High Order Numerical Methods for the Riesz Derivatives and the Space Riesz Fractional Differential Equation

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Outline

- Fractional calculus: partial but important
- Motivation
- Numerical methods for Riesz fractional derivatives
- Numerical methods for space Riesz fractional differential equation
- Conclusions
- Acknowledgements
Fractional calculus: partial but important

Calculus = integration + differentiation

Fractional calculus = fractional integration + fractional differentiation

Fractional integration (or fractional integral)

Mainly one: Riemann-Liouville (RL) integral

Fractional derivatives

More than 6: not mutually equivalent.

RL and Caputo derivatives are mostly used.
Fractional calculus: partial but important

Definitions of RL and Caputo derivatives:

\[ \text{RL } D_{0,t}^\alpha x(t) = \frac{d^m}{dt^m} D_{0,t}^{-(m-\alpha)} x(t), \quad m - 1 < \alpha < m \in \mathbb{Z}^+. \]

\[ \text{C } D_{0,t}^{\alpha} x(t) = D_{0,t}^{-(m-\alpha)} \frac{d^m}{dt^m} x(t), \quad m - 1 < \alpha < m \in \mathbb{Z}^+. \]

\[ \text{C } D_{0,t}^{\alpha} x(t) = \text{RL } D_{0,t}^{\alpha} [x(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} x^{(k)}(0)], \quad m - 1 < \alpha < m \in \mathbb{Z}^+. \]

The involved fractional integral is in the sense of RL.

\[ D_{0,t}^{-\alpha} x(t) = Y_\alpha \ast x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau) d\tau, \quad \alpha \in \mathbb{R}^+. \]
Once mentioned it, fractional calculus is taken for granted to be the mathematical generalization of calculus.

But this is not true!!!

For $\alpha$ - th $(m-1 < \alpha < m \in Z^+)$ Caputo derivative, fix $t$

$$\lim_{\alpha \to (m-1)^+} cD_{0,t}^\alpha x(t) = x^{(m-1)}(t) - x^{(m-1)}(0),$$

$$\lim_{\alpha \to m^-} cD_{0,t}^\alpha x(t) = x^{(m)}(t).$$

Classical derivative is not the special case of the Caputo derivative.
On the other hand, see an example below:

Investigate a function in $[0, 1 + \varepsilon]$,

$$x(t) = \begin{cases} 1 - t, & 0 \leq t \leq 1, \\ t - 1, & 1 < t \leq 1 + \varepsilon, \varepsilon > 0. \end{cases}$$

$\text{RL} D_{0,t}^\alpha x(t) \exists$ in $(0,1+\varepsilon]$, $\alpha \in (0,1)$.

$x'(t) \exists$ in $[0,1) \cup (1,1+\varepsilon]$.

The existence intervals of two kind of derivatives for the same function is not the same, neither relation of inclusion.

**RL derivative can not** be regarded as the mathematical generalization of the** classical derivative.**
Therefore, fractional calculus, closely related to classical calculus, is not the direct generalization of classical calculus in the sense of rigorous mathematics.

For details, see the following review article:
Where is fractional calculus?

It is here.

Example

\( x(t) = t^{-\alpha}, 0 < \alpha \leq 1 / 2, \) is not a solution to

\[
\begin{cases}
\frac{dx}{dt} = f(x, t), & \text{arbitrarily given,} \\
x(0) = x_0, & \text{arbitrarily given.}
\end{cases}
\]

But it is a solution to

\[
\begin{cases}
_{RL}D_{0,t}^{\alpha} x(t) = f(x, t), & \text{for some } f(x, t), \\
_{RL}D_{0,t}^{\alpha-1} x(t) \bigg|_{t=0} = x_0, & \text{for some } x_0.
\end{cases}
\]
Another Example

\[ x(t) = \begin{cases} 
1, & \quad t = \frac{q}{p} \in (0,1], \quad (p,q) = 1, \\
0, & \quad \text{others in } [0,1],
\end{cases} \]

is not a solution to

\[
\begin{cases} 
\frac{dx}{dt} = f(x,t), \quad \text{arbitrarily given}, \\
x(0) = x_0, \quad \text{arbitrarily given.}
\end{cases}
\]

But it is a solution to

\[
(*) \quad \begin{cases} 
_{RL}D_0^\alpha x(t) = 0, \quad \alpha \in (0,1), \\
_{RL}D_0^{\alpha-1} x(t) \big|_{x=0} = 0.
\end{cases}
\]

Actually, \( x = 0 \) a.e. solves eqn \( (*) \).

