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# On the distinguished role of the Mittag-Leffler and Wright functions in fractional calculus

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# 1. Introduction to the Mittag-Leffler functions

Since a few decades the special transcendental function known as Mittag-Leffler function has attracted an increasing attention of researchers because of its key role in treating problems related to integral and differential equations of fractional order.

Since its introduction by the Swedish mathematician Mittag-Leffler at the beginning of the last century up to the 1990's, this function was seldom considered by mathematicians and applied scientists.

From a mathematical point of view we recall the paper by Hille and Tamarkin (1930) on the solutions of the Abel-Integral equation of the second kind, the Handbook of the Bateman project, Erdélyi (1955), where it was restricted in the chapter devoted to miscellaneous functions, and the books by Davis (1936), by Sansone and Gerretsen (1960), by Dzherbashyan (1966) (unforunately in Russian) and finally by Samko, Kilbas and Marichev (1987-1993).

For applications we recall a few papers by Cole (1933), Cole and Cole (1941, 1942), Gross (1947) and Caputo and Mainardi (1971a,1971b) where the Mittag-Leffler function was adopted to represent the responses in dielectric and viscoelastic media.

In the 1960's the Mittag-Leffler function started to exit from the realm of miscellaneous functions because it was considered as a special case of the general class of Fox $H$ functions, that can exhibit an arbitrary number of parameters in their integral Mellin-Barnes representation, see e.g. the books by Kiryakova (1994), Kilbas and Saigo (2004), Mathai et al. (2010). However, in our opinion, this classification in a more general framework has, to some extent, obscured the relevance and the applicability of this function in applied science. In fact most mathematical models are based on a small number of parameters, say 1 or 2 or 3, so that a general theory may be confusing whereas the adoption of a generalized Mittag-Leffler function with 2 or 3 indices may be sufficient, see e.g. Beghin and Orsingher (2010), Tomovski et al. (2010), Capelas et al. (2011), Sandev et al. (2012).

Multi-index Mittag-Leffler functions have been introduced as well, see e.g. Kiryakova (2010), Kiryakova and Luchko (2010), Kilbas et al. (2013), but their extensive use has not yet been pointed out in applied sciences.

Nowadays it is well recognized that the Mittag-Leffler function plays a fundamental role in Fractional Calculus even if with a single parameter (as originally introduced by Mittag-Leffler) just to be worth of being referred to as the *Queen Function of Fractional Calculus*, see Mainardi and Gorenflo (2007).

On this respect we just point out a few of reviews and treatises on Fractional Calculus (in order of publication time): Gorenflo and Mainardi (1997), Podlubny (1999), Hilfer (2000), Kilbas et al. (2006), Magin (2006), Mathai and Haubold (2008), Mainardi (2010), Diethelm (2010), Klafter et al (2012), Baleanu et al (2012).

## 2. The Mittag-Lefler function in fractional relaxation processes

The Mittag-Leffler function is defined by the following power series, convergent in the whole complex plane,

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}\,, \quad \alpha > 0\,, \quad z \in \mathbb{C}\,. \tag{2.1}$$

We recognize that it is an entire function providing a simple generalization of the exponential function to which it reduces for $\alpha = 1$. We also note that for the convergence of the power series in (2.1) the parameter $\alpha$ may be complex provided that $\Re(\alpha) > 0$.

The most interesting properties of the Mittag-Leffler function are associated with its asymptotic expansions as $z \to \infty$ in various sectors of the complex plane. For detailed asymptotic analysis, which includes the smooth transition across the Stokes lines, the interested reader is referred to Wong and Zhao (2002).

In this paper we limit ourselves to the Mittag-Leffler function of order $\alpha \in (0, 1)$ on the negative real semi-axis where is known to be completely monotone (CM) due a classical result by Pollard (1948), who proved a conjecture by Feller (based on the probability theory).

Let us recall that a function $\phi(t)$ with $t \in \mathbb{R}^+$ is called a completely monotone (CM) function if it is non negative, of class $C^\infty$, and $(-1)^n \phi^{(n)}(t) \geq 0$ for all $n \in \mathbb{N}$. Then a function $\psi(t)$ with $t \in \mathbb{R}^+$ is called a Bernstein function if it is non negative, of class $C^\infty$, with a CM first derivative.

These functions play fundamental roles in linear hereditary mechanics to represent relaxation and creep processes, see e.g. Mainardi (2010).

For mathematical details we refer the interested reader to the survey paper by Miller and Samko (2001) and to the recent book by Schilling et al. (2010).

In particular we are interested to the function

$$e_\alpha(t) := E_\alpha(-t^\alpha) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{\alpha n}}{\Gamma(\alpha n + 1)}\,, \quad t > 0\,, \quad 0 < \alpha \le 1\,, \tag{2.2}$$

that provides the solution to the fractional relaxation equation, see Gorenflo and Mainardi (1997), Mainardi and Gorenflo (2007), Mainardi (2010).

For readers' convenience let us briefly outline the topic concerning the generalization via fractional calculus of the first-order differential equation governing the phenomenon of (exponential) relaxation.

Recalling (in non-dimensional units) the initial value problem

$$\frac{du}{dt} = -u(t)\,, \quad t \ge 0\,, \quad \text{with} \quad u(0^+) = 1 \tag{2.3}$$

whose solution is

$$u(t) = \exp\left(-t\right), \tag{2.4}$$

the following two alternatives with $\alpha \in (0,1)$ are offered in the literature:

$$\frac{du}{dt} = -D_t^{1-\alpha}\, u(t)\,, \quad t \geq 0\,, \quad \text{with} \quad u(0^+) = 1\,, \tag{2.5a}$$

$$_*D_t^{\alpha}\, u(t) = -u(t)\,, \quad t \geq 0\,, \quad \text{with} \quad u(0^+) = 1\,. \tag{2.5b}$$

where $D_t^{1-\alpha}$ and $_*D_t^{\alpha}$ denote the fractional derivative of order $1-\alpha$ in the Riemann-Liouville sense and the fractional derivative of order $\alpha$ in the Caputo sense, respectively.

For a generic order $\mu \in (0,1)$ and for a sufficiently well-behaved function $f(t)$ with $t \in \mathbb{R}^+$ the above derivatives are defined as follows, see e.g. Gorenflo and Mainardi (1997), Podlubny (1999):

$$D_t^{\mu}\, f(t) = \frac{1}{\Gamma(1-\mu)}\, \frac{d}{dt} \left[ \int_0^t \frac{f(\tau)}{(t-\tau)^{\mu}}\, d\tau \right]\,, \tag{2.6a}$$

$$_*D_t^{\mu}\, f(t) = \frac{1}{\Gamma(1-\mu)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\mu}}\, d\tau\,. \tag{2.6b}$$

Between the two derivatives we have the relationship

$$_{*}D_t^{\mu}\, f(t) = D_t^{\mu}\, f(t) - f(0^+)\, \frac{t^{-\mu}}{\Gamma(1-\mu)} = D_t^{\mu}\, \left[f(t) - f(0^+)\right] . \tag{2.7}$$

Both derivatives in the limit $\mu \to 1^-$ reduce to the standard first derivative but for $\mu \to 0^+$ we have

$$D_t^{\mu} f(t) \to f(t)\,, \quad {}_{*}D_t^{\mu} f(t) = f(t) - f(0^+)\,, \quad \mu \to 0^+\,. \tag{2.8}$$

In analogy to the standard problem (2.3), we solve the problems (2.5a) and (2.5b) with the Laplace transform technique, using the rules pertinent to the corresponding fractional derivatives.

The problems (a) and (b) are equivalent since the Laplace transform of the solution in both cases comes out as

$$\widetilde{u}(s) = \frac{s^{\alpha-1}}{s^{\alpha}+1}\,, \tag{2.9}$$

that yields our function

$$u(t) = e_{\alpha}(t) := E_{\alpha}(-t^{\alpha})\,. \tag{2.10}$$

The Laplace transform pair

$$e_\alpha(t) \div \frac{s^{\alpha-1}}{s^\alpha+1}, \quad \alpha > 0, \tag{2.11}$$

can be proved by transforming term by term the power series representation of $e_\alpha(t)$ in the R.H.S of (2.2).

Furthermore, by anti-transforming the R.H.S of (2.11) by using the complex Bromwich formula, and taking into account for $0 < \alpha < 1$ the contribution from branch cut on the negative real semi-axis, we get

$$e_\alpha(t) = \int_0^\infty \mathrm{e}^{-rt} K_\alpha(r)\, dr\,, \tag{2.12}$$

where, for the Titchmarsh formula,

$$K_\alpha(r) = \mp \frac{1}{\pi} \,\mathrm{Im}\left\{ \left. \frac{s^{\alpha-1}}{s^\alpha+1} \right|_{s=r\,e^{\pm i\pi}} \right\}$$

$$= \frac{1}{\pi} \frac{r^{\alpha-1} \sin(\alpha\pi)}{r^{2\alpha} + 2\,r^\alpha \cos(\alpha\pi) + 1} \geq 0\,. \tag{2.13}$$

Since $K_\alpha(r)$ is non negative for all $r$ in the integral, the above formula proves that $\mathrm{e}_\alpha(t)$ is CM function in view of the Bernstein theorem. This theorem provides a necessary and sufficient condition for a CM function as a real Laplace transform of a non negative measure.

However, the CM property of $e_\alpha(t)$ can also be seen as a consequence of the result by Pollard because the transformation $x = t^\alpha$ is a Bernstein function for $\alpha \in (0, 1)$.

In fact it is known that a CM function can be obtained by composing a CM with a Bernstrein function based on the following theorem:

*Let $\phi(t)$ be a CM function and let $\psi(t)$ be a Bernstein function, then $\phi[\psi(t)]$ is a CM function.*

As a matter of fact, $K_\alpha(r)$ provides an interesting spectral representation of $e_\alpha(t)$ in frequencies. With the change of variable $\tau = 1/r$ we get the corresponding spectral representation in relaxation times, namely

$$e_\alpha(t) = \int_0^\infty \mathrm{e}^{-t/\tau} H_\alpha(\tau)\, d\tau\,,\;\; H_\alpha(\tau) = \tau^2\, K_\alpha(1/\tau)\,, \qquad (2.14)$$

that can be interpreted as a continuous distributions of elementary (i.e. exponential) relaxation processes.

We also note that (surprisingly) we get the identity between the two spectral distributions

$$K_\alpha(r) = H_\alpha(\tau)\,. \qquad (2.15)$$

This kind of universal/scaling property, pointed out in Linear Viscoelasticity by the author, Mainardi (2010), seems a peculiar one for our Mittag-Leffler function $e_\alpha(t)$.

In Fig 1 we show $K_\alpha(r)$ for some values of the parameter $\alpha$. Of course for $\alpha = 1$ the Mittag-Leffler function reduces to the exponential function $\exp(-t)$ and the corresponding spectral distribution is the Dirac delta generalized function centred at $r = 1$, namely $\delta(r-1)$.

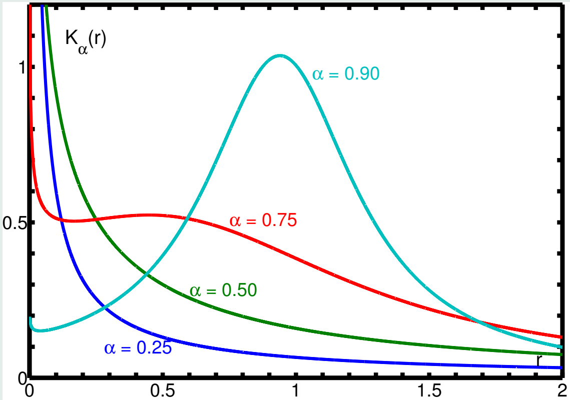


**Fig.1** Plots of the spectral function $K_\alpha(r)$ for $\alpha = 0.25, 0.50, 0.75, 0.90$ in the frequency range $0 \le r \le 2$.

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# 3. Asymptotic approximations to the Mittag-Lefler function

In Fig 2 we show some plots of $e_\alpha(t)$ for some values of the parameter $\alpha$. It is worth to note the different rates of decay of $e_\alpha(t)$ for small and large times. In fact the decay is very fast as $t \to 0^+$ and very slow as $t \to +\infty$.

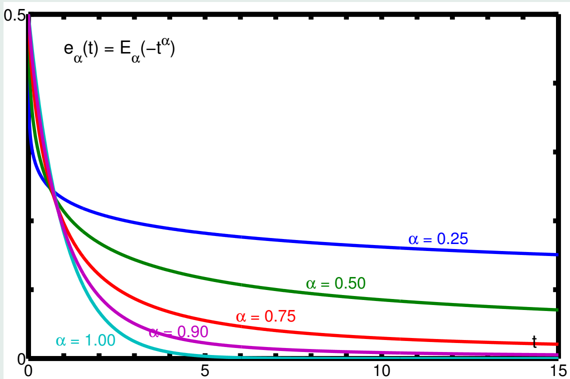


**Fig.2** Plots of the Mittag-Leffler function $e_\alpha(t)$ for $\alpha = 0.25, 0.50, 0.75, 0.90, 1.$ in the time range $0 \leq t \leq 15$

## The two common asymptotic approximations

It is common to point out that the function $e_\alpha(t)$ matches for $t \to 0^+$ with a stretched exponential with an infinite negative derivative, whereas as $t \to \infty$ with a negative power law. The short time approximation is derived from the convergent power series representation (2.2). In fact,

$$e_\alpha(t) = 1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + \cdots \sim \exp\left[-\frac{t^\alpha}{\Gamma(1+\alpha)}\right], \; t \to 0\,. \quad (3.1)$$

The long time approximation is derived from the asymptotic power series representation of $e_\alpha(t)$ that turns out to be, see Erdélyi (1955),

$$e_\alpha(t) \sim \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^{-\alpha n}}{\Gamma(1-\alpha n)}\,, \; t \to \infty\,, \quad (3.2)$$

so that, at the first order,

$$e_\alpha(t) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,, \; t \to \infty\,. \quad (3.3)$$

It is common to say that the function $e_\alpha(t)$ interpolates the stretched exponential due its very fast decay for short times, whereas interpolates the negative power law due to its very slow decay for long times. Then, we get the two asymptotic representations as $t \to 0$ and $t \to \infty$ commonly found in the literature:

$$\mathrm{e}_\alpha(t) \sim \begin{cases} e_\alpha^0(t) := \exp\left[-\dfrac{t^\alpha}{\Gamma(1+\alpha)}\right], & t \to 0\,; \\ e_\alpha^\infty(t) := \dfrac{t^{-\alpha}}{\Gamma(1-\alpha)} = \dfrac{\sin(\alpha\pi)}{\pi}\,\dfrac{\Gamma(\alpha)}{t^\alpha}, & t \to \infty\,. \end{cases} \tag{3.4}$$

The stretched exponential has been adopted in order to replace the divergent expression $1 - t^\alpha/\Gamma(1+\alpha)$ obtained from (3.1), asymptotically equivalent to $e_\alpha(t)$ as $t \to 0$. Of course we have the inequality

$$e_\alpha^0(t) \le e_\alpha^\infty(t)\,, \quad t \ge 0\,, \quad 0 < \alpha < 1\,. \tag{3.5}$$

In Figs 3-7 LEFT we compare for $\alpha = 0.25, 0.5, 0.75, 0.90, 0.99$ in logarithmic scales the function $e_\alpha(t)$ and its asymptotic representations, the stretched exponential $e^0_\alpha(t)$ valid for $t \to 0$ (dashed line) and the power law $e^\infty_\alpha(t)$ valid for $t \to \infty$ (dotted line). We have chosen the time range $10^{-5} \le t \le 10^{+5}$.

