DISCONTINUOUS GALERKIN METHODS FOR FRACTIONAL DIFFUSION PROBLEMS

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Abstract.

Part 1 We propose a time-stepping discontinuous Galerkin method for the numerical solution of fractional sub-diffusion problems of the form \( \partial_t u - \partial_t^{-\alpha} \nabla^2 u = f(t) \) with \(-1 < \alpha < 0\). We derive generic \( hp \)-version error estimates after proving the well-posedness of the approximate solution. By employing geometrically refined time-steps and linearly increasing approximation orders, we show exponential rates of convergence in the number of temporal degrees of freedom for solutions with singular behavior near \( t = 0 \) caused by the weakly singular kernel. Moreover, for \( h \)-version DG approximations on appropriate graded meshes near \( t = 0 \), we proved that the error is of order \( O(k^{p+1+\frac{\alpha}{2}}) \), where \( k \) is the maximum time-step size and \( p \geq 1 \) is the degree of the time-stepping discontinuous Galerkin solution.

Part 2 I this part I talk about the use of the hybridizable discontinuous Galerkin method for numerically solving our fractional diffusion problem. For exact time-marching, we derive optimal algebraic error estimates assuming that the exact solution is sufficiently regular. Thus, if for each time \( t \in [0, T] \) the approximations are taken to be piecewise polynomials of degree \( r \geq 0 \) on the spatial domain \( \Omega \), the approximations to \( u \) in the \( L_\infty(0, T; L_2(\Omega)) \)-norm and to \( -\nabla u \) in the \( L_\infty(0, T; L_2(\Omega)) \)-norm are proven to converge with the rate \( h^{r+1} \), where \( h \) is the maximum diameter of the elements of the mesh. Moreover, for \( r \geq 1 \) and quasi-uniform meshes, we obtain a superconvergence result which allows us to compute, in an elementwise manner, a new approximation for \( u \) converging with a rate faster than \( \sqrt{\log(Th^{-2/(\alpha+1)})} h^{r+2} \), for quasi-uniform meshes. These results hold uniformly in \( \alpha \) on any closed subinterval of \((-1, 0)\) provided the exact solution is smooth.

I end my talk with a series of numerical simulations.

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