Attention: fractional is very possibly a powerful tool for nonsmooth functions.
For **Riemann-Liouville derivative**, high order methods were very possibly proposed by Lubich (SIAM J. Math. Anal., 1986)

\[
\mathcal{D}_{a,x_n}^{\alpha} f(x_n) = h^{-\alpha} \sum_{j=0}^{n} \omega_{n-j}^{\alpha} f(x_j) + h^{-\alpha} \sum_{j=0}^{s} \omega_{n,j} f(x_j) + O(h^p).
\]

For **Caputo derivative**, high order schemes were constructed by Li, Chen, Ye (J. Comput. Phys. Phys., 2011)

**In this talk, we focus on high order algorithms for Riesz fractional derivatives and space Riesz fractional differential equation.**
The **Riesz fractional derivative** is defined on \((a, b)\) as follows,

\[
\frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} = -\Psi_\alpha \left( RL\, D^\alpha_{a,x} + RL\, D^\alpha_{x,b} \right) u(x, t),
\]

in which,

\[
\Psi_\alpha = \frac{1}{2} \sec \left( \frac{\pi \alpha}{2} \right), \quad n-1 < \alpha < n \in \mathbb{Z}^+,
\]

\[
RL\, D^\alpha_{a,x} u(x, t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_a^x \frac{u(\xi, t)}{(x-\xi)^{\alpha-n+1}} d\xi,
\]

\[
RL\, D^\alpha_{x,b} u(x, t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_x^b \frac{u(\xi, t)}{\xi-x)^{\alpha-n+1}} d\xi.
\]
The space Riesz fractional differential equation reads as

\[
\frac{\partial u(x,t)}{\partial t} = K \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + f(x,t), \quad 1 < \alpha < 2, \quad a < x < b, \quad 0 < t \leq T
\]

\[
\begin{align*}
\left. D^{\alpha-2}_{RL} u(x,t) \right|_{x=a} &= \phi(t), \quad 0 \leq t \leq T, \\
\left. D^{\alpha-2}_{RL} u(x,t) \right|_{x=b} &= \varphi(t), \quad 0 \leq t \leq T, \\
u(x,0) &= \psi(x), \quad a \leq x \leq b.
\end{align*}
\]

The above boundary value conditions are of Dirichlet type. Neumann or Rubin boundary value conditions can be similarly proposed.

Unsuitable initial or boundary value conditions: ill-posed.
Numerical methods for Riesz fractional derivatives

Already existed (typical) numerical schemes:

Let $h$ be the step size with $x_m = a + mh$, $m = 0, 1, \cdots, M$, and $t_n = n\tau$, $n = 0, 1, \cdots, N$, where $h = (b - a) / M$, $\tau = T / N$.

1) The first order scheme

$$\frac{\partial^\alpha u(x_m, t)}{\partial |x|^\alpha} = -\frac{\Psi^\alpha}{h^\alpha} \left( \sum_{k=0}^{m} \varpi_k^{(\alpha)} u(x_{m-k}, t) + \sum_{k=0}^{M-m} \varpi_k^{(\alpha)} u(x_{m+k}, t) \right) + O(h),$$

in which $\varpi_k^{(\alpha)} = (-1)^k \binom{\alpha}{k} = \frac{(-1)^k}{\Gamma(1+k) \Gamma(1+\alpha-k)} \Gamma(1+\alpha)$. 


2) The shifted first order scheme

\[
\frac{\partial^\alpha u(x_m,t)}{\partial |x|^\alpha} = -\Psi_\alpha \left( \sum_{k=0}^{m} \omega_k^{(\alpha)} u(x_{m-k+1},t) + \sum_{k=0}^{M-m} \omega_k^{(\alpha)} u(x_{m+k-1},t) \right) + O(h),
\]

in which \( \omega_k^{(\alpha)} = (-1)^k \binom{\alpha}{k} = \frac{(-1)^k \Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1+\alpha-k)}. \)
Numerical methods for Riesz fractional derivatives