In the RIGHT we have shown the plots of the relative errors (in absolute values)

$$\frac{|e^0_\alpha(t) - e_\alpha(t)|}{e_\alpha(t)}, \quad \frac{|e^\infty_\alpha(t) - e_\alpha(t)|}{e_\alpha(t)}, \tag{3.6}$$

pointing out a line at an error 1% under which the approximations can be considered reliable.

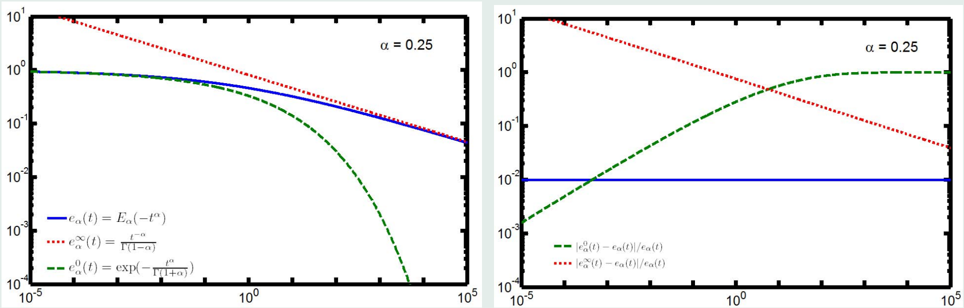


**Fig.3** Approximations $e_\alpha^0(t)$ (dashed line) and $e_\alpha^\infty(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and relative errors (RIGHT) in $10^{-5} \leq t \leq 10^{+5}$ for $\alpha = 0.25$.

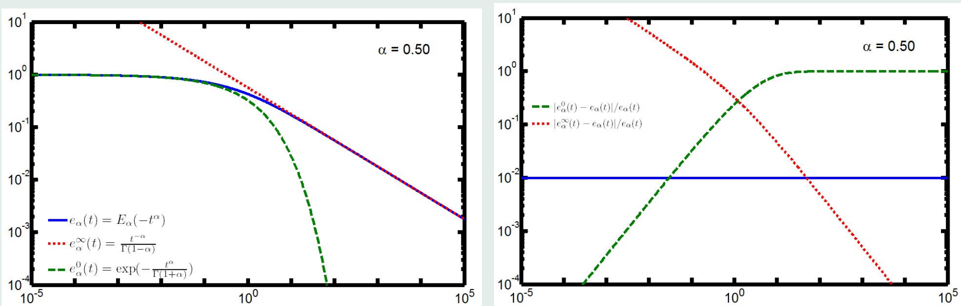


**Fig.4** Approximations $e_\alpha^0(t)$ (dashed line) and $e_\alpha^\infty(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and relative errors (RIGHT) in $10^{-5} \leq t \leq 10^{+5}$ for $\alpha = 0.50$.

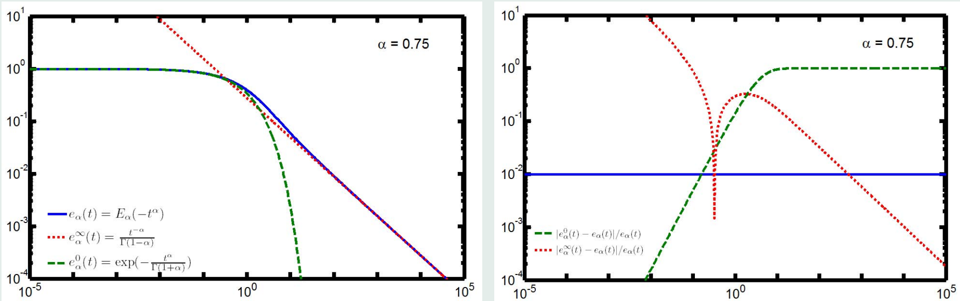


**Fig.5** Approximations $e_\alpha^0(t)$ (dashed line) and $e_\alpha^\infty(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and relative errors (RIGHT) in $10^{-5} \le t \le 10^{+5}$ for $\alpha = 0.75$.

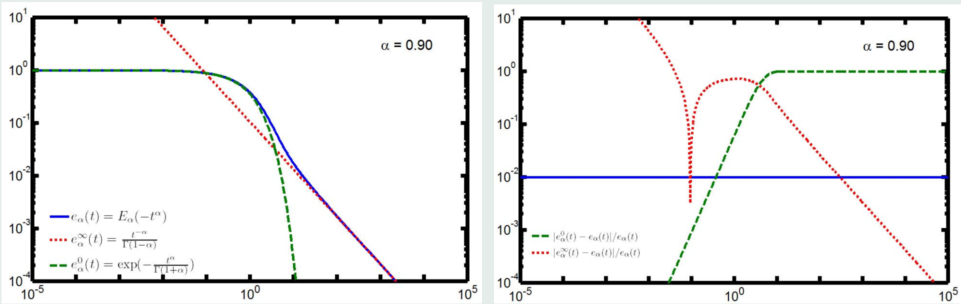


**Fig.6** Approximations $e_\alpha^0(t)$ (dashed line) and $e_\alpha^\infty(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and relative errors (RIGHT) in $10^{-5} \le t \le 10^{+5}$ for $\alpha = 0.90$.

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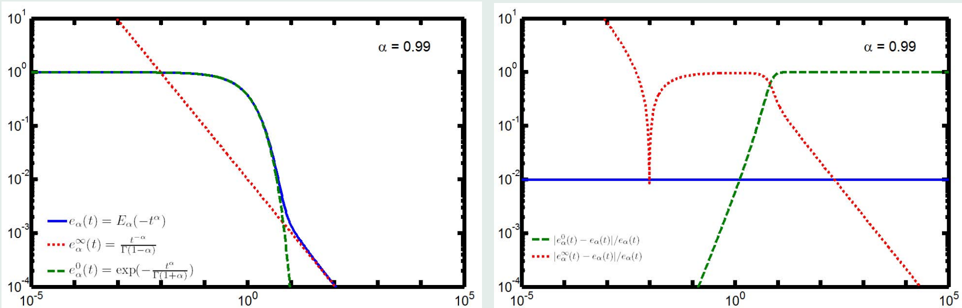


**Fig.7** Approximations $e_\alpha^0(t)$ (dashed line) and $e_\alpha^\infty(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and relative errors (RIGHT) in $10^{-5} \le t \le 10^{+5}$ for $\alpha = 0.99$.

We note from Figs 3-7 that, whereas the plots of $e_\alpha^0(t)$ remain always under the corresponding ones of $e_\alpha(t)$, the plots of $e_\alpha^\infty(t)$ start upper those of $e_\alpha(t)$ but, at a certain point, an intersection may occur so changing the sign of the relative errors.

## The two rational asymptotic approximations

We now propose a new set of CM functions approximating $e_\alpha(t)$: $\{f_\alpha(t), g_\alpha(t)\}$, alternative to $\{e_\alpha^0(t), e_\alpha^\infty(t)\}$, obtained as the first Padè approximants [0/1] to the power series in $t^\alpha$ (2.2) and (3.2), respectively. For more details on the theory of Padè Approximants we refer e.g. to Baker (1975). We thus obtain the following rational functions in $t^\alpha$:

$$f_\alpha(t) := \frac{1}{1 + \frac{t^\alpha}{\Gamma(1+\alpha)}} \sim 1 - \frac{t^\alpha}{\Gamma(1+\alpha)} \sim e_\alpha(t)\,,\ t \to 0\,, \quad (3.7)$$

$$g_\alpha(t) := \frac{1}{1 + t^\alpha \Gamma(1-\alpha)} \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \sim e_\alpha(t)\,,\ t \to \infty\,. \quad (3.8)$$

Now we prove the inequality

$$g_\alpha(t) \le f_\alpha(t)\,, \quad t \ge 0\,, \quad 0 < \alpha < 1\,, \quad (3.9)$$

as a straightforward consequence of the reflection formula of the gamma function.

In fact, recalling the definitions (3.7)-(3.8) we have for $\forall t \geq 0$ and $\alpha \in (0,1)$:

$$g_\alpha(t) \leq f_\alpha(t) \iff \Gamma(1-\alpha) \geq \frac{1}{\Gamma(1+\alpha)}$$

$$\iff$$

$$\Gamma(1-\alpha)\,\Gamma(1+\alpha) = \frac{\pi\alpha}{\sin(\pi\alpha)} \geq 1\,.$$

In Figs 8-12 LEFT we compare for $\alpha = 0.25, 0.5, 0.75, 0.90, 0.99$ in logarithmic scales the function $e_\alpha(t)$ and its rational asymptotic representations, $f_\alpha(t)$ valid for $t \to 0$ (dashed line) and $g_\alpha(t)$ valid for $t \to \infty$ (dotted line). We have chosen the time range $10^{-5} \leq t \leq 10^{+5}$.

In the RIGHT we have shown the plots of the relative errors (no longer in absolute values)

$$\frac{f_\alpha(t) - e_\alpha(t)}{e_\alpha(t)}\,, \quad \frac{e_\alpha(t) - g_\alpha(t|}{e_\alpha(t)}\,, \tag{3.10}$$

pointing out a line at an error 1% under which the approximation can be considered reliable.

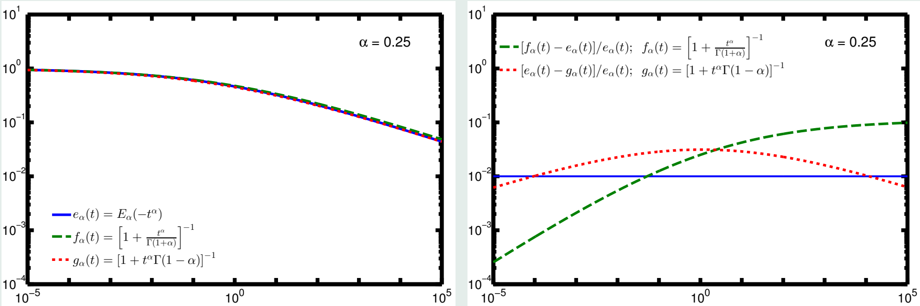


**Fig.8** Approximations $f_\alpha(t)$ (dashed line) and $g_\alpha(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and relative errors (RIGHT) in $10^{-5} \leq t \leq 10^{+5}$ for $\alpha = 0.25$.

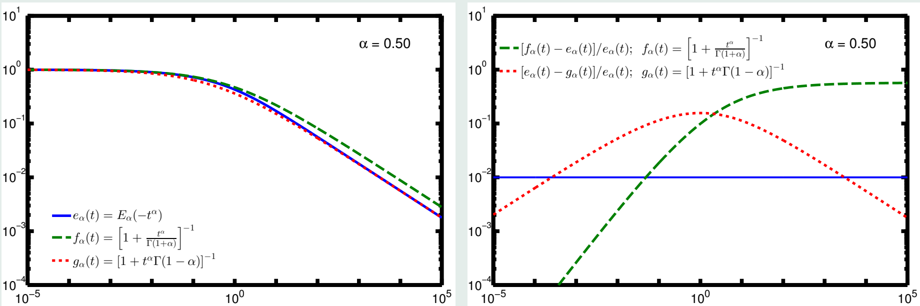


**Fig.9** Approximations $f_\alpha(t)$ (dashed line) and $g_\alpha(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and relative errors (RIGHT) in $10^{-5} \leq t \leq 10^{+5}$ for $\alpha = 0.50$.

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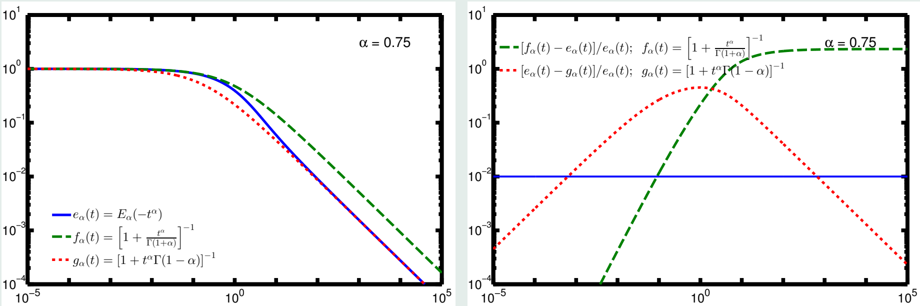


**Fig.10** Approximations $f_\alpha(t)$ (dashed line) and $g_\alpha(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and relative errors (RIGHT) in $10^{-5} \leq t \leq 10^{+5}$ for $\alpha = 0.75$.

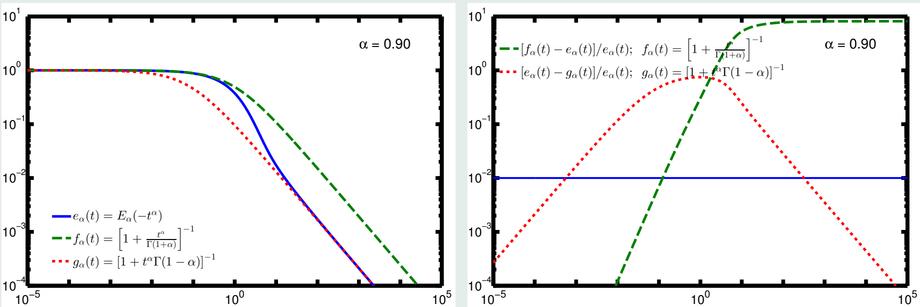


**Fig.11** Approximations $f_\alpha(t)$ (dashed line) and $g_\alpha(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and relative errors (RIGHT) in $10^{-5} \leq t \leq 10^{+5}$ for $\alpha = 0.90$.

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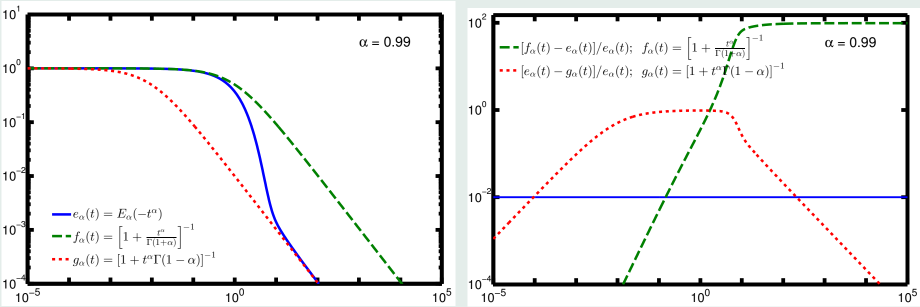


**Fig.12** Approximations $f_\alpha(t)$ (dashed line) and $g_\alpha(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and relative errors (RIGHT) in $10^{-5} \le t \le 10^{+5}$ for $\alpha = 0.99$.

