Fractional acoustic wave equations from mechanical and thermal constitutive equations

Sverre Holm
Outline

• 2 Perspectives
• Non-fractional wave equations
  – Low- and high-frequency regions
• Fractional viscoelastic models
  – Kelvin-Voigt
  – Zener
• Fractional heat flow model
2 Perspectives

1. Attenuation Power Laws
   – Wave Equations
2. Constitutive Equations

Physics or mathematics?
1. Power Laws: P- and S- loss, $\omega^y$

- Similar power laws
- Longitudinal, pressure:
  - Granite: $y \approx 1$
  - Liver: $y \approx 1.3$
- Shear:
  - YIG: $y=2$
    (Yttrium indium garnet)
  - Granite: $y \approx 1$

Data for shear and longitudinal wave loss which show power-law dependence over four decades of frequency.
1. Wave Equations with Power Law Solutions

\[ \nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} + \tau \frac{\partial}{\partial t} (-\nabla^2)^{y/2} u = 0 \]

\[ \nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} + \tau \frac{\partial}{\partial t} (-\nabla^2)^{y/2} u + \eta (-\nabla^2)^{(y+1)/2} u = 0 \]

\[ \nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} - \frac{2\alpha_0}{\cos(\pi y/2)} \frac{\partial^{y+1} u}{\partial t^{y+1}} = 0 \]

\[ \nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} - \frac{2\alpha_0}{c_0 \cos(\pi y/2)} \frac{\partial^{y+1} u}{\partial t^{y+1}} - \frac{\alpha_0^2}{\cos^2(\pi y/2)} \frac{\partial^2 u}{\partial t^2} = 0 \]
1. Wave Equations with Power Law Solutions

\[ \nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} + \tau_\sigma \frac{\partial^\alpha}{\partial t^\alpha} \nabla^2 u = 0 \]

\[ \nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} + \tau_\sigma \frac{\partial^\alpha}{\partial t^\alpha} \nabla^2 u - \frac{\tau_\epsilon}{c_0^2} \frac{\partial^{\alpha+2} u}{\partial t^{\alpha+2}} = 0 \]

\[ \nabla^2 p - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} + \tau_\sigma \frac{\partial^\alpha}{\partial t^\alpha} \nabla^2 p + \tau_{th} \frac{\partial^{2-\gamma}}{\partial t^{2-\gamma}} \nabla^2 p = -\frac{\beta_{NL} \partial^2 p^2}{\rho_0 c_0^4 \partial t^2} \]
3. Constitutive Equations

Non-fractional

\[ \sigma(t) = E \left[ \epsilon(t) + \tau_\sigma \frac{\partial \epsilon(t)}{\partial t} \right] \]

\[ \sigma(t) + \tau_\epsilon \frac{\partial \sigma(t)}{\partial t} = E \left[ \epsilon(t) + \tau_\sigma \frac{\partial \epsilon(t)}{\partial t} \right] \]

\[ q = -\kappa \nabla T \]

Fractional

\[ \sigma(t) = E \left[ \epsilon(t) + \tau_\sigma^{\alpha} \frac{\partial^{\alpha} \epsilon(t)}{\partial t^{\alpha}} \right] \]

\[ \sigma(t) + \tau_{\epsilon^{\beta}} \frac{\partial^{\beta} \sigma(t)}{\partial t^{\beta}} = E \left[ \epsilon(t) + \tau_\sigma^{\alpha} \frac{\partial^{\alpha} \epsilon(t)}{\partial t^{\alpha}} \right] \]

\[ q(t) = -\tau_{th}^{1-\gamma} \kappa \Gamma^{\gamma-1} \nabla T(t) \]
Kelvin-Voigt constitutive equation

- Stress, $\sigma$ vs strain $\epsilon$

$$\sigma(t) = E \left[ \epsilon(t) + \tau_\sigma \frac{\partial \epsilon(t)}{\partial t} \right]$$

- $E$ : elastic modulus
- $\tau_\sigma$ : ratio of viscosity and elasticity – retardation time

- Hooke + Newton = spring + damper
Viscous wave equation

1. Conservation of mass
2. Conservation of momentum
3. Kelvin-Voigt constitutive equation =>

\[ \nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} + \tau \sigma \frac{\partial}{\partial t} \nabla^2 u = 0 \]

- Dispersion equation, insert displacement
\[ u = \exp\{i(\omega t - kx)\} \]

\[ k^2 - \frac{\omega^2}{c_0^2} + \tau \sigma \omega k^2 = 0 \]
Attenuation, phase/group velocity - 2 regions

- Never a single power law
- Causal since constitutive equation was causal
Application dependent cut-off

- **Low frequency region:**
  - Pressure waves in water, $\tau_\sigma \approx 1$ ns
  - $\omega \tau_\sigma \ll 1$ for frequencies less than about 1 GHz
  - Application: Sonar (1. order model for water)