3) The L2 numerical scheme

If $1 < \alpha < 2$, then one has

\[
\frac{\partial^\alpha u(x_m, t)}{\partial |x|^\alpha} = -\frac{\Psi_\alpha}{\Gamma(3-\alpha)h^\alpha} \left\{ \frac{(1-\alpha)(2-\alpha)u(x_0, t)}{m^\alpha} + \frac{(2-\alpha)[u(x_1, t) - u(x_0, t)]}{m^{\alpha-1}} \right. \\
+ \sum_{k=0}^{m-1} d_k^{(\alpha)} \left[ u(x_{m-k+1}, t) - 2u(x_{m-k}, t) + u(x_{m-k-1}, t) \right] \\
+ \frac{(1-\alpha)(2-\alpha)u(x_M, t)}{(M-m)^\alpha} + \frac{(2-\alpha)[u(x_M, t) - u(x_{M-1}, t)]}{m^{\alpha-1}} \\
+ \sum_{k=0}^{M-m-1} d_k^{(\alpha)} \left[ u(x_{m+k+1}, t) - 2u(x_{m+k}, t) + u(x_{m+k+1}, t) \right] \right\} + O(h),
\]

in which $d_k^{(\alpha)} = (k+1)^{2-\alpha} - k^{2-\alpha}$, $k = 0, 1, \ldots, \max(m-1, M-m-1)$. 
4) The second order numerical scheme based on the spline interpolation method

\[
\frac{\partial^{\alpha} u(x_m, t)}{\partial |x|^\alpha} = \frac{-\Psi_{\alpha}}{\Gamma(4-\alpha) h^\alpha} \sum_{k=0}^{M} z^{(\alpha)}_{m,k} u(x_k, t) + O(h^2),
\]

in which \( z^{(\alpha)}_{m,k} \) is defined below,

\[
z^{(\alpha)}_{m,k} = \begin{cases} 
\overline{z}^{(\alpha)}_{m,k}, & k < m-1, \\
\overline{z}^{(\alpha)}_{m,m-1} + \tilde{z}^{(\alpha)}_{m,m-1}, & k = m-1, \\
\overline{z}^{(\alpha)}_{m,m} + \tilde{z}^{(\alpha)}_{m,m}, & k = m, \\
\overline{z}^{(\alpha)}_{m,m+1} + \tilde{z}^{(\alpha)}_{m,m+1}, & k = m+1, \\
\tilde{z}^{(\alpha)}_{m,k}, & k > m+1.
\end{cases}
\]
Numerical methods for Riesz fractional derivatives

\[ \bar{Z}_{m,k}(\alpha) = \begin{cases} \bar{c}_{m-1,k} - 2\bar{c}_{m,k} + \bar{c}_{m+1,k} , & k \leq m-1, \\ -2\bar{c}_{m,k} + \bar{c}_{m+1,k} , & k = m, \\ \bar{c}_{m+1,k} , & k = m+1, \\ 0 , & k > m+1, \end{cases} \]

in which

\[ \bar{c}_{j,k} = \begin{cases} (j-1)^{3-\alpha} - j^{2-\alpha} (j-3+\alpha) , & k = 0, \\ (j-k+1)^{3-\alpha} - 2(j-k)^{3-\alpha} + (j-k-1)^{3-\alpha} , & 1 \leq k \leq j-1, \\ 1 , & k = j; \end{cases} \]
Numerical methods for Riesz fractional derivatives

\[
\tilde{z}^{(\alpha)}_{m,k} = \begin{cases} 
0, & k < m-1, \\
\tilde{c}_{m-1,m-1}, & k = m-1, \\
-2\tilde{c}_{m,m} + \tilde{c}_{m-1,m}, & k = m, \\
\tilde{c}_{m-1,k} - 2\tilde{c}_{m,k} + \tilde{c}_{m+1,k}, & m+1 \leq k \leq M, 
\end{cases}
\]

in which

\[
\tilde{c}_{j,k} = \begin{cases} 
1, & k = j \\
(k - j + 1)^{3-\alpha} - 2(k - j)^{3-\alpha} + (k - j - 1)^{3-\alpha}, & j+1 \leq k \leq M-1 \\
(3-\alpha-M+j)(M-j)^{2-\alpha} + (M-j-1)^{3-\alpha}, & k = M 
\end{cases}
\]

with \( j = m-1, m, m+1. \)
Numerical methods for Riesz fractional derivatives

5) The second order scheme based on the fractional centered difference method

\[
\frac{\partial^\alpha u(x_m, t)}{\partial |x|^\alpha} = -\frac{1}{h^\alpha} \Delta_h^\alpha u(x_m, t) + O(h^2).
\]