As a matter of fact, from the plots in Figs 8-12, we recognize, for the time range and for $\alpha \in (0,1)$ considered by us, the relevant inequality

$$g_\alpha(t) \le e_\alpha(t) \le f_\alpha(t)\,, \tag{3.11}$$

that is $g_\alpha(t)$ and $f_\alpha(t)$ provide lower and upper bounds to our Mittag-Leffler function $e_\alpha(t)$ . This of course can be seen as a conjecture that we leave as an open problem to be proved (or disproved) by specialists of CM functions.

We also see that whereas the short time approximation $f_\alpha(t)$ turns out to be good only for small times, the long time approximation $g_\alpha(t)$ is good (surprisingly) also for short times, falling down only in an intermediate time range.

In fact from the RIGHT of Figs 8-12 we can estimate the ranges of validity when the relative error is less than 1%. We could further discuss on them.

Concerning Padè Approximants (PA), we like to cite the paper by Freed et al. (2002), where an appendix is devoted to the table of PA for the Mittag-Leffler function $E_\alpha(-x)$, $x \geq 0$.

We point out the fact that our first PA's $[0/1]$ are constructed from two different series (in positive and negative powers of $t^\alpha$) both resulting CM functions whereas the successive PA's of higher order can exhibit oscillations for $t > 0$.

# Conclusions

We have discussed some noteworthy properties of the Mittag-Leffler function $E_\alpha(-t^\alpha)$ with $0 < \alpha < 1$ in the range $t > 0$.

Being a completely monotone (CM) function, because of the Bermstein theorem this function admits non negative frequency and time spectra. We have pointed out that these two spectra are equal, so providing a universal scaling property.

Furthermore, in view of a numerical approximation we have compared two different sets of approximating CM functions, asymptotically equivalent to $E_\alpha(-t^\alpha)$ for small and large times: the former commonly used in the literature, the latter probably new.

The last set is constituted by two simple rational functions that provide upper and lower bounds, at least in our numerical examples on a large time range.

We are allowed to the conjecture that this bounding property is always valid for any $t > 0$ and for any $\alpha \in (0, 1)$: an open problem offered to specialists of CM functions.

Last but not the least, we note that our rational function approximating $E_\alpha(-t^\alpha)$ for large times provides (surprisingly) a good numerical approximation also for short times, so failing only in an intermediate range.

## Acknowledgments

The author is grateful to Marco Di Cristina and Andrea Giusti for their valuable help in plotting the Mittag-Leffler function. They have used the MATLAB routine by Podlubny (2006) that is essentially based on the MATHEMATICA routine by Gorenflo et al. (2002).

# 4. The generalized Mittag-Leffler functions

## Definitions

The 3-parameter Mittag-Leffler function

$$E^{\gamma}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!\Gamma(\alpha n+\beta)}\, z^n\,, \tag{4.1}$$

$$\mathrm{Re}\{\alpha\} > 0,\ \mathrm{Re}\{\beta\} > 0,\ \mathrm{Re}\{\gamma\} > 0\,,$$

$$(\gamma)_n = \gamma(\gamma+1)\ldots(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}\,.$$

For $\gamma = 1$ we recover the 2-parameter Mittag-Leffler function

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}\,, \tag{4.2}$$

and for $\gamma = \beta = 1$ we recover the standard Mittag-Leffler function

$$E_{\alpha}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}\,. \tag{4.3}$$

## The Laplace transform pairs

Let us now consider the relevant formulas of Laplace transform pairs related to the above three functions, already known in the literature for $\alpha, \beta > 0$ and $0 < \gamma \leq 1$, when the independent variable $at^\alpha$ is real where $t > 0$ is interpreted as time and $a$ is a certain constant.

Let us start with the most general function. Substituting the series representation of the Prabhakar generalized Mittag-Leffler function in the Laplace transformation yields the identity

$$\int_0^\infty \mathrm{e}^{-st}\, t^{\beta-1}\, E^\gamma_{\alpha,\beta}(at^\alpha)\,\mathrm{d}t = s^{-\beta} \sum_{n=0}^\infty \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} \left(\frac{a}{s}\right). \quad (4.4)$$

On the other hand (binomial series)

$$(1+z)^{-\gamma} = \sum_{n=0}^\infty \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma-n)n!}\, z^n = \sum_{n=0}^\infty (-1)^n \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)n!}\, z^n\,. \quad (4.5)$$

Comparison of Eq. (4.4) and Eq. (4.5) yields the Laplace transform pair

$$t^{\beta-1}\, E^{\gamma}_{\alpha,\beta}\,(at^{\alpha}) \div \frac{s^{-\beta}}{(1-as^{-\alpha})^{\gamma}} = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-a)^{\gamma}}\,. \tag{4.6}$$

In particular we get the known Laplace transform pairs, see e.g. [Mainardi (2010), Podlubny (1999)],

$$t^{\beta-1}\, E_{\alpha,\beta}(at^{\alpha}) \div \frac{s^{\alpha-\beta}}{s^{\alpha}-a} = \frac{s^{-\beta}}{1-as^{-\alpha}}\,, \tag{4.7}$$

$$E_{\alpha}(at^{\alpha}) \div \frac{s^{\alpha-1}}{s^{\alpha}-a} = \frac{s^{-1}}{1-as^{-\alpha}}\,. \tag{4.8}$$

In [Mainardi (2010), Capelas et al (2011)] we proved

$$\xi_G(t) := t^{\beta-1}\, E^{\gamma}_{\alpha,\beta}\,(-t^{\alpha}) \div \widetilde{\xi}_G(s) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}+1)^{\gamma}}\,, \tag{4.9}$$

is CM iff

$$0 < \alpha, \beta, \gamma \le 1 \quad \text{with} \quad \alpha\,\gamma \le \beta\,. \tag{4.10}$$

Here $\xi_G(t)$ denotes the response function in dielectric theory following our paper [Capelas et al (2011)]

The Bernstein theorem ensures that our response function $\xi_G(t)$, being CM, can be expressed with a *spectral distribution function* $K^{\gamma}_{\alpha,\beta}(r) \geq 0$ such that

$$\xi_G(t) = \int_0^\infty \mathrm{e}^{-rt}\, K^{\gamma}_{\alpha,\beta}(r)\, \mathrm{d}r\,, \tag{4.11}$$

where

$$K^{\gamma}_{\alpha,\beta}(r) := \frac{r^{-\beta}}{\pi} \mathrm{Im}\left\{ \mathrm{e}^{i\pi\beta} \left( \frac{r^{\alpha} + \mathrm{e}^{-i\pi\alpha}}{r^{\alpha} + 2\cos\pi\alpha + r^{-\alpha}} \right)^{\gamma} \right\}. \tag{4.12}$$

We note that for $\beta = \gamma = 1$ we obtain from (4.12) the spectral distribution function of the classical Mittag-Leffler function outlined in (4.11).

## Mathematical models for dielectric relaxation

It is well recognized that relaxation phenomena in dielectrics deviate more or less strongly from the classical Debye law for which the Laplace transform pair for complex susceptibility ($s = -i\omega$) and response function ($t \geq 0$) reads in an obvious notation,

$$\widetilde{\xi}_{\rm D}(s) = \frac{1}{1+s} \div \xi_{\rm D}(t) = {\rm e}^{-t} \,. \qquad (4.13)$$

Here, for the sake of simplicity, we have assumed the frequency $\omega$ and the time $t$ normalized with respect to a characteristic frequency $\omega_D$ and a corresponding relaxation time $\tau_D = 1/\omega_D$.

In the literature a number of laws have been proposed to describe the non-Debye (or anomalous) relaxation phenomena in dielectrics, of which the most relevant ones are referred to
**Cole – Cole (C-C)**,
**Davidson – Cole (D-C)**,
**Havriliak – Negami (H-N)**
see e.g., [Jonscher (1983), Jonscher (1996)]
and [Uchaikin and Sibatov (2013)].

The Laplace transform pair in Eq. (4.9) with the condition (4.10) on the 3-order parameters can be considered as the pair $\xi_G(t)$ $(t \geq 0)$ and $\widetilde{\xi}_G(s)$ $(s = i\omega)$ for a possible new mathematical model of the response function and the complex susceptibility in the framework of a general relaxation theory of dielectrics, according to [Capelas et al (2011)].

We first show how the three classical models referred to Cole–Cole (C-C), Davidson–Cole (D-C) and Havriliak–Negami (H-N) are contained in our general model when $\alpha\gamma = \beta$:

$$\alpha\gamma = \beta \begin{cases} 0 < \alpha < 1\,,\ \beta = \alpha\,,\ \gamma = 1\ :\ \text{C-C}\ \{\alpha\}\,, \\ \alpha = 1\,,\ \beta = \gamma\,,\ 0 < \gamma < 1\ :\ \text{D-C}\ \{\gamma\}\,, \\ 0 < \alpha < 1\,,\ 0 < \gamma < 1\ :\ \text{H-N}\ \{\alpha, \gamma\}\,. \end{cases} \tag{4.14}$$

## The Cole–Cole relaxation model

[Cole and Cole (1941), Cole and Cole (1942)]

The C-C relaxation model is a non-Debye relaxation model depending on one parameter, say $\alpha$ $(0 < \alpha < 1)$, that for $\alpha = 1$ reduces to the standard Debye model. We have for $0 < \alpha < 1$:

$$\widetilde{\xi}_{\text{C-C}}(s) = \frac{1}{1+s^\alpha} \div \xi_{\text{C-C}}(t) = t^{\alpha-1}\, E^1_{\alpha,\alpha}(-t^\alpha) = -\frac{d}{dt} E_\alpha(-t^\alpha)\,. \tag{4.15}$$

## The Davidson–Cole relaxation model

[Davidson and Cole (1951)]

The D-C relaxation model is a non-Debye relaxation model depending on one parameter, say $\gamma$ $(0 < \gamma < 1)$, see , that for $\gamma = 1$ reduces to the standard Debye model. The corresponding complex susceptibility $(s = -i\omega)$ and response function read for $0 < \gamma < 1$:

$$\widetilde{\xi}_{\text{D-C}}(s) = \frac{1}{(1+s)^\gamma} \div \xi_{\text{D-C}}(t) = t^{\gamma-1}\, E^\gamma_{1,\gamma}(-t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}\, \mathrm{e}^{-t}\,. \tag{4.16}$$

## The Havriliak–Negami relaxation model

[Havriliak and Negami (1966), Havriliak and Negami (1967)]

The H-N relaxation model is a non-Debye relaxation model depending on two parameters, say $\alpha$ $(0 < \alpha < 1)$ and $\gamma$ $(0 < \gamma < 1)$, that for $\alpha = \gamma = 1$ reduces to the standard Debye model. The corresponding complex susceptibility and response function read with $0 < \alpha, \gamma < 1$:

$$\widetilde{\xi}_{\text{H-N}}(s) = \frac{1}{(1+s^\alpha)^\gamma} \div \xi_{\text{H-N}}(t) = t^{\alpha\gamma-1}\, E^{\gamma}_{\alpha,\alpha\gamma}(-t^\alpha)\,. \quad (4.17)$$

We recognize that the H-N relaxation model for $\gamma = 1$ reduces to the C-C model, while for $\alpha = 1$ to the D-C model. We also note that whereas for the C-C and H-N models the corresponding response functions decay like a certain negative power of time, the D-C response function exhibits an exponential decay.

# 5. Introduction to the Wright functions

Here we provide a survey of the high transcendental functions related to the Wright special functions.

Like the functions of the Mittag-Leffler type, the functions of the Wright type are known to play fundamental roles in various applications of the fractional calculus. This is mainly due to the fact that they are interrelated with the Mittag-Leffler functions through Laplace and Fourier transformations.

We start providing the definitions in the complex plane for the general Wright function and for two special cases that we call auxiliary functions. Then we devote particular attention to the auxiliary functions in the real field, because they admit a probabilistic interpretation related to the fundamental solutions of certain evolution equations of fractional order. These equations are fundamental to understand phenomena of anomalous diffusion or intermediate between diffusion and wave propagation.

At the end we add some historical and bibliographical notes.

# 6. The Wright function $W_{\lambda,\mu}(z)$

The Wright function, that we denote by $W_{\lambda,\mu,}(z)$ is so named in honour of E. Maitland Wright, the eminent British mathematician, who introduced and investigated this function in a series of notes starting from 1933 in the framework of the asymptotic theory of partitions, see [Wright (1933); 1935a; 1935b].The function is defined by the series representation, convergent in the whole $z$-complex plane,

$$\boxed{W_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\lambda n+\mu)}\,, \quad \lambda > -1\,, \quad \mu \in \mathbb{C}\,,} \qquad (F.1)$$

so $W_{\lambda,\mu}(z)$ is an *entire function*. Originally Wright assumed $\lambda > 0$, and, only in 1940, he considered $-1 < \lambda < 0$, see [Wright 1940]. We note that in the handbook of the Bateman Project [Erdelyi et al. Vol. 3, Ch. 18], presumably for a misprint, $\lambda$ is restricted to be non negative.

## The integral representation

$$\boxed{W_{\lambda,\mu}(z) = \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\sigma + z\sigma^{-\lambda}}\,\frac{d\sigma}{\sigma^{\mu}}\,,\ \lambda > -1\,,\ \mu \in \mathbb{C}\,,} \quad (F.2)$$

where $Ha$ denotes the Hankel path. The equivalence of the series and integral representations is easily proved using the Hankel formula for the Gamma function

$$\frac{1}{\Gamma(\zeta)} = \int_{Ha} \mathrm{e}^{u}\, u^{-\zeta}\, du\,,\ \zeta \in \mathbb{C}\,,$$

and performing a term-by-term integration. In fact,

$$W_{\lambda,\mu}(z) = \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\sigma + z\sigma^{-\lambda}}\,\frac{d\sigma}{\sigma^{\mu}} = \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\sigma}\left[\sum_{n=0}^{\infty}\frac{z^n}{n!}\,\sigma^{-\lambda n}\right]\frac{d\sigma}{\sigma^{\mu}}$$

$$= \sum_{n=0}^{\infty}\frac{z^n}{n!}\left[\frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\sigma}\,\sigma^{-\lambda n-\mu}\,d\sigma\right] = \sum_{n=0}^{\infty}\frac{z^n}{n!\,\Gamma[\lambda n + \mu]}\,.$$

It is possible to prove that the Wright function is entire of order $1/(1+\lambda)$, hence of exponential type if $\lambda \ge 0$. The case $\lambda = 0$ is trivial since $W_{0,\mu}(z) = \mathrm{e}^{z}/\Gamma(\mu)$.

## Asymptotic expansions.

For the detailed asymptotic analysis in the whole complex plane for the Wright functions, the interested reader is referred to Wong and Zhao (1999a),(1999b), who have considered the two cases $\lambda \geq 0$ and $-1 < \lambda < 0$ separately, including a description of Stokes' discontinuity and its smoothing.

