau) d\tau, \quad (t > a). \]

\[ 0R_a^\alpha y(t) = 0D_t^\alpha \left( (1-H(t-a))y(t) \right) \]
First “large step” in $[0, a]$

\[ 0 D_t^\alpha y(t) = f(y(t), t), \quad (t > 0), \]
\[ y(0) = 0, \]

\[ a D_t^\alpha y(t) = f(y(t), t) - 0 R_a^\alpha y(t), \quad (t > a), \]
\[ y(a) = y_a, \]

Auxiliary function:

\[ y(t) = u(t) + y_a, \]

Second “large step” in $[a, b]$

\[ a D_t^\alpha u(t) = f(u(t) + y_a, t) - 0 R_a^\alpha y(t) - y_a, \quad (t > a), \]
\[ u(a) = 0. \]
Method of “large steps”: example (1)

\[ 0 D_t^{1/2} y(t) + y(t) = \frac{t^{1.5}}{\Gamma(1.5)} + t, \quad (t > 0), \]
\[ y(0) = 0. \]

Exact solution: \( y(t) = t. \)

First “large step”: interval \([0, 1]\):

```matlab
clear all
h = 0.01;
t = 0:h:1;
N = 1/h + 1;
M = zeros(N,N);
M = ban(0.5, N, h) + eye(N,N);
F = (t.^0.5/gamma(1.5) + t);
M = eliminator(N, [1])*M*eliminator(N, [1]);
F = eliminator(N, [1])*F;
Y = M\F;
Y0 = [0; Y];
plot(t,Y0,'b')
set(gca, 'xlim', [0 2], 'ylim', [0 2])
grid on, hold on
```
Method of “large steps”: example (2)

Second “large step”: interval $[1, 2]$

$$0D_t^{1/2} y(t) = 1D_t^{1/2} y(t) + \frac{1}{\Gamma(0.5)} \int_0^1 \frac{y'(\tau) d\tau}{(t-\tau)^{1/2}}, \quad (t > 1)$$

$$1D_t^{1/2} y(t) + y(t) = \frac{t^{1.5}}{\Gamma(1.5)} + t - \frac{1}{\Gamma(0.5)} \int_0^1 \frac{d\tau}{(t-\tau)^{1/2}} \quad (t > 1).$$

$$1D_t^\alpha y(t) + y(t) = \frac{t^{1.5}}{\Gamma(1.5)} + t - \frac{2t^{0.5}}{\Gamma(0.5)} + \frac{2(t-1)^{0.5}}{\Gamma(0.5)}; \quad (t > 1)$$

$$y(1) = 1.$$

$$y(t) = u(t) + 1,$$
Method of “large steps”: example (3)

The problem to solve in $[1, 2]$: 

\[ 1D_t^\alpha u(t) + u(t) = \frac{t^{1.5}}{\Gamma(1.5)} + t - \frac{2t^{0.5}}{\Gamma(0.5)} + \frac{2(t - 1)^{0.5}}{\Gamma(0.5)} - 1; \quad (t > 1) \]

\[ u(1) = 0. \]

Using the matrix approach

```matlab
clear all
h = 0.01;
t = 1:h:2;
N = 1/h + 1;
M = zeros(N,N);
M = bvan(0.5, N, h) + eye(N,N);
F = (t.^-(0.5)/gamma(1.5)) + t - 2*t.^-(0.5)/gamma(0.5) ...
+ 2*(t-1).^-(0.5)/gamma(0.5) - 1;'
M = eliminator(N,[1])*M*eliminator(N,[1])';
F = eliminator(N,[1])*F;
U = M \\ F;
U0 = [0; U];
YO = U0 + 1;
plot(t, YO, 'g')```

Monday, June 3, 2013
Method of “large steps” and the problem of initialization

C. Lorenzo and T. Hartley raised the question about initialization of fractional derivatives. Their motivation was to use or recover the information about the process $y(t)$ in the interval $(0, a)$, if we consider fractional derivatives of $y(t)$ in $(a, \infty)$.

NOTE: in the second “large step” in the considered sample problem we used, in fact, the proper initialization of the fractional derivative in the interval $(1, 2)$ based on the known behavior of $y(t)$ in $(0, 1)$.
Finale
The Matrix Approach

- Uniform approach to CO, VO, and DO differentiation and integration.
- Easy, algorithmic, modular, ready to use.
- Allows solution of ODEs, including nonlinear problems.
- Allows solution of partial fractional differential equations.
- Allows consideration and solution of fractional differential equations with delays.
- Allows numerical solution of FDEs with a mixture of left-sided, right-sided, symmetric, CO, VO, DO fractional derivatives...
- Can be used on non-equidistant grids and in combination with the new method of “large steps”.
- On the road:
  - using sparse matrices;
  - using parallel computations with the MATLAB Parallel Toolbox;
  - applications to non-equidistantly sampled processes;
  - and more...

Monday, June 3, 2013
1. The idea

\[ \ldots, \frac{d^{-2} f}{dt^{-2}}, \frac{d^{-1} f}{dt^{-1}}, f, \frac{df}{dt}, \frac{d^2 f}{dt^2}, \ldots \]

2. Data to models

\[ y = y_0 E_{\alpha,1}(a t^\alpha) \quad \frac{C}{0} D_t^\alpha y(t) - k y(t) = 0, \quad y(0) = y_0 \]

3. Numerical solution

\[ 0 D_t^\alpha U - a^2 0 D_x^\beta U = F \]

\[ \left\{ B_n^\alpha \otimes E_m - a^2 E_n \otimes R_m^\beta \right\} u_{nm} = f_{nm} \]
Credits

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Thank you!