The so called fractional centered difference operator is defined by

\[
\Delta_h^\alpha u(x, t) = \sum_{k=-\infty}^{\infty} \frac{(-1)^k \Gamma(\alpha + 1)}{\Gamma\left(\frac{\alpha}{2} - k + 1\right) \Gamma\left(\frac{\alpha}{2} + k + 1\right)} u(x - kh, t).
\]
Numerical methods for Riesz fractional derivatives

6) The second order scheme based on the weighted and shifted Grünwald-Letnikov (GL) formula

\[
\frac{\partial^\alpha u(x_m, t)}{\partial |x|^\alpha} = -\frac{\Psi_\alpha}{h^\alpha} \left( \nu_1 \sum_{k=0}^{m+\ell_1} \omega_k^{(\alpha)} u(x_{m-k+\ell_1}, t) + \nu_2 \sum_{k=0}^{m+\ell_2} \omega_k^{(\alpha)} u(x_{m-k+\ell_2}, t) \right)
\]

\[
+ \nu_1 \sum_{k=0}^{M-m+\ell_1} \omega_k^{(\alpha)} u(x_{m+k-\ell_1}, t) + \nu_2 \sum_{k=0}^{M-m+\ell_2} \omega_k^{(\alpha)} u(x_{m+k-\ell_2}, t)
\]

\[
+ O(h^2),
\]

in which

\[
\nu_1 = \frac{\alpha - 2\ell_2}{2(\ell_1 - \ell_2)}, \quad \nu_2 = \frac{2\ell_1 - \alpha}{2(\ell_1 - \ell_2)},
\]

\(\ell_1, \ell_2\) are two integers.
7) The third order scheme based on the weighted and shifted GL formula

\[
\frac{\partial^\alpha u(x_m, t)}{\partial |x|^\alpha} = \frac{\Psi}{h^\alpha} \left( \kappa_1 \sum_{m=0}^{m+\ell_1} \sigma_k^{(\alpha)} u(x_{m-k+\ell_1}, t) + \kappa_2 \sum_{k=0}^{m+\ell_2} \sigma_k^{(\alpha)} u(x_{m-k+\ell_2}, t) + \kappa_3 \sum_{k=0}^{m+\ell_3} \sigma_k^{(\alpha)} u(x_{m-k+\ell_3}, t) + O(h^3) \right)
\]

in which, \( \kappa_1 = \frac{12 \ell_2 \ell_3 - (6 \ell_2 + 6 \ell_3 + 1) \alpha + 3 \alpha^2}{12 \left( \ell_2 \ell_3 - \ell_1 \ell_2 - \ell_1 \ell_3 + \ell_1^2 \right)} \),

\( \kappa_2 = \frac{12 \ell_1 \ell_3 - (6 \ell_1 + 6 \ell_3 + 1) \alpha + 3 \alpha^2}{12 \left( \ell_1 \ell_3 - \ell_1 \ell_2 - \ell_2 \ell_3 + \ell_2^2 \right)} \),

\( \kappa_3 = \frac{12 \ell_1 \ell_2 - (6 \ell_1 + 6 \ell_2 + 1) \alpha + 3 \alpha^2}{12 \left( \ell_1 \ell_2 - \ell_1 \ell_3 - \ell_2 \ell_3 + \ell_3^2 \right)} \),

\( \ell_1, \ell_2, \ell_3 \) are three integers.
In our work, we construct **new three kinds of second order schemes and a fourth order numerical scheme.** We first derive second order schemes.

**Lemma 1.** Assume that \( u(x,t) \) with respect to \( x \) sufficiently smooth. For arbitrarily different numbers \( p, q \) and \( s \), we have

\[
\begin{align*}
  u(x_s,t) &= \frac{(x_s - x_q)u(x_p,t) + (x_p - x_s)u(x_q,t)}{x_p - x_q} \\
  &+ O\left(\left\|\left(x_q - x_s\right)(x_p - x_s)\right\|\right).
\end{align*}
\]
Lemma 2. For any positive number $\alpha$, we have

$$\sum_{k=0}^{\infty} \omega_k^{(\alpha)} = 0,$$

where $\omega_k^{(\alpha)} = (-1)^k \binom{\alpha}{k} = \frac{(-1)^k \Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1+\alpha-k)}$.