In the second case, that, as a matter of fact is the most interesting for us, we set $\lambda = -\nu \in (-1, 0)$, and we recall the asymptotic expansion originally obtained by Wright himself, that is valid in a suitable sector about the negative real axis as $|z| \to -\infty$,

$$\begin{aligned} W_{-\nu,\mu}(z) &= Y^{1/2-\mu}\, \mathrm{e}^{-Y} \left[ \sum_{m=0}^{M-1} A_m\, Y^{-m} + O(|Y|^{-M}) \right], \\ Y = Y(z) &= (1-\nu)\, (-\nu^{\nu}\, z)^{1/(1-\nu)}, \end{aligned} \tag{F.3}$$

where the $A_m$ are certain real numbers.

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## Generalization of the Bessel functions.

For $\lambda = 1$ and $\mu = \nu + 1 \geq 0$ the Wright functions turn out to be related to the well known Bessel functions $J_\nu$ and $I_\nu$ by the identities:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu W_{1,\nu+1}\left(-\frac{z^2}{4}\right), \qquad (F.4)$$

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu W_{1,\nu+1}\left(\frac{z^2}{4}\right). \qquad (F.5)$$

In view of this property some authors refer to the Wright function as the *Wright generalized Bessel function* (misnamed also as the *Bessel-Maitland function*) and introduce the notation

$$J_\nu^{(\lambda)}(z) := \left(\frac{z}{2}\right)^\nu W_{\lambda,\nu+1}\left(-\frac{z^2}{4}\right) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(\lambda n + \nu + 1)}, \qquad (F.6)$$

with $\lambda > 0$ and $\nu > -1$. In particular $J_\nu^{(1)}(z) := J_\nu(z)$.

## Recurrence relations

Some of the properties, that the Wright functions share with the most popular Bessel functions, were enumerated by Wright himself.

Hereafter, we quote some relevant relations from the handbook of Bateman Project Handbook, see [Erdelyi 1955, Vol. 3, Ch. 18]:

$$\lambda z \, W_{\lambda,\lambda+\mu}(z) = W_{\lambda,\mu-1}(z) + (1-\mu)\, W_{\lambda,\mu}(z)\,, \qquad (F.7)$$

$$\frac{d}{dz} W_{\lambda,\mu}(z) = W_{\lambda,\lambda+\mu}(z)\,. \qquad (F.8)$$

We note that these relations can easily be derived from the series or integral representations, (F.1) or (F.2).

# 7. The auxiliary functions of the Wright type

In his first analysis of the time fractional diffusion equation the Author [Mainardi 1993-1994], aware of the Bateman project but not of 1940 paper by Wright, introduced the two (Wright-type) entire **auxiliary functions**,

$$\boxed{F_\nu(z) := W_{-\nu,0}(-z)\,,\ 0 < \nu < 1\,,} \qquad (F.9)$$

and

$$\boxed{M_\nu(z) := W_{-\nu,1-\nu}(-z)\,,\ 0 < \nu < 1\,,} \qquad (F.10)$$

inter-related through

$$\boxed{F_\nu(z) = \nu\, z\, M_\nu(z)\,.} \qquad (F.11)$$

As a matter of fact the functions $F_\nu(z)$ and $M_\nu$ are particular cases of $W_{\lambda,\mu}(z)$ by setting $\lambda = -\nu$ and $\mu = 0$, $\mu = 1$, respectively.

## Series representations

$$
\begin{aligned}
F_\nu(z) &:= \sum_{n=1}^{\infty} \frac{(-z)^n}{n!\,\Gamma(-\nu n)}\,,\\
&:= \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{n!}\,\Gamma(\nu n+1)\,\sin(\pi\nu n)\,,
\end{aligned}
\qquad (F.11)
$$

and

$$
\begin{aligned}
M_\nu(z) &:= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma[-\nu n+(1-\nu)]}\,,\\
&:= \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!}\,\Gamma(\nu n)\,\sin(\pi\nu n)\,,
\end{aligned}
\qquad (F.12)
$$

The second series representations in (F.11)-(F.12) have been obtained by using the well-known reflection formula for the Gamma function,

$$\Gamma(\zeta)\,\Gamma(1-\zeta) = \pi/\sin\pi\zeta\,.$$

## The integral representations

$$\boxed{F_\nu(z) := \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\,\sigma - z\sigma^\nu}\, d\sigma\,,} \qquad (F.14)$$

$$\boxed{M_\nu(z) := \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\,\sigma - z\sigma^\nu}\, \frac{d\sigma}{\sigma^{1-\nu}}\,,} \qquad (F.15)$$

We note that the relation (F.11), $F_\nu(z) = \nu z M_\nu(z)$, can be obtained directly from (F.12)-(F.13) with an integration by parts. In fact,

$$M_\nu(z) = \int_{Ha} \mathrm{e}^{\,\sigma - z\sigma^\nu}\, \frac{d\sigma}{\sigma^{1-\nu}} = \int_{Ha} \mathrm{e}^{\,\sigma} \left(-\frac{1}{\nu z}\,\frac{d}{d\sigma}\mathrm{e}^{\,-z\sigma^\nu}\right) d\sigma$$

$$= \frac{1}{\nu z}\int_{Ha} \mathrm{e}^{\,\sigma - z\sigma^\nu}\, d\sigma = \frac{F_\nu(z)}{\nu z}\,.$$

As usual, the equivalence of the series and integral representations is easily proved using the Hankel formula for the Gamma function and performing a term-by-term integration.

## Special cases

Explicit expressions of $F_\nu(z)$ and $M_\nu(z)$ in terms of known functions are expected for some particular values of $\nu$. In [Mainardi & Tomirotti 1994] the Authors have shown that for $\nu = 1/q$, where $q \geq 2$ is a positive integer, the auxiliary functions can be expressed as a sum of $(q-1)$ simpler entire functions. In the particular cases $q = 2$ and $q = 3$ we find

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \left(\frac{1}{2}\right)_m \frac{z^{2m}}{(2m)!} = \frac{1}{\sqrt{\pi}} \exp\left(-z^2/4\right), \tag{F.16}$$

and

$$M_{1/3}(z) = \frac{1}{\Gamma(2/3)} \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)_m \frac{z^{3m}}{(3m)!} - \frac{1}{\Gamma(1/3)} \sum_{m=0}^{\infty} \left(\frac{2}{3}\right)_m \frac{z^{3m+1}}{(3m+1)!} \tag{F.17}$$

$$= 3^{2/3}\,\mathrm{Ai}\left(z/3^{1/3}\right),$$

where *Ai* denotes the *Airy function*.

Furthermore, it can be proved that $M_{1/q}(z)$ satisfies the differential equation of order $q-1$

$$\frac{d^{q-1}}{dz^{q-1}} M_{1/q}(z) + \frac{(-1)^q}{q} z \, M_{1/q}(z) = 0 \,, \qquad (F.18)$$

subjected to the $q-1$ initial conditions at $z=0$, derived from (F.15),

$$M_{1/q}^{(h)}(0) = \frac{(-1)^h}{\pi} \Gamma[(h+1)/q] \, \sin[\pi \, (h+1)/q] \,, \qquad (F.19)$$

with $h = 0, \, 1, \, \ldots \, q-2$. We note that, for $q \geq 4$, Eq. (F.18) is akin to the *hyper-Airy* differential equation of order $q-1$, see *e.g.* [Bender & Orszag 1987].

We point out that the most relevant applications of our auxiliary functions, are when the variable is real. More precisely, from now on, we will consider functions that are defined either on the positive real semi-axis $\mathbb{R}^+$ or on all of $\mathbb{R}$ in a symmetric way. We agree to denote the variable with $r$ or $x$ or $t$ when it is restricted to $\mathbb{R}^+$ and with $|x|$ for all of $\mathbb{R}$.

## The asymptotic representation of $M_\nu(r)$ as $r \to \infty$

of the function $M_\nu(r)$ as $r \to \infty$. Choosing as a variable $r/\nu$ rather than $r$, the computation of the requested asymptotic representation by the saddle-point approximation yields, see [Mainardi & Tomirotti 1994],

$$M_\nu(r/\nu) \sim a(\nu)\, r^{(\nu - 1/2)/(1-\nu)} \exp\left[-b(\nu)\, r^{1/(1-\nu)}\right], \qquad (F.20)$$

where

$$a(\nu) = \frac{1}{\sqrt{2\pi\,(1-\nu)}} > 0\,, \quad b(\nu) = \frac{1-\nu}{\nu} > 0\,. \qquad (F.21)$$

The above evaluation is consistent with the first term in Wright's asymptotic expansion (F.3) after having used the definition (F.10).

We point out that in the limit $\nu \to 1^-$ the function $M_\nu(r)$ tends to the Dirac generalized function $\delta(r-1)$.

## Plots of $M_\nu(|x|)$

To gain more insight of the effect of the parameter $\nu$ on the behaviour close to and far from the origin, we will adopt both linear and logarithmic scale for the ordinates.

In Figs. F.1 and F.2 we compare the plots of the $M_\nu(|x|)$-Wright functions in $|x| \le 5$ for some rational values in the ranges $\nu \in [0, 1/2]$ and $\nu \in [1/2, 1]$, respectively. Thus in Fig. F.1 we see the transition from $\exp(-|x|)$ for $\nu = 0$ to $1/\sqrt{\pi}\exp(-x^2)$ for $\nu = 1/2$, whereas in Fig. F.2 we see the transition from $1/\sqrt{\pi}\exp(-x^2)$ to the delta functions $\delta(x \pm 1)$ for $\nu = 1$.

In plotting $M_\nu(|x|)$ at fixed $\nu$ for sufficiently large $|x|$ the asymptotic representation (F.20)-(F.21) is useful since, as $|x|$ increases, the numerical convergence of the series in (F.15) becomes poor and poor up to being completely inefficient: henceforth, the matching between the series and the asymptotic representation is relevant. However, as $\nu \to 1^-$, the plotting remains a very difficult task because of the high peak arising around $x = \pm 1$.

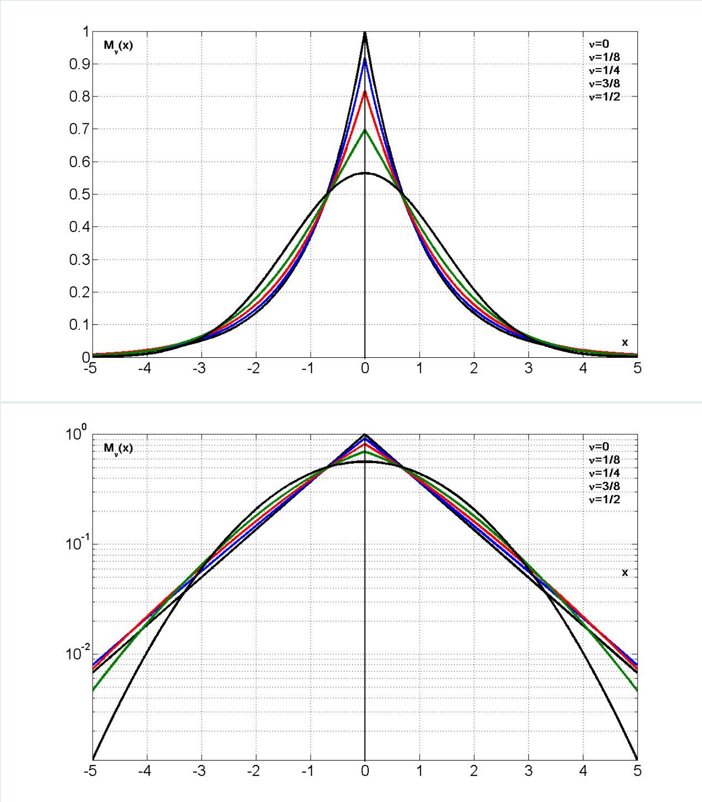


**Fig. F1** - Plots of $M_\nu(|x|)$ with $\nu = 0, 1/8, 1/4, 3/8, 1/2$ for $|x| \le 5$; top: linear scale, bottom: logarithmic scale.

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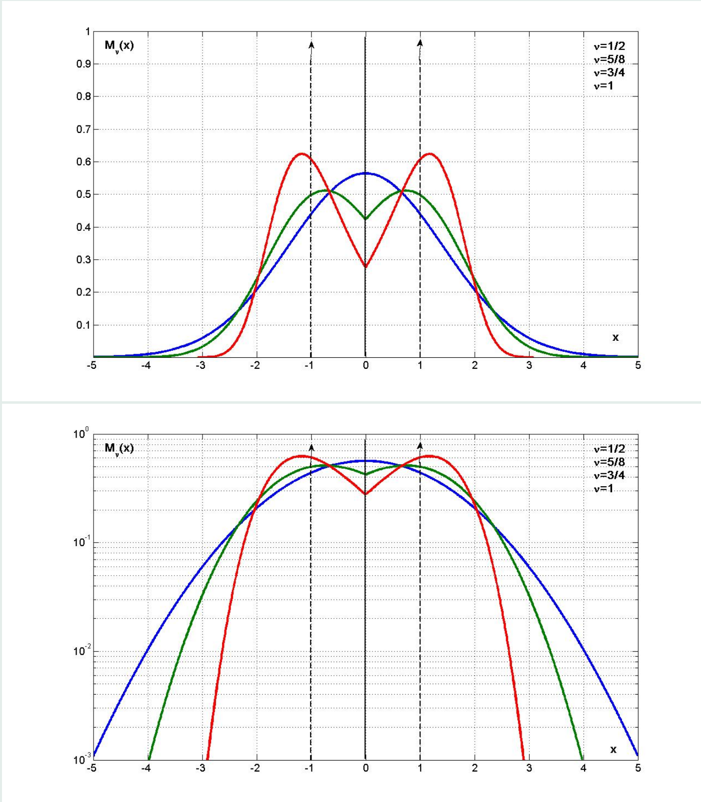


**Fig. F2** - Plots of $M_\nu(|x|)$ with $\nu = 1/2\,,\,5/8\,,\,3/4\,,\,1$ for $|x| \leq 5$: top: linear scale; bottom: logarithmic scale)

## The Laplace transform pairs

Let us consider the Laplace transform of the Wright function using the following notation

$$W_{\lambda,\mu}(\pm r) \div \mathcal{L}\,[W_{\lambda,\mu}(\pm r); s] := \int_0^\infty \mathrm{e}^{\,-s\,r}\, W_{\lambda,\mu}(\pm r)\, dr\,,$$

where $r$ denotes a non negative real variable, *i.e.* $0 \le r < +\infty$ , and $s$ is the Laplace complex parameter.

When $\lambda > 0$ the series representation of the Wright function can be transformed term-by-term. In fact, for a known theorem of the theory of the Laplace transforms, see *e.g.* [Doetsch (197], the Laplace transform of an entire function of exponential type can be obtained by transforming term-by-term the Taylor expansion of the original function around the origin. In this case the resulting Laplace transform turns out to be analytic and vanishing at infinity.