Let

$$\mu_h^\alpha \left( u \left( x - jh, t \right) \right) = \frac{u(x,t) + f(x - 2jh,t)}{2^\alpha}.$$
Define

\[ A_C \Delta_h^\alpha u(x, t) = \mu_h^\alpha \left( C \Delta_h^\alpha u(x, t) \right), \]

where

\[ C \Delta_h^\alpha u(x, t) = \sum_{k=0}^\infty \varpi_k^{(\alpha)} u \left( x - \left( k - \frac{\alpha}{2} \right) h, t \right). \]

Then one has

\[ A_C \Delta_h^\alpha u(x, t) = \frac{1}{2^\alpha} \sum_{k=0}^\infty \varpi_k^{(\alpha)} u \left( x - (2k - \alpha) h, t \right). \]
Theorem 1. Let $u(x,t)$ and the Fourier transform of the $RL D_{-\infty,x}^{\alpha+2} u(x,t)$ with respect to $x$ both be in $L_1(R)$, then

$$\frac{\Delta_h^\alpha u(x,t)}{h^\alpha} = RL D_{-\infty,x}^\alpha u(x,t) + O(h^2), \text{ i.e.,}$$

$$RL D_{-\infty,x}^{\alpha} u(x,t) = \frac{1}{(2h)^\alpha} \sum_{k=0}^{\infty} \sigma_k^{(x)} u(x - (2k - \alpha)h,t) + O(h^2).$$
Numerical methods for Riesz fractional derivatives

By choices of $x_s$, $x_p$ and $x_q$, one has new three kinds of second order schemes:

\[
\frac{\partial^{\alpha} u(x_m, t_n)}{\partial |x|^\alpha} = -\frac{\Psi_{\alpha}}{(2h)^\alpha} \left[ \left(1 - \frac{\alpha}{2}\right) \sum_{k=0}^{[m/2]} \omega_k^{(\alpha)} u(x_{m-2k}, t) \right]
\]

\[
+ \frac{\alpha}{2} \sum_{k=0}^{[m/2]} \omega_k^{(\alpha)} u(x_{m-2(k-1)}, t) + \left(1 - \frac{\alpha}{2}\right) \sum_{k=0}^{[M-m]/2} \omega_k^{(\alpha)} u(x_{m+2k}, t) \quad (I)
\]

\[
+ \frac{\alpha}{2} \sum_{k=0}^{[M-m]/2} \omega_k^{(\alpha)} u(x_{m+2(k-1)}, t) \bigg] + O(h^2),
\]
Numerical methods for Riesz fractional derivatives

\[
\frac{\partial^\alpha u(x_m, t_n)}{\partial |x|^\alpha} = -\frac{\Psi^\alpha}{(2 \lambda)^\alpha} \left[ \left( 1 + \frac{\alpha}{2} \right) \sum_{k=0}^{m-1} \omega_k^{(\alpha)} u(x_{m-2k}, t) - \frac{\alpha}{2} \sum_{k=0}^{m/2-1} \omega_k^{(\alpha)} u(x_{m-2(k+1)}, t) \right] + O(h^2),
\]

(II)

\[
\frac{\partial^\alpha u(x_m, t_n)}{\partial |x|^\alpha} = -\frac{\Psi^\alpha}{(2 \lambda)^\alpha} \left[ \left( \frac{1}{2} + \frac{\alpha}{4} \right) \sum_{k=0}^{m/2} \omega_k^{(\alpha)} u(x_{m-2(k-1)}, t) + \left( \frac{1}{2} - \frac{\alpha}{4} \right) \sum_{k=0}^{m/2-1} \omega_k^{(\alpha)} u(x_{m-2(k+1)}, t) \right] + O(h^2).
\]

(III)
New kinds of third order numerical schemes. Skipped...