As a consequence, we obtain the Laplace transform pair

$$\boxed{W_{\lambda,\mu}(\pm r) \div \frac{1}{s} E_{\lambda,\mu}\left(\pm\frac{1}{s}\right), \; \lambda > 0\,, \; |s| > 0\,,} \qquad (F.22)$$

where $E_{\lambda,\mu}$ denotes the Mittag-Leffler function in two parameters. The proof is straightforward noting that

$$\sum_{n=0}^{\infty} \frac{(\pm r)^n}{n!\,\Gamma(\lambda n + \mu)} \div \frac{1}{s} \sum_{n=0}^{\infty} \frac{(\pm 1/s)^n}{\Gamma(\lambda n + \mu)}\,,$$

and recalling the series representation of the Mittag-Leffler function,

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}\,, \; \alpha > 0\,, \;\; \beta \in \mathbb{C}\,.$$

For $\lambda \to 0^+$ Eq. (F.22) provides the Laplace transform pair

$$W_{0^+,\mu}(\pm r) = \frac{\mathrm{e}^{\pm r}}{\Gamma(\mu)} \div \frac{1}{\Gamma(\mu)} \frac{1}{s \mp 1} = \frac{1}{s} E_{0,\mu}\left(\pm \frac{1}{s}\right), \quad |s| > 1, \tag{F.23}$$

where, to remain in agreement with (F.22), we have formally put

$$E_{0,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu)} := \frac{1}{\Gamma(\mu)} E_0(z) := \frac{1}{\Gamma(\mu)} \frac{1}{1-z}, \quad |z| < 1.$$

We recognize that in this special case the Laplace transform exhibits a simple pole at $s = \pm 1$ while for $\lambda > 0$ it exhibits an essential singularity at $s = 0$.

For $-1 < \lambda < 0$ the Wright function turns out to be an entire function of order greater than 1, so that care is required in establishing the existence of its Laplace transform, which necessarily must tend to zero as $s \to \infty$ in its half-plane of convergence.

For the sake of convenience we limit ourselves to derive the Laplace transform for the special case of $M_\nu(r)$ ; the exponential decay as $r \to \infty$ of the *original* function provided by (F.20) ensures the existence of the *image* function. From the integral representation (F.13) of the $M_\nu$ function we obtain

$$M_\nu(r) \div \frac{1}{2\pi i} \int_0^\infty \mathrm{e}^{-s\,r} \left[ \int_{Ha} \mathrm{e}^{\sigma - r\sigma^\nu} \frac{d\sigma}{\sigma^{1-\nu}} \right] dr$$

$$= \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\sigma} \sigma^{\nu-1} \left[ \int_0^\infty \mathrm{e}^{-r(s+\sigma^\nu)} \, dr \right] d\sigma = \frac{1}{2\pi i} \int_{Ha} \frac{\mathrm{e}^{\sigma} \sigma^{\nu-1}}{\sigma^\nu + s} \, d\sigma \, .$$

Then, by recalling the integral representation of the Mittag-Leffler function,

$$E_\alpha(z) = \frac{1}{2\pi i} \int_{Ha} \frac{\zeta^{\alpha-1}\, \mathrm{e}^{\zeta}}{\zeta^\alpha - z}\, d\zeta\,,\ \alpha > 0\,,\ z \in \mathbb{C}\,,$$

we obtain the Laplace transform pair

$$\boxed{M_\nu(r) := W_{-\nu,1-\nu}(-r) \div E_\nu(-s)\,,\ 0 < \nu < 1\,.} \qquad (F.24)$$

In this case, transforming term-by-term the Taylor series of $M_\nu(r)$ yields a series of negative powers of $s$, that represents the asymptotic expansion of $E_\nu(-s)$ as $s \to \infty$ in a sector around the positive real axis.

We note that (F.24) contains the well-known Laplace transform pair, see *e.g.* [Doetsch 1974],

$$M_{1/2}(r) := \frac{1}{\sqrt{\pi}} \exp\left(-r^2/4\right) \div E_{1/2}(-s) := \exp\left(s^2\right) \operatorname{erfc}(s)\ ,$$

valid $\forall s \in \mathbb{C}$

Analogously, using the more general integral representation (F.2) of the standard Wright function, we can prove that in the case $\lambda = -\nu \in (-1, 0)$ and $\mathrm{Re}(\mu) > 0$, we get

$$W_{-\nu,\mu}(-r) \div E_{\nu,\mu+\nu}(-s)\,,\; 0 < \nu < 1\,. \qquad (F.25)$$

In the limit as $\nu \to 0^+$ (thus $\lambda \to 0^-$) we formally obtain the Laplace transform pair

$$W_{0^-,\mu}(-r) := \frac{e^{-r}}{\Gamma(\mu)} \div \frac{1}{\Gamma(\mu)}\,\frac{1}{s+1} := E_{0,\mu}(-s)\,. \qquad (F.26)$$

Therefore, as $\lambda \to 0^{\pm}$, and $\mu = 1$ we note a sort of continuity in the results (F.23) and (F.26) since

$$W_{0,1}(-r) := e^{-r} \div \frac{1}{(s+1)} = \begin{cases} (1/s)\,E_0(-1/s)\,, & |s| > 1\,; \\ E_0(-s)\,, & |s| < 1\,. \end{cases} \qquad (F.27)$$

We here point out the relevant *Laplace transform pairs related to the auxiliary functions of argument* $r^{-\nu}$, see [Mainardi (1994); (1996a); (1996b)],

$$\boxed{\frac{1}{r}\,F_\nu\,(1/r^\nu) = \frac{\nu}{r^{\nu+1}}\,M_\nu\,(1/r^\nu)\;\div\;\mathrm{e}^{\,-s^\nu}\,,\;0<\nu<1\,.} \qquad (F.28)$$

$$\boxed{\frac{1}{\nu}\,F_\nu\,(1/r^\nu) = \frac{1}{r^\nu}\,M_\nu\,(1/r^\nu)\;\div\;\frac{\mathrm{e}^{\,-s^\nu}}{s^{1-\nu}}\,,\;0<\nu<1\,.} \qquad (F.29)$$

We recall that the Laplace transform pairs in (F.28) were formerly considered by [Pollard (1946)], who provided a rigorous proof based on a formal result by [Humbert (1945)]. Later [Mikusinski (1959)] got a similar result based on his theory of operational calculus, and finally, albeit unaware of the previous results, [Buchen & Mainardi (1975)] derived the result in a formal way. We note, however, that all these Authors were not informed about the Wright functions.

Hereafter we like to provide two independent proofs of (F.28) carrying out the inversion of $\exp(-s^\nu)$, either by the complex Bromwich integral formula or by the formal series method. Similarly we can act for the Laplace transform pair (F.29).

For the complex integral approach we deform the Bromwich path $Br$ into the Hankel path $Ha$, that is equivalent to the original path, and we set $\sigma = sr$. Recalling (F.13)-(F.14), we get

$$\mathcal{L}^{-1}\left[\exp\left(-s^\nu\right)\right] = \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{\,sr\,-\,s^\nu}\,ds = \frac{1}{2\pi i\, r}\int_{Ha} \mathrm{e}^{\,\sigma\,-\,(\sigma/r)^\nu}\,d\sigma$$

$$= \frac{1}{r}\,F_\nu\left(1/r^\nu\right) = \frac{\nu}{r^{\nu+1}}\,M_\nu\left(1/r^\nu\right)\,.$$

Expanding in power series the Laplace transform and inverting term by term we formally get, after recalling (F.12)-(F.13):

$$\mathcal{L}^{-1}\left[\exp\left(-s^\nu\right)\right] = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\,\mathcal{L}^{-1}\left[s^{\nu n}\right] = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\,\frac{r^{-\nu n-1}}{\Gamma(-\nu n)}$$

$$= \frac{1}{r}\,F_\nu\left(1/r^\nu\right) = \frac{\nu}{r^{\nu+1}}\,M_\nu\left(1/r^\nu\right)\,.$$

We note the relevance of Laplace transforms (F.24) and (F.28) in pointing out the non-negativity of the Wright function $M_\nu(x)$ and the complete monotonicity of the Mittag-leffler functions $E_\nu(-x)$ for $x > 0$ and $0 < \nu < 1$. In fact, since $\exp\,(-s^\nu)$ denotes the Laplace transform of a probability density (precisely , the extremal Lévy stable density of index $\nu$, see [Feller (1971)]), the L.H.S. of (F.28) must be non-negative, and so also the L.H.S of F(24). As a matter of fact the Laplace transform pair (F.24) shows, replacing $s$ by $x$, that the spectral representation of the Mittag-Leffler function $E_\nu(-x)$ is expressed in terms of the $M$-Wright function $M_\nu(r)$, that is:

$$E_\nu(-x) = \int_0^\infty \mathrm{e}^{-rx}\, M_\nu(r)\, dr\,,\ 0 < \nu < 1\,,\ x \geq 0\,. \quad (F.30)$$

We now recognize that Eq. (F.30) is consistent with a result derived by [Pollard (1948)].

It is instructive to compare the spectral representation of $E_\nu(-x)$ with that of the function $E_\nu(-t^\nu)$. We recall

$$E_\nu(-t^\nu) = \int_0^\infty \mathrm{e}^{-rt}\, K_\nu(r)\, dr\,,\; 0 < \nu < 1\,,\; t \geq 0\,, \qquad (F.31)$$

with *spectral function*

$$K_\nu(r) = \frac{1}{\pi}\, \frac{r^{\nu-1}\, \sin(\nu\pi)}{r^{2\nu} + 2\, r^\nu\, \cos(\nu\pi) + 1} = \frac{1}{\pi\, r}\, \frac{\sin(\nu\pi)}{r^\nu + r^{-\nu} + 2\, \cos(\nu\pi)}\,. \qquad (F.32)$$

The relationship between $M_\nu(r)$ and $K_\nu(r)$ is worth to be explored. Both functions are non-negative, integrable and normalized in $\mathbb{R}^+$, so they can be adopted in probability theory as density functions.

Whereas the transition $K_\nu(r) \to \delta(r-1)$ for $\nu \to 1$ is easy to be detected numerically in view of the explicit representation (F.32), the analogous transition $M_\nu(r) \to \delta(r-1)$ is quite a difficult matter in view of its series and integral representations. In this respect see the figure hereafter.

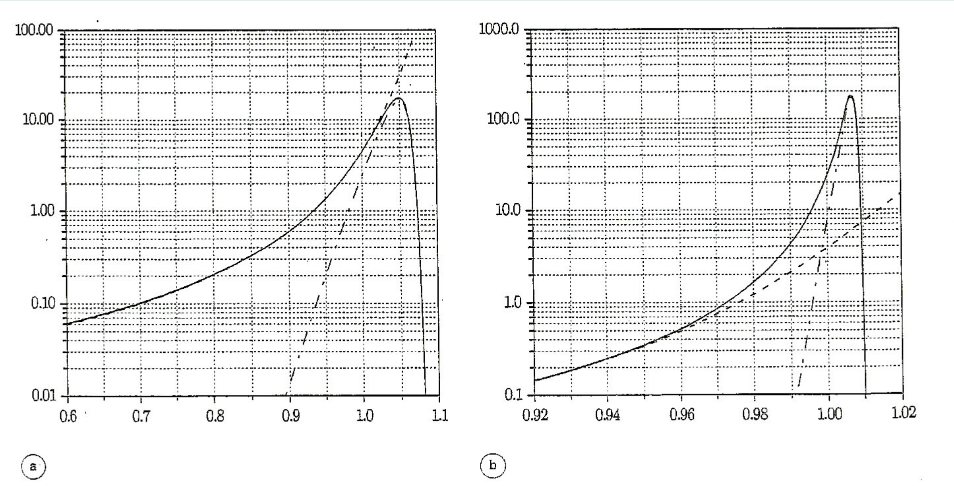


**Fig. 6** - Plots of $M_\nu(r)$ with $\nu = 1 - \epsilon$ around the maximum $r \approx 1$.

Here we compare the cases (a) $\epsilon = 0.01$, (b) $\epsilon = 0.001$, obtained by Pipkin's method (continuous line), 100 terms-series (dashed line) and the standard saddle-point method (dashed-dotted line).

# Notes on the auxiliary Wright functions

In early nineties, in his former analysis of fractional equations interpolating diffusion and wave-propagation, the present Author, see *e.g.* [Mainardi (WASCOM 1993)], has introduced the functions of the Wright type

$$F_\nu(z) := W_{-\nu,0}(-z)\,, \quad M_\nu(z) := W_{-\nu,1-\nu}(-z)$$

with $0 < \nu < 1$, inter-related through $F_\nu(z) = \nu\, z\, M_\nu(z)$ to characterize the solutions for typical boundary value problems.

Being in that time only aware of the Bateman project where the parameter $\lambda$ of the Wright function $W_{\lambda,\mu}(z)$ was erroneously restricted to non-negative values, the Author thought to have extended the original Wright function, in an original way, calling $F_\nu$ and $M_\nu$ *auxiliary functions.*

Presumably for this reason the function $M_\nu$ is referred as the *Mainardi function* in the book by Podlubny (Academic Press 1999) and in some papers including *e.g.* Balescu (CSF 2007).

It was Professor Stanković, during the presentation of the paper [Mainardi-Tomirotti (TMSF1994] at the Conference *Transform Methods and Special Functions, Sofia 1994*, who informed the Author that this extension for $-1 < \lambda < 0$ was already made just by Wright himself in 1940 (following his previous papers in 1930's). In his paper devoted to the 80-th birthday of Prof. Stanković, see [Mainardi-Gorenflo-Vivoli (FCAA 2005)], the Author took the occasion to renew his personal gratitude to Prof. Stanković for this earlier information that has induced him to study the original papers by Wright and work (also in collaboration) on the functions of the Wright type for further applications, see *e.g.* [Gorenflo-Luchko-Mainardi (1999); (2000)] and [Mainardi-Pagnini (2003)].

For more mathematical details on the functions of the Wright type, the reader may be referred to the article by [Kilbas-Saigo-Trujillo (FCAA 2002)] and references therein. For the numerical point of view we like to point out the recent paper by [Luchko (FCAA 2008)], where algorithms are provided for computation of the Wright function on the real axis with prescribed accuracy.

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# 8. Probability and fractional diffusion

For certain **stochastic processes of renewal type**, functions of Mittag-Leffler and Wright type can be adopted as **probability distributions**, see Mainardi-Gorenflo-Vivoli (FCAA 2005), where they are compared.

Here, we restrict our attention to the $M$-Wright functions with support both $\mathbb{R}^+$ and all of $\mathbb{R}$ (in symmetric way) that play fundamental roles in **stochastic processes of fractional diffusion**

The exponential decay for $x \to +\infty$ pointed out in Eqs (F.20)-(F.21) ensures that $M_\nu(x)$ is absolutely integrable in $\mathbb{R}^+$ and in $\mathbb{R}$. By recalling the Laplace transform pair (F.24) related to the Mittag-Leffler function, we get

$$\int_0^{+\infty} M_\nu(x)\, dx = \frac{1}{2}\int_{-\infty}^{+\infty} M_\nu(|x|)\, dx = E_\nu(0) = 1\,. \qquad (F.33)$$

Being non-negative, $M_\nu(x)$ and $\frac{1}{2}M_\nu(|x|)$ can be interpreted as probability density functions in $\mathbb{R}^+$ and in $\mathbb{R}$, respectively. More generally, we can compute from (F.24) all the moments in $\mathbb{R}^+$, for $n = 1, 2 \ldots,$ as follows

$$\int_0^{+\infty} x^n \, M_\nu(x)\, dx = \lim_{s\to 0} (-1)^n \frac{d^n}{ds^n} E_\nu(-s) = \frac{\Gamma(n+1)}{\Gamma(\nu n + 1)} \,. \tag{F.34}$$

More generally than Eqs. (F.33)-(F.34) we can derive all moments of order $\delta > -1$ as sketched hereafter :

$$\int_0^{+\infty} x^\delta \, M_\nu(x)\, dx = \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\,\sigma} \left[ \int_0^{+\infty} \mathrm{e}^{\,-\sigma^\nu x} \, x^\delta \, dx \right] \frac{d\sigma}{\sigma^{1-\nu}}$$

$$= \frac{n!}{2\pi i} \int_{Ha} \frac{\mathrm{e}^{\,\sigma}}{\sigma^{\nu\delta+1}} \, d\sigma = \frac{\Gamma(\delta+1)}{\Gamma(\nu\delta+1)} \,,$$

where the exchange between the two integrals turns out to be legitimate.

**The Fourier transform of the $M$-Wright function.**

$$\mathcal{F}\left[\tfrac{1}{2}M_\nu(|x|)\right] := \frac{1}{2}\int_{-\infty}^{+\infty} e^{i\kappa x}\, M_\nu(|x|)\, dx = \int_0^{\infty} \cos(\kappa x)\, M_\nu(x)\, dx = E_{2\nu}(-\kappa^2)\,.$$

$$\int_0^{\infty} \cos(\kappa x)\, M_\nu(x)\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{\kappa^{2n}}{(2n)!} \int_0^{\infty} x^{2n}\, M_\nu(x)\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{\kappa^{2n}}{\Gamma(2\nu n + 1)} = E_{2\nu,1}(-\kappa^2)\,.$$

$$\int_0^{\infty} \sin(\kappa x)\, M_\nu(x)\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{\kappa^{2n+1}}{(2n+1)!} \int_0^{\infty} x^{2n+1}\, M_\nu(x)\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{\kappa^{2n+1}}{\Gamma(2\nu n + 1 + \nu)} = \kappa E_{2\nu,1+\nu}(-\kappa^2)\,.$$

## Relations with Lévy stable distributions

The term stable has been assigned by the French mathematician Paul Lévy, who in the 1920's years started a systematic research in order to generalize the celebrated *Central Limit Theorem* to probability distributions with infinite variance. For stable distributions we can assume the following

DEFINITION:

**If two independent real random variables with the same shape or type of distribution are combined linearly and the distribution of the resulting random variable has the same shape, the common distribution (or its type, more precisely) is said to be stable.**

The restrictive condition of stability enabled Lévy (and then other authors) to derive the **canonic form** for the Fourier transform of the densities of these distributions. Such transform in probability theory is known as **characteristic function**.

Here we follow the parameterization in [Feller (1952);(1971)] revisited in [Gorenflo & Mainardi (FCAA 1998)] and in [Mainardi-Luchko-Pagnini (FCAA 2001].

Denoting by $L_\alpha^\theta(x)$ a generic stable density in $\mathbb{R}$, where $\alpha$ is the **index of stability** and and $\theta$ the asymmetry parameter, improperly called **skewness**, its characteristic function reads:

$$L_\alpha^\theta(x) \div \widehat{L}_\alpha^\theta(\kappa) = \exp\left[-\psi_\alpha^\theta(\kappa)\right], \quad \psi_\alpha^\theta(\kappa) = |\kappa|^\alpha \, \mathrm{e}^{i(\mathrm{sign}\,\kappa)\theta\pi/2}, \quad (F.35)$$

$$0 < \alpha \leq 2, \; |\theta| \leq \min\{\alpha, 2-\alpha\}.$$

We note that the allowed region for the parameters $\alpha$ and $\theta$ turns out to be a diamond in the plane $\{\alpha, \theta\}$ with vertices in the points $(0,0)$, $(1,1)$, $(1,-1)$, $(2,0)$, that we call the **Feller-Takayasu diamond**, see Fig. F.4. For values of $\theta$ on the border of the diamond (that is $\theta = \pm\alpha$ if $0 < \alpha < 1$, and $\theta = \pm(2-\alpha)$ if $1 < \alpha < 2$) we obtain the so-called **extremal stable densities**.

We note the **symmetry relation** $L_\alpha^\theta(-x) = L_\alpha^{-\theta}(x)$, so that a stable density with $\theta = 0$ is symmetric

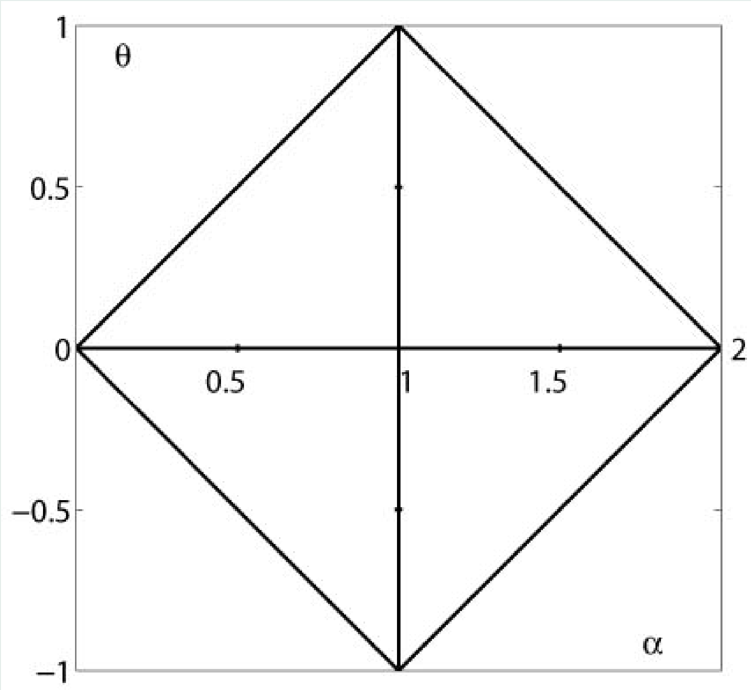


Figure 1: The Feller-Takayasu diamond for Lévy stable densities.

Stable distributions have noteworthy properties of which the interested reader can be informed from the existing literature. Here-after we recall some peculiar **Properties**:

- **The class of stable distributions possesses its own domain of attraction**, see *e.g.* [Feller (1971)].

- **Any stable density is unimodal and indeed bell-shaped**, *i.e.* its $n$-th derivative has exactly $n$ zeros in $\mathbb{R}$, see [Gawronski (1984)].

- **The stable distributions are self-similar and infinitely divisible**. These properties derive from the canonic form (F.35) through the scaling property of the Fourier transform.

**Self-similarity** means

$$L_\alpha^\theta(x,t) \div \exp\left[-t\psi_\alpha^\theta(\kappa)\right] \Longleftrightarrow L_\alpha^\theta(x,t) = t^{-1/\alpha}\, L_\alpha^\theta(x/t^{1/\alpha})], \tag{F.36}$$

whre $t$ is a positive parameter. If $t$ is time, then $L_\alpha^\theta(x,t)$ is a spatial density evolving on time with self-similarity.

**Infinite divisibility** means that for every positive integer $n$, the characteristic function can be expressed as the $n$th power of some characteristic function, so that any stable distribution can be expressed as the $n$-fold convolution of a stable distribution of the same type. Indeed, taking in (F.35) $\theta = 0$, without loss of generality, we have

$$\mathrm{e}^{-t|\kappa|^\alpha} = \left[\mathrm{e}^{-(t/n)|\kappa|^\alpha}\right]^n \Longleftrightarrow L_\alpha^0(x,t) = \left[L_\alpha^0(x,t/n)\right]^{*n}, \tag{F.37}$$

where

$$\left[L_\alpha^0(x,t/n)\right]^{*n} := L_\alpha^0(x,t/n) * L_\alpha^0(x,t/n) * \cdots * L_\alpha^0(x,t/n)$$

is the multiple Fourier convolution in $\mathbb{R}$ with $n$ identical terms.

Only in special cases we get well-known probability distributions.

For $\alpha = 2$ (so $\theta = 0$), we recover the **Gaussian pdf**, that turns out to be the only stable density with finite variance, and more generally with finite moments of any order $\delta \geq 0$. In fact

$$L_2^0(x) = \frac{1}{2\sqrt{\pi}} \mathrm{e}^{-x^2/4} \,.$$

All the other stable densities have finite absolute moments of order $\delta \in [-1, \alpha)$.

For $\alpha = 1$ and $|\theta| < 1$, we get

$$L_1^\theta(x) = \frac{1}{\pi} \frac{\cos(\theta\pi/2)}{[x + \sin(\theta\pi/2)]^2 + [\cos(\theta\pi/2)]^2} \,,$$

which for $\theta = 0$ includes the **Cauchy-Lorentz pdf**,

$$L_1^0(x) = \frac{1}{\pi} \frac{1}{1 + x^2} \,.$$

In the limiting cases $\theta = \pm 1$ for $\alpha = 1$ we obtain the **singular Dirac pdf's**

$$L_1^{\pm 1}(x) = \delta(x \pm 1) \,.$$

In general we must recall the power series expansions provided in [Feller (1971)]. We restrict our attention to $x > 0$ since the evaluations for $x < 0$ can be obtained using the symmetry relation.

The convergent expansions of $L_\alpha^\theta(x)$ $(x > 0)$ turn out to be for $0 < \alpha < 1\,, \quad |\theta| \le \alpha$ :

$$L_\alpha^\theta(x) = \frac{1}{\pi\, x} \sum_{n=1}^{\infty} (-x^{-\alpha})^n \, \frac{\Gamma(1+n\alpha)}{n!} \, \sin\left[\frac{n\pi}{2}(\theta - \alpha)\right] \,; \tag{F.38}$$

for $1 < \alpha \le 2\,, \quad |\theta| \le 2-\alpha$ :

$$L_\alpha^\theta(x) = \frac{1}{\pi\, x} \sum_{n=1}^{\infty} (-x)^n \, \frac{\Gamma(1+n/\alpha)}{n!} \, \sin\left[\frac{n\pi}{2\alpha}(\theta - \alpha)\right] \,. \tag{F.39}$$

From the series in (F.38) and the symmetry relation we note that the extremal stable densities for $0 < \alpha < 1$ are unilateral, precisely vanishing for $x > 0$ if $\theta = \alpha$, vanishing for $x < 0$ if $\theta = -\alpha$. In particular the unilateral extremal densities $L_\alpha^{-\alpha}(x)$ with $0 < \alpha < 1$ have as Laplace transform $\exp\,(-s^\alpha)$.

From a comparison between the series expansions in (F.38)-(F.39) and in (F.14)-(F.15), we recognize that for $x > 0$ **the auxiliary functions of the Wright type are related to the extremal stable densities** as follows, see [Mainardi-Tomirotti (1997)],

$$L_\alpha^{-\alpha}(x) = \frac{1}{x}\, F_\alpha(x^{-\alpha}) = \frac{\alpha}{x^{\alpha+1}}\, M_\alpha(x^{-\alpha})\,, \quad 0 < \alpha < 1\,, \quad (F.40)$$

$$L_\alpha^{\alpha-2}(x) = \frac{1}{x}\, F_{1/\alpha}(x) = \frac{1}{\alpha}\, M_{1/\alpha}(x)\,, \quad 1 < \alpha \le 2\,. \quad (F.41)$$

In Eqs. (F.40)-(F.41), for $\alpha = 1$, the skewness parameter turns out to be $\theta = -1$, so we get the singular limit $L_1^{-1}(x) = M_1(x) = \delta(x-1)$.

More generally, all (regular) stable densities, given in Eqs. (F.38)-(F.39), were recognized to belong to the class of Fox $H$-functions, as formerly shown by [Schneider (LNP 1986)], see also [Mainardi-Pagnini-Saxena (2003)].

## The M-Wright functions in two variables: pdf's for time-fractional diffusion and time-fractional drift

We now consider the $M$-Wright functions as spatial probability densities (in $x \geq 0$) evolving in time ($t \geq 0$) with self-similarity, that is

$$M_\nu(x,t) := t^{-\nu}\, M_\nu(xt^{-\nu})\,, \quad x,t \geq 0 \quad 0<\nu<1\,. \qquad (F.42)$$

The absolute moments of order $\delta \geq -1$ are all finite, resulting from (F.34)

$$\int_0^{+\infty} x^\delta\, M_\nu(x,t)\, dx = \frac{\Gamma(\delta+1)}{\Gamma(\nu\delta+1)}\, t^{\nu\delta}\,. \qquad (F.43)$$

From Eq. (F.29) we derive the Laplace transform of $M_\nu(x,t)$ with respect to $t \in \mathbb{R}^+$,

$$\mathcal{L}\left\{M_\nu(x,t); t \to s\right\} = s^{\nu-1}\, \mathrm{e}^{\,-xs^\nu}\,. \qquad (F.43)$$

From Eq. (F.24) we derive the Laplace transform of $M_\nu(x,t)$ with respect to $x \in \mathbb{R}^+$,

$$\mathcal{L}\left\{M_\nu(x,t); x \to s\right\} = E_\nu\left(-st^\nu\right)\,. \qquad (F.44)$$

We now consider the *symmetric* spatial probability densities (in all of $\mathbb{R}$) evolving in time obtained by taking

$$\frac{1}{2} M_\nu(|x|,t)\,, \quad -\infty < x < +\infty\,,\ t \geq 0\,, \quad 0 < \nu < 1\,, \quad (F.45)$$

whose absolute moments of order $\delta > -1$ are given by the RHS in (F.43). The even moments of order $\delta = 2n$ are

$$\frac{1}{2}\int_{-\infty}^{+\infty} x^{2n}\, M_\nu(|x|,t)\,dx = \frac{\Gamma(2n+1)}{\Gamma(2\nu n+1)}\, t^{2n\nu}\,, \quad t \geq 0\,, \quad (F.46)$$

and the characteristic function (that is Fourier transform with respect to $x \in \mathbb{R}$) is

$$\mathcal{F}\left\{\frac{1}{2} M_\nu(|x|,t); x \to \kappa\right\} = E_{2\nu}\left(-\kappa^2 t^{2\nu}\right)\,. \quad (F.47)$$

In particular we have,

$$\begin{cases} \int_0^\infty \cos(\kappa x)\, M_\nu(x,t)\,dx &= E_{2\nu,1}(-\kappa^2\, t^{2\nu})\,, \\ \int_0^\infty \sin(\kappa x)\, M_\nu(x,t)\,dx &= \kappa\, t^\nu\, E_{2\nu,1+\nu}(-\kappa^2\, t^{2\nu})\,. \end{cases} \quad (F.48)$$

It is worthwhile to note that for $\nu = 1/2$ we recover the **Gaussian density** evolving in time with variance $\sigma^2 = 2t$ consistently with **normal diffusion**:

$$\frac{1}{2} M_{1/2}(|x|,t) = \frac{1}{2\sqrt{\pi} t^{1/2}} \, \mathrm{e}^{\,-x^2/(4t)} \,, \qquad (F.49)$$

$$\sigma^2(t) := \tfrac{1}{2} \int_{-\infty}^{+\infty} x^2 \, M_{1/2}(|x|,t) \, dx = 2t \,.$$

For generic $M$-Wright functions in two variables we have the **composition rule**, see [Mainardi-Pagnini-Gorenflo (2003)]

*Let $M_\lambda(x;t)$, $M_\mu(x;t)$ and $M_\nu(x;t)$ be $M$-Wright functions of orders $\lambda, \mu, \nu \in (0,1)$ respectively, we have for any $x, t \geq 0$:*

$$M_\nu(x,t) = \int_0^\infty M_\lambda(x;\tau) \, M_\mu(\tau;t) \, d\tau \,, \quad \textit{with} \;\; \nu = \lambda \, \mu \,. \qquad (F.50)$$

Following the PhD thesis by A. Mura (2008), the above **composition formula** (F.50) was used to define self-similar stochastic processes (with stationary increments), that properly generalize the most popular Gaussian processes.