Now directly derive a fourth order scheme.
Theorem 2. Let $u(x,t)$ lie in $C^7(R)$ whose partial derivatives up to order seven with respect to $x$ belong to $\mathcal{L}_1(R)$. Set

$$L_\theta u(x,t) = \sum_{k=-\infty}^{\infty} g_k^{(\alpha)} u\left(x-(k+\theta)h,t\right), \quad \theta = -1, 0, 1,$$

in which

$$g_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha}{2} - k + 1\right) \Gamma\left(\frac{\alpha}{2} + k - 1\right)}.$$

Then one has

$$\partial_t^\alpha u(x,t) = \frac{1}{h^\alpha} \left[ \frac{\alpha}{24} L_{-1} u(x,t) + \frac{\alpha}{24} L_1 u(x,t) - (1 + \frac{\alpha}{12}) L_0 u(x,t) \right] + O(h^4).$$
Based on Theorem 2, a fourth order scheme can be established as

\[
\frac{\partial^\alpha u(x_m, t)}{\partial |x|^{\alpha}} = \frac{\alpha}{24h^\alpha} \sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u(x_{m-(k+1)}, t) \\
+ \frac{\alpha}{24h^\alpha} \sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u(x_{m-(k-1)}, t) \\
- \left(1 + \frac{\alpha}{12}\right) \frac{1}{h^\alpha} \sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u(x_{m-k}, t) + O(h^4),
\]

where

\[
g_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha + 1)}{\Gamma\left(\frac{\alpha - k + 1}{2}\right) \Gamma\left(\frac{\alpha + k - 1}{2}\right)}.\]
Recall Equ (1).

\[
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = K \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + f(x,t), & 1 < \alpha < 2, a < x < b, 0 < t \leq T \\
RL D_{a,x; t}^{\alpha-2} u(x,t) |_{x=a} = \phi(t), & 0 \leq t \leq T, \\
RL D_{x,b; t}^{\alpha-2} u(x,t) |_{x=b} = \varphi(t), & 0 \leq t \leq T, \\
u(x,0) = \psi(x), & 0 \leq x \leq L.
\end{cases}
\]
Let $u_m^n$ be the approximation solution of $u(x_m, t_n)$, one has the following finite difference scheme for Equ (1).

$$u_m^n - \mu_1 \sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u_{m-k-1}^n - \mu_1 \sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u_{m-k+1}^n + \mu_2 \sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u_{m-k}^n =$$

$$u_m^{n-1} + \mu_1 \sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u_{m-k-1}^{n-1} + \mu_1 \sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u_{m-k+1}^{n-1} - \mu_2 \sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u_{m-k}^{n-1} + \frac{\tau}{2} f_m^n + \frac{\tau}{2} f_m^{n-1}, \quad (2)$$

where $\mu_1 = \frac{\alpha \tau K}{48h^\alpha}, \mu_2 = \frac{\tau K}{2h^\alpha} \left(1 + \frac{\alpha}{12}\right)$. 
Set $U^n = \left( u^n_1, u^n_2, \ldots, u^n_{M-1} \right)^T$, $F^n = (f^n_1, f^n_2, \ldots, f^n_{M-1})$.

Then system (2) can be written in a compact form:

$$\left( I + H \right) U^n = \left( I - H \right) U^{n-1} + \frac{\tau}{2} F^n + \frac{\tau}{2} F^{n-1},$$

where $I$ is an identity matrix with order $M - 1$, $H = \mu_2 G - \mu_1 G^+ - \mu_1 G^-$, $G, G^+, G^-$ are omitted here.
Numerical method for space Riesz fractional differential equation

Theorem 3. (Stability)
Difference scheme (3) (or (2)) is unconditionally stable.

Theorem 4. (Convergence)
\[ |u(x_m, t_k) - u_m^k| \leq C \left( \tau^2 + h^4 \right). \]
Example 1

Consider the function \( u(x) = x^2 (1 - x)^2, \ x \in [0,1] \).

Its exact Riesz fractional derivative is

\[
\frac{\partial^\alpha u(x)}{\partial |x|^\alpha} = -\sec \left( \frac{\pi}{2} \alpha \right) \left\{ \frac{1}{\Gamma(3 - \alpha)} \left[ x^{2-\alpha} + (1 - x)^{2-\alpha} \right] - \frac{6}{\Gamma(4 - \alpha)} \left[ x^{3-\alpha} + (1 - x)^{3-\alpha} \right] + \frac{12}{\Gamma(5 - \alpha)} \left[ x^{4-\alpha} + (1 - x)^{4-\alpha} \right] \right\}. 
\]
Numerical Examples

We use the second order numerical formulas (I), (II), and (III) to compute the given function, the absolute errors and convergence orders at $x=0.5$ are shown in Tables 1-3.