These processes are governed by stretched time-fractional diffusion equations, and are suitable to describe phenomena of **anomalous diffusion**, both slow and fast. see [Mura-Pagnini (2008)], [Mura-Taqqu-Mainardi (2008)], [Mura-Mainardi (2009)].

They are referred to as
**Generalized grey Brownian Motions**
and include both
**Gaussian Processes** (standard Brownian motion, fractional Brownian motion)
and **non-Gaussian Processes** (Schneider's grey Brownian motion),
see later.

## The time-fractional diffusion equation

There exist three equivalent forms of the time-fractional diffusion equation of a single order, two with fractional derivative and one with fractional integral, provided we refer to the standard initial condition $u(x,0) = u_0(x)$.

Taking a real number $\beta \in (0,1)$, the time-fractional diffusion equation of order $\beta$ in the Riemann-Liouville sense reads

$$\frac{\partial u}{\partial t} = K_\beta \, D_t^{1-\beta} \, \frac{\partial^2 u}{\partial x^2} \,, \qquad (F.51)$$

in the Caputo sense reads

$$_*D_t^\beta u = K_\beta \, \frac{\partial^2 u}{\partial x^2} \,, \qquad (F.52)$$

and in integral form

$$u(x,t) = u_0(x) + K_\beta \, \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \, \frac{\partial^2 u(x,\tau)}{\partial x^2} \, d\tau \,. \qquad (F.53)$$

where $K_\beta$ is a sort of fractional diffusion coefficient of dimensions $[K_\beta] = [L]^2 [T]^{-\beta} = cm^2/sec^\beta$.

The fundamental solution (or **Green function**) $\mathcal{G}_\beta(x,t)$ for the equivalent Eqs. (F.51) - (F..53), that is the solution corresponding to the initial condition

$$\mathcal{G}_\beta(x,0^+) = u_0(x) = \delta(x) \qquad (F.54)$$

can be expressed in terms of the $M$-Wright function

$$\mathcal{G}_\beta(x,t) = \frac{1}{2}\frac{1}{\sqrt{K_\beta}\, t^{\beta/2}}\, M_{\beta/2}\left(\frac{|x|}{\sqrt{K_\beta}\, t^{\beta/2}}\right) . \qquad (F..55)$$

The corresponding variance can be promptly obtained

$$\sigma^2_\beta(t) := \int_{-\infty}^{+\infty} x^2\, \mathcal{G}_\beta(x,t)\, dx = \frac{2}{\Gamma(\beta+1)}\, K_\beta\, t^\beta\, . \qquad (F.56)$$

As a consequence, for $0 < \beta < 1$ the variance is consistent with a process of **slow diffusion** with similarity exponent $H = \beta/2$.

**Appendix A** The fundamental solution $\mathcal{G}_\beta(x,t)$ for the time-fractional diffusion equation can be obtained by applying in sequence the Fourier and Laplace transforms to any form chosen among Eqs. (F.51)-(F.53). Let us devote our attention to the integral form (F.53) using non-dimensional variables by setting $K_\beta = 1$ and adopting the notation $J_t^\beta$ for the fractional integral. Then, our Cauchy problem reads

$$\mathcal{G}_\beta(x,t) = \delta(x) + J_t^\beta \frac{\partial^2 \mathcal{G}_\beta}{\partial x^2}(x,t)\,. \qquad (A.1)$$

In the Fourier-Laplace domain, after applying formula for the Laplace transform of the fractional integral and observing $\widehat{\delta}(\kappa) \equiv 1$, we get

$$\widetilde{\widehat{G}}_\beta(\kappa,s) = \frac{1}{s} - \frac{\kappa^2}{s^\beta}\,\widetilde{\widehat{G}}_\beta(\kappa,s)\,,$$

from which

$$\widetilde{\widehat{\mathcal{G}}}_\beta(\kappa,s) = \frac{s^{\beta-1}}{s^\beta + \kappa^2}\,,\; 0 < \beta \le 1\,,\; \mathrm{Re}(s) > 0\,,\; \kappa \in \mathbb{R}\,. \qquad (A.2)$$

**Strategy (S1):** Recalling the Fourier transform pair

$$\frac{a}{b+\kappa^2} \stackrel{\mathcal{F}}{\leftrightarrow} \frac{a}{2b^{1/2}} \, \mathrm{e}^{-|x| b^{1/2}}, \quad a, b > 0\,, \qquad (A.3)$$

and setting $a = s^{\beta-1}$, $b = s^\beta$, we get

$$\widetilde{\mathcal{G}_\beta}(x,s) = \frac{1}{2} s^{\beta/2-1} \, \mathrm{e}^{-|x| s^{\beta/2}}\,. \qquad (A.4)$$

**Strategy (S2):** Recalling the Laplace transform pair

$$\frac{s^{\beta-1}}{s^\beta + c} \stackrel{\mathcal{L}}{\leftrightarrow} E_\beta(-ct^\beta)\,, \quad c > 0\,, \qquad (A.5)$$

and setting $c = \kappa^2$, we have

$$\widehat{G_\beta}(\kappa, t) = E_\beta(-\kappa^2 t^\beta)\,. \qquad (A.6)$$

Both strategies lead to the result

$$\mathcal{G}_\beta(x,t) = \frac{1}{2} M_{\beta/2}(|x|, t) = \frac{1}{2}\, t^{-\beta/2}\, M_{\beta/2}\left(\frac{|x|}{t^{\beta/2}}\right)\,, \qquad (A.7)$$

consistent with Eq. (F.55).

## The time-fractional drift equation

Let us finally note that the $M$-Wright function does appear also in the fundamental solution of the time-fractional drift equation. Writing this equation in non-dimensional form and adopting the Caputo derivative we have

$$ {}_*D_t^{\beta} u(x,t) = -\frac{\partial}{\partial x} u(x,t)\,, \quad -\infty < x < +\infty\,,\; t \geq 0\,, \quad (B.1)$$

where $0 < \beta < 1$ and $u(x,0^+) = u_0(x)$. When $u_0(x) = \delta(x)$ we obtain the fundamental solution (Green function) that we denote by $\mathcal{G}^*_\beta(x,t)$. Following the approach of Appendix A, we show that

$$ \mathcal{G}^*_\beta(x,t) = \begin{cases} t^{-\beta}\, M_\beta\left(\dfrac{x}{t^\beta}\right), & x > 0\,, \\ 0\,, & x < 0\,, \end{cases} \quad (B.2)$$

that for $\beta = 1$ reduces to the right running pulse $\delta(x-t)$ for $x > 0$.

In the Fourier-Laplace domain, after applying the formula for the Laplace transform of the Caputo fractional derivative and observing $\widehat{\delta}(\kappa) \equiv 1$, we get

$$s^{\beta}\,\widehat{\widetilde{G^*_\beta}}(\kappa, s) - s^{\beta-1} = +i\kappa\,\widehat{\widetilde{G^*_\beta}}(\kappa, s)\,,$$

from which

$$\widehat{\widetilde{\mathcal{G}^*_\beta}}(\kappa, s) = \frac{s^{\beta-1}}{s^{\beta} - i\kappa}\,, \; 0 < \beta \le 1\,, \quad \Re(s) > 0\,, \; \kappa \in \mathbb{R}\,. \quad (B.3)$$

Like in Appendix A, to determine the Green function $\mathcal{G}^*_\beta(x,t)$ in the space-time domain we can follow two alternative strategies related to the order in carrying out the inversions in (B.3).

(S1) : invert the Fourier transform getting $\widetilde{\mathcal{G}_\beta}(x, s)$ and then invert the remaining Laplace transform;
(S2) : invert the Laplace transform getting $\widehat{G^*_\beta}(\kappa, t)$ and then invert the remaining Fourier transform.

**Strategy (S1):** Recalling the Fourier transform pair

$$\frac{a}{b - i\kappa} \overset{\mathcal{F}}{\leftrightarrow} \frac{a}{b}\,\mathrm{e}^{\,-xb}\,, \quad a, b > 0\,,\; x > 0\,, \qquad (B.4)$$

and setting $a = s^{\beta-1}$, $b = s^{\beta}$, we get

$$\widetilde{\widehat{\mathcal{G}^*_\beta}}(x, s) = s^{\beta-1}\,\mathrm{e}^{\,-xs^\beta}\,. \qquad (B.5)$$

**Strategy (S2):** Recalling the Laplace transform pair

$$\frac{s^{\beta-1}}{s^\beta + c} \overset{\mathcal{L}}{\leftrightarrow} E_\beta(-ct^\beta)\,, \quad c > 0\,, \qquad (B.6)$$

and setting $c = -i\kappa$, we have

$$\widehat{G^*_\beta}(\kappa, t) = E_\beta(i\kappa t^\beta)\,. \qquad (B.7)$$

Both strategies lead to the result (B.2).

In view of Eq. (F.40) we also recall that the $M$-Wright function is related to the unilateral **extremal stable density** of index $\beta$, so we write our Green function as

$$\mathcal{G}^*_\beta(x, t) = \frac{t}{\beta}\, x^{-1-1/\beta}\, L_\beta^{-\beta}\left(t x^{-1/\beta}\right)\,, \qquad (B.8)$$

## The stretched time-fractional diffusion equation

In the time fractional diffusion equation (F.51), where we have adopted the R-L derivative, let us stretch the time variable by replacing $t$ with $t^{\alpha/\beta}$ where $0 < \alpha < 2$ and $0 < \beta \leq 1$. We have

$$\frac{\partial u}{\partial t^{\alpha/\beta}} = K_{\alpha\,\beta}\, D_{t^{\alpha/\beta}}^{1-\beta}\, \frac{\partial^2 u}{\partial x^2}\,, \tag{F.57}$$

namely

$$\frac{\partial u}{\partial t} = \frac{\alpha}{\beta} t^{\alpha/\beta-1}\, K_{\alpha\,\beta}\, D_{t^{\alpha/\beta}}^{1-\beta}\, \frac{\partial^2 u}{\partial x^2}\,, \tag{F.58}$$

where $K_{\alpha\,\beta}$ is a sort of stretched diffusion coefficient of dimensions $[K_{\alpha\,\beta}] = [L]^2[T]^{-\alpha} = cm^2/sec^\alpha$ that reduces to $K_\alpha$ if $\beta = 1$ and to $K_\beta$ if $\alpha = \beta$.

Integration of Eq. (F.58) gives the corresponding integral equation , see [Mura-Pagnini (2008)]

$$u(x,t) = u_0(x) + K_{\alpha\,\beta}\,\frac{1}{\Gamma(\beta)}\,\frac{\alpha}{\beta}\int_0^t \tau^{\alpha/\beta-1}\,(t^{\alpha/\beta} - \tau^{\alpha/\beta})^{\beta-1}\,\frac{\partial^2 u(x,\tau)}{\partial x^2}\,d\tau\,, \quad (F.59)$$

whose Green function $\mathcal{G}_{\alpha\,\beta}(x,t)$ is

$$\mathcal{G}_{\alpha\,\beta}(x,t) = \frac{1}{2}\,\frac{1}{\sqrt{K_{\alpha\,\beta}}\,t^{\alpha/2}}\,M_{\beta/2}\left(\frac{|x|}{\sqrt{K_{\alpha\beta}}\,t^{\alpha/2}}\right)\,, \quad (F.60)$$

with variance

$$\sigma^2_{\alpha,\beta}(t) := \int_{-\infty}^{+\infty} x^2\,\mathcal{G}_{\alpha,\beta}(x,t)\,dx = \frac{2}{\Gamma(\beta+1)}\,K_{\alpha\,\beta}\,t^{\alpha}\,. \quad (F.61)$$

As a consequence, the resulting process turns out to be self-similar with Hurst exponent $H = \alpha/2$ and a variance law consistent both with slow diffusion if $0 < \alpha < 1$ and fast diffusion if $1 < \alpha < 2$.

We note that the parameter $\beta$ does explicitly enter in the variance law (5.21) only in the determination of the multiplicative constant.

## Fractional diffusion processes with stationary increments

We have seen that any Green function associated to the diffusion-like equations considered in the previous Section can be interpreted as the time-evolving one-point *pdf* of certain self-similar stochastic processes. However, in general, it is not possible to define a *unique* (self-similar) stochastic process because the determination of any multi-point probability distribution is required, see e.g. [Mura-Taqqu-Mainardi (2008)].

In other words, starting from a master equation which describes the dynamic evolution of a probability density function $f(x,t)$, it is always possible to define an equivalence class of stochastic processes with the same marginal density function $f(x,t)$. All these processes provide suitable stochastic representations for the starting equation.

It is clear that additional requirements may be stated in order to uniquely select the probabilistic model.

For instance, considering Eq. (F.58), the additional requirement of stationary increments, as shown by Mura et al., can lead to a class $\{B_{\alpha,\beta}(t),\ t \geq 0\}$, called **"generalized" grey Brownian motion** ($ggBm$), which, **by construction**, is made up of self-similar processes with stationary increments and Hurst exponent $H = \alpha/2$. Thus $\{B_{\alpha,\beta}(t),\ t \geq 0\}$ is a special class of $H\!-\!sssi$ processes[1], which provide non-Markovian stochastic models for anomalous diffusion, both of slow type $(0 < \alpha < 1)$ and fast type $(1 < \alpha < 2)$.

The $ggBm$ includes some well known processes, so that it defines an interesting general theoretical framework. The fractional Brownian motion ($fBm$) appears for $\beta = 1$, the grey Brownian motion ($gBm$), defined by Schneider (1990), corresponds to the choice $\alpha = \beta$, with $0 < \beta < 1$; finally, the standard Brownian motion ($Bm$) is recovered by setting $\alpha = \beta = 1$ We should note that only in the particular case of $Bm$ the corresponding process is Markovian.

[1] According to a common terminology, $H\!-\!sssi$ stands for $H$-self-similar-stationary-increments, see for details Taqqu (2003).

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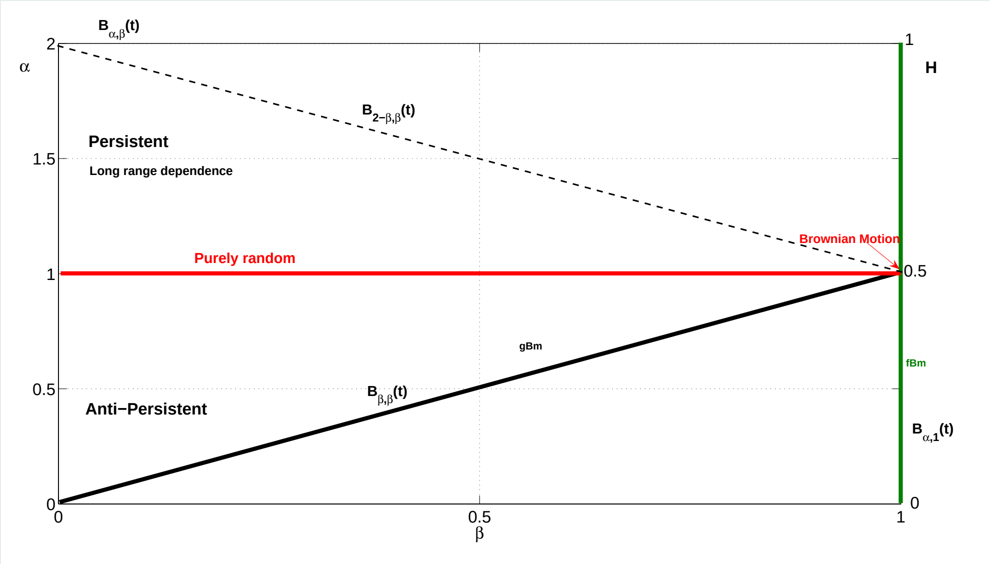
In Figure we present a diagram that allows to identify the elements of the $ggBm$ class.




**Parametric class of generalized grey Brownian motion**

The top region $1 < \alpha < 2$ corresponds to the domain of fast diffusion with *long-range dependence* [2] In this domain the increments of the process $B_{\alpha,\beta}(t)$ are positively correlated, so that the trajectories tend to be more regular (*persistent*). It should be noted that long-range dependence is associated to a non-Markovian process which exhibits long-memory properties.

The horizontal line $\alpha = 1$ corresponds to processes with uncorrelated increments, which model various phenomena of normal diffusion.

For $\alpha = \beta = 1$ we recover the Gaussian process of the standard Brownian motion.

The Gaussian process of the fractional Brownian motion is identified by the vertical line $\beta = 1$.

[2] A self-similar process with stationary increments is said to possess long-range dependence if the autocorrelation function of the increments tends to zero like a power function and such that it does not result integrable, see for details Taqqu (2003).

The bottom region $0 < \alpha < 1$ corresponds to the domain of slow diffusion. The increments of the corresponding process $B_{\alpha,\beta}(t)$ turn out to be negatively correlated and this implies that the trajectories are strongly irregular (*anti-persistent motion*); the increments form a stationary process which does not exhibit long-range dependence.

Finally, the diagonal line $(\alpha = \beta)$ represents the Schneider grey Brownian motion $(gBm)$.

Here we want to define the *ggBm* by making use of the Kolmogorov extension theorem and the properties of the $M$-Wright function. The generalized grey Brownian motion $B_{\alpha,\beta}(t)$ is a stochastic process defined in a certain probability space such that its finite-dimensional densities are given by

$$f_{\alpha,\beta}(x_1, x_2, \ldots, x_n; \gamma_{\alpha,\beta}) = \frac{(2\pi)^{-\frac{n-1}{2}}}{\sqrt{2\Gamma(1+\beta)^n \det \gamma_{\alpha,\beta}}} \int_0^\infty \frac{1}{\tau^{n/2}} M_{1/2}\left(\frac{\xi}{\tau^{1/2}}\right) M_\beta(\tau) d\tau, \tag{F.62}$$

with

$$\xi = \left(2\Gamma(1+\beta)^{-1} \sum_{i,j=1}^{n} x_i {\gamma_{\alpha,\beta}}^{-1}(t_i, t_j) x_j\right)^{1/2}, \tag{F.63}$$

and covariance matrix, with $i, j = 1, \ldots, n$

$$\gamma_{\alpha,\beta}(t_i, t_j) = \frac{1}{\Gamma(1+\beta)} (t_i^\alpha + t_j^\alpha - |t_i - t_j|^\alpha) . \tag{F.64}$$

Using the composition rule (F.50), the finite dimensional density (F.62) for $n = 1$ reduces to:

$$\begin{aligned} f_{\alpha,\beta}(x,t) &= \frac{1}{\sqrt{4t^\alpha}} \int_0^\infty M_{1/2}\left(|x|t^{-\alpha/2}, \tau\right) M_\beta(\tau,1)\, d\tau \\ &= \frac{1}{2} t^{-\alpha/2} M_{\beta/2}(|x|t^{-\alpha/2}) \end{aligned} \qquad (F.65)$$

This means that the marginal density function of the $ggBm$ is indeed the fundamental solution (F.60) of the stretched time-fractional diffusion equations (F.57)-(F.58) with $K_{\alpha\beta} = 1$.

Moreover, because $M_1(\tau) = \delta(\tau - 1)$, for $\beta = 1$, putting $\gamma_{\alpha,1} \equiv \gamma_\alpha$, we have that Eq. (F.62) provides the Gaussian density of the fractional Brownian motion,

$$f_{\alpha,1}(x_1, x_2, \ldots, x_n; \gamma_{\alpha,1}) = \frac{(2\pi)^{-\frac{n-1}{2}}}{\sqrt{2 \det \gamma_\alpha}} M_{1/2}\left(\left(2 \sum_{i,j=1}^{n} x_i \gamma_\alpha^{-1}(t_i,t_j) x_j\right)^{1/2}\right). \qquad (F.66)$$

By the definition used above, it is clear that, fixed $\beta$, the stochastic process $B_{\alpha,\beta}(t)$ is characterized only by its covariance structure, as shown by Mura et al,.

In other words, the *ggBm*, which is not Gaussian in general, is an example of a process defined only through its first and second moments, which indeed is a remarkable property of Gaussian processes.

Consequently, the **generalized grey Brownian motion** appears to be a direct generalization of the **Gaussian processes**, in the same way as the $M$-Wright function is a generalization of the Gaussian function.

# 9. Essentials of Fractional Calculus in $\mathbb{R}^+$

The **Riemann-Liouville fractional integral** of order $\mu > 0$ is defined as

$$ {}_tJ^\mu f(t) := \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau)\, d\tau \,, \quad \mu > 0 \,, \qquad (FC.1)$$

$$\Gamma(\mu) := \int_0^\infty \mathrm{e}^{-u} u^{\mu-1}\, du \,, \quad \Gamma(n+1) = n! \quad \textbf{Gamma function}.$$

By convention ${}_tJ^0 = I$ (Identity operator). We can prove

$$ {}_tJ^\mu \, {}_tJ^\nu = {}_tJ^\nu \, {}_tJ^\mu = {}_tJ^{\mu+\nu} \,, \; \mu, \nu \ge 0 \,, \quad \textbf{semigroup property} \qquad (FC.2)$$

$$ {}_tJ^\mu \, t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\mu)} t^{\gamma+\mu} \,, \quad \mu \ge 0 \,, \quad \gamma > -1 \,, \quad t > 0 \,. \qquad (FC.3)$$

The **fractional derivative** of order $\mu > 0$ in the **Riemann-Liouville** sense is defined v as the operator ${}_tD^\mu$

$$ {}_tD^\mu \, {}_tJ^\mu = I \,, \quad \mu > 0 \,. \qquad (FC.4) $$

If $m$ denotes the positive integer such that $m-1 < \mu \leq m$, we recognize from Eqs. (FC.2) and (FC.4)

$$ {}_tD^\mu \, f(t) := {}_tD^m \, {}_tJ^{m-\mu} \, f(t) \,, \qquad (FC.5) $$

hence

$$ {}_tD^\mu f(t) = \begin{cases} \dfrac{d^m}{dt^m} \left[ \dfrac{1}{\Gamma(m-\mu)} \displaystyle\int_0^t \dfrac{f(\tau)\, d\tau}{(t-\tau)^{\mu+1-m}} \right], & m-1 < \mu < m, \\ \dfrac{d^m}{dt^m} f(t), & \mu = m. \end{cases} \qquad (FC.5') $$

For completion ${}_tD^0 = I$. The semigroup property is no longer valid but

$$ {}_tD^\mu \, t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\mu)} \, t^{\gamma-\mu} \,, \quad \mu \geq 0 \,, \quad \gamma > -1 \,, \quad t > 0 \,. \qquad (FC.6) $$

However, the property ${}_tD^\mu = {}_tJ^{-\mu}$ is not generally valid!

The **fractional derivative** of order $\mu \in (m-1, m]$ $(m \in \mathbb{N})$ in the **Caputo** sense is defined as the operator ${}_tD_*^\mu$ such that

$$ {}_tD_*^\mu\, f(t) := {}_tJ^{m-\mu}\, {}_tD^m\, f(t)\,, \qquad (FC.7) $$

hence

$$ {}_tD_*^\mu\, f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\mu)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)\, d\tau}{(t-\tau)^{\mu+1-m}}\,, & m-1 < \mu < m, \\ \dfrac{d^m}{dt^m} f(t)\,, & \mu = m. \end{cases} \qquad (FC.7') $$

Thus, when the order is not integer the two fractional derivatives differ in that the derivative of order $m$ does not generally commute with the fractional integral.

We point out that the **Caputo fractional derivative** satisfies the relevant property of being zero when applied to a constant, and, in general, to any power function of non-negative integer degree less than $m$, if its order $\mu$ is such that $m-1 < \mu \le m$.

Gorenflo and Mainardi (1997) have shown the essential relationships between the two fractional derivatives (when both of them exist),

$$ {}_tD_*^\mu f(t) = \begin{cases} {}_tD^\mu \left[ f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\, \frac{t^k}{k!} \right], \\ {}_tD^\mu f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)\, t^{k-\mu}}{\Gamma(k-\mu+1)}, \end{cases} \quad m-1 < \mu < m\,. \qquad (FC.8) $$

In particular, if $m = 1$ we have

$$ {}_tD_*^\mu f(t) = \begin{cases} {}_tD^\mu \left[ f(t) - f(0^+) \right], \\ {}_tD^\mu f(t) - \frac{f(0^+)\, t^{-\mu}}{\Gamma(1-\mu)}, \end{cases} \quad 0 < \mu < 1\,. \qquad (FC.9) $$

The **Caputo fractional derivative**, represents a sort of regularization in the time origin for the **Riemann-Liouville fractional derivative**. We note that for its existence all the limiting values $f^{(k)}(0^+) := \lim_{t\to 0^+} f^{(k)}(t)$ are required to be finite for $k = 0, 1, 2 \ldots m-1$.

We observe the different behaviour of the two fractional derivatives at the end points of the interval $(m-1, m)$ namely when the order is any positive integer: whereas ${}_tD^{\mu}$ is, with respect to its order $\mu$, an operator continuous at any positive integer, ${}_tD_*^{\mu}$ is an operator left-continuous since

$$\begin{cases} \lim\limits_{\mu \to (m-1)^+} {}_tD_*^{\mu} f(t) = f^{(m-1)}(t) - f^{(m-1)}(0^+) \,, \\ \lim\limits_{\mu \to m^-} {}_tD_*^{\mu} f(t) = f^{(m)}(t) \,. \end{cases} \qquad (FC.10)$$

We also note for $m-1 < \mu \le m$,

$${}_tD^{\mu} f(t) = {}_tD^{\mu} g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j \, t^{\mu-j} \,, \quad (FC.11)$$

$${}_tD_*^{\mu} f(t) = {}_tD_*^{\mu} g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j \, t^{m-j} \,. \quad (FC.12)$$

In these formulae the coefficients $c_j$ are arbitrary constants.

We point out the major utility of the Caputo fractional derivative in treating initial-value problems for physical and engineering applications where initial conditions are usually expressed in terms of integer-order derivatives. This can be easily seen using the **Laplace transformation**.

Writing the Laplace transform of a sufficiently well-behaved function $f(t)$ $(t \geq 0)$ as

$$\mathcal{L}\{f(t); s\} = \widetilde{f}(s) := \int_0^\infty \mathrm{e}^{-st} f(t)\, dt \,,$$

the known rule for the ordinary derivative of integer order $m \in \mathbb{N}$ is

$$\mathcal{L}\{{}_tD^m f(t); s\} = s^m \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{m-1-k} f^{(k)}(0^+) \,, \quad m \in \mathbb{N} \,,$$

where

$$f^{(k)}(0^+) := \lim_{t \to 0^+} {}_tD^k f(t) \,.$$

For the **Caputo derivative** of order $\mu \in (m-1, m]$ $(m \in \mathbb{N})$ we have

$$\begin{aligned} &\mathcal{L}\{ {}_tD_*^{\mu} f(t); s\} = s^{\mu} \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{\mu-1-k} f^{(k)}(0^+)\,, \\ &f^{(k)}(0^+) := \lim_{t\to 0^+} {}_tD^k f(t)\,. \end{aligned} \qquad (FC.13)$$

The corresponding rule for the **Riemann-Liouvile derivative** of order $\mu$ is

$$\begin{aligned} &\mathcal{L}\{ {}_tD_t^{\mu} f(t); s\} = s^{\mu} \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{m-1-k} g^{(k)}(0^+)\,, \\ &g^{(k)}(0^+) := \lim_{t\to 0^+} {}_tD^k g(t)\,, \quad g(t) := {}_tJ^{m-\mu} f(t)\,. \end{aligned} \qquad (FC.14)$$

Thus the rule (FC.14) is more cumbersome to be used than (FC.13) since it requires initial values concerning an extra function $g(t)$ related to the given $f(t)$ through a fractional integral.

However, when all the limiting values $f^{(k)}(0^+)$ are finite and the order is not integer, we can prove by that all $g^{(k)}(0^+)$ vanish so that the formula (FC.14) simplifies into

$$\mathcal{L}\left\{ {}_tD^\mu \, f(t); s \right\} = s^\mu \, \widetilde{f}(s) \,, \quad m-1 < \mu < m \,. \qquad (FC.15)$$

For this proof it is sufficient to apply the Laplace transform to Eq. (FC.8), by recalling that

$$\mathcal{L}\left\{ t^\nu; s \right\} = \Gamma(\nu+1)/s^{\nu+1} \,, \quad \nu > -1 \,, \qquad (FC.16)$$

and then to compare (FC.13) with (FC.14).

For more details on the theory and applications of fractional calculus we recommend to consult in addition to the well-known books by Samko, Kilbas & Marichev (1993), Miller & Ross (1993), and Podlubny (1999), those appeared in the last few years, by West, Bologna & Grigolini (2003), Zaslavsky (2005), Kilbas, Srivastava & Trujillo (2006), Mainardi(2010), Diethelm (2010), Baleanu, Diethelm, Scalas & Trujillo (2012), Uchaikin (2013).

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