Table 1: The absolute errors and convergence orders of the Example 1 by numerical formula (21)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$h$</th>
<th>the absolute error</th>
<th>the convergence order</th>
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<td>$\frac{1}{40}$</td>
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## Numerical Examples

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Table 2: The absolute errors and convergence orders of the Example 1 by numerical formula (22)

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### Numerical Examples

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Table 3: The absolute errors and convergence orders of the Example 1 by numerical formula (23)

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Example 2
Consider the following Riesz fractional differential equation
\[
\frac{\partial u(x,t)}{\partial t} = K \frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} + f(x,t), \quad 0 < x < 1, \quad 0 < t \leq 1
\]
where
\[
f(t) = (2\alpha + 1)t^{2\alpha}x^4(1-x)^4 + t^{2\alpha+1} \sec\left(\frac{\pi}{2}\alpha\right) \left\{ \frac{12}{\Gamma(5-\alpha)} \left[ x^{4-\alpha} + (1-x)^{4-\alpha} \right] \\
- \frac{240}{\Gamma(6-\alpha)} \left[ x^{5-\alpha} + (1-x)^{5-\alpha} \right] + \frac{2160}{\Gamma(7-\alpha)} \left[ x^{6-\alpha} + (1-x)^{6-\alpha} \right] \\
- \frac{10080}{\Gamma(8-\alpha)} \left[ x^{7-\alpha} + (1-x)^{7-\alpha} \right] + \frac{20160}{\Gamma(9-\alpha)} \left[ x^{8-\alpha} + (1-x)^{8-\alpha} \right] \right\}
\]
together with the initial condition \( u(x,0) = 0 \) and homogenous boundary value conditions.
The exact solution is \( u(x,t) = t^{2\alpha+1}x^4(1-x)^4 \).
The following table shows the maximum error, time and space convergence orders which confirms with our theoretical analysis.

Table 4: The maximum errors, temporal and spatial convergence orders of the Example 2 by difference scheme (37).

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### Numerical Examples

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Numerical Examples

Figs. 6.1 and 6.2 show the comparison of the analytical and numerical solutions with $\alpha = 1.1$ at $t = 0.2$, $\alpha = 1.9$ at $t = 0.8$.

Figure 6.1: Comparison between the analytical solution and the numerical solution at $t = 0.2$ with $\alpha = 1.1$ in Example 2. ($\tau = \frac{1}{50}, h = \frac{1}{50}$)
Figure 6.2: Comparison between the analytical solution and the numerical solution at $t = 0.8$ with $\alpha = 1.9$ in Example 2. ($\tau = \frac{1}{100}$, $h = \frac{1}{80}$)
Numerical Examples

Figs. 6.3-6.6 display the numerical and analytical solution surface with $\alpha = 1.1$ and $\alpha = 1.8$

Figure 6.3: The numerical solution surface when $\alpha = 1.2$ in Example 2. ($\tau = \frac{1}{500}, h = \frac{1}{100}$)
Figure 6.4: The analytical solution surface when $\alpha = 1.2$ in Example 2.
Figure 6.5: The numerical solution surface when $\alpha = 1.8$ in Example 2. ($\tau = \frac{1}{50}, h = \frac{1}{80}$)
Figure 6.6: The analytical solution surface when $\alpha = 1.8$ in Example 2.
1) **Proper** and **exact** applications of fractional calculus

2) Long-term integrations at each step, induced by **historical dependencies** and/or **long-range interactions**, require more computational time and storage capacity.

Therefore the effective and economical numerical methods are needed.

3) Fractional calculus + Stochastic process
Stochastics and nonlocality may better characterize our complex world.

So numerical fractional stochastic differential equations have been placed on the agenda.

And more......
Conclusions?

Not yet, but Go Fractional with some lists.
Conclusions

Numerical calculations:

Conclusions

**Fractional dynamics:**


**Review article**
Acknowledgements

Q & A!

Thank Professor George Em Karniadakis for cordial invitation.

Thank you all for coming.