- **High frequency region:**
  - Shear waves in human tissue, $\tau_\sigma \approx 0.1$ sec
  - $\omega \tau_\sigma \gg 1$ for frequencies larger than about 10 Hz
  - Application: Elastography
Losses in a fluid

\[ \nabla^2 p - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} + \frac{\delta}{c_0^4} \frac{\partial^3 p}{\partial t^3} = 0 \]

- **Loss**: \( \delta = \delta_{me} + \delta_{th} = \frac{1}{\rho_0} \left( \zeta + \frac{4}{3} \eta \right) + \kappa \frac{1}{\rho_0} \left( \frac{1}{c_v} - \frac{1}{c_p} \right) \)
- **Mechanical**: 
  - \( \eta, \zeta \) - shear and bulk viscosity coefficients
- **Thermal**: 
  - \( \kappa \) - thermal conductivity
  - \( c_v, c_p \) - heat capacity per unit of mass at constant volume / pressure.
- Loss terms are additive, but only for low frequencies, Bhatia, Ultrasonic absorption, 1967 (ch. 4.5)
Fractional Kelvin-Voigt

• Stress, $\sigma$ vs strain $\varepsilon$ - Fractional Kelvin-Voigt

$$\sigma(t) = E \left[ \varepsilon(t) + \tau^\alpha_\sigma \frac{\partial^\alpha \varepsilon(t)}{\partial t^\alpha} \right]$$


$- \alpha=1$ is normal viscoelastic case (figure)
A physical fractional device

- Capacitor soakage, dielectric absorption
- $i = C d^a u / dt^a$, $a \lesssim 1$
- $Z = u / i = 1 / C(j\omega)^a$
- Example in video:
  - 220 uF/63 Volt
  - 10 Volts for 60 sec
  - Shorted for 6 sec
  - [http://www.youtube.com/watch?v=vhHog_yCQ4Q](http://www.youtube.com/watch?v=vhHog_yCQ4Q)
Wave equation from fractional K-V

• Conservation laws =>

\[ \nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} + \tau^\alpha \frac{\partial^\alpha}{\partial t^\alpha} \nabla^2 u = 0 \]


– Dispersion equation, \( u = \exp\{i(\omega t - kx)\} \)

• imaginary part of \( k \) is loss: \( k^2 - \omega^2/c_0^2 + (\tau\sigma i\omega)^\alpha k^2 = 0. \)

Attenuation, fractional Kelvin-Voigt

\[ y = \alpha + 1 \quad \text{and} \quad y = 1 - \alpha / 2 \]

Elastography
shear waves,
\( \omega \tau \gg 1 \)
0/0.5\(<y<1\)

Ultrasound
compression waves
\( \omega \tau \ll 1, \, 1<y<2 \)

\[ \alpha = 0.1, \, 0.3, \, 0.7, \, \text{and} \, 1 \]

\[ \alpha = 1: \text{viscoelastic case} \]
Attenuation, fractional Zener

\[ y = \alpha + 1, \ y = 1 - \alpha / 2, \ y = 1 - \alpha \]

1. Low frequency
2. Medium frequency = High frequency for Kelvin-Voigt model
3. High frequency
   \[ \alpha = \beta = 0.1, \ 0.3, \ 0.7, \ \text{and} \ 1 \]
MR Elastography, brain

- Experimental and model wave speed and damping

Modifying Fourier’s heat law

  - From \( q = -\kappa \nabla T \)
  - To \( q(t) = -\int_{-\infty}^{t} K(t - \tau) \nabla T(\tau) d\tau. \)
Fractional Fourier law

• Power law memory kernel \((1 < \gamma \leq 2)\):

\[
K(t - \tau) = \frac{\chi}{\Gamma(\gamma - 1)}(t - \tau)^{\gamma - 2}
\]

• Gives fractional Fourier law (fract. integral)

\[
q(t) = -\chi I^{\gamma - 1} \nabla T(t)
\]


– \(\gamma \to 1 \Rightarrow\) standard Fourier law (kernel \(\to\) impulse)

• Fractional diffusion:

\[
\frac{1}{\alpha} \frac{\partial^\gamma T}{\partial t^\gamma} = \nabla^2 T
\]
Fract. Kelvin-Voigt + Fourier law:

- Fract. Westervelt eq of 1. form, decoupled:
  \[
  \nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} + \frac{\delta_{me}}{c_0^2} \frac{\partial^\alpha}{\partial t^\alpha} \nabla^2 u + \frac{\delta_{th}}{c_0^2} \frac{\partial^{2-\gamma}}{\partial t^{2-\gamma}} \nabla^2 u = -\frac{\beta}{\rho_0 c_0^4} \frac{\partial^2 p^2}{\partial t^2}
  \]
  \[
  \delta_{me} = \frac{\tau^{\alpha-1}}{\rho_0} \left( \zeta + \frac{4}{3} \eta \right) \quad \delta_{th} = \frac{\kappa_t^{1-\gamma}}{\rho_0} \left( \frac{1}{c_v} - \frac{1}{c_p} \right)
  \]

- Fract. Westervelt of 2. form, coupled \( \gamma=2-\alpha \):
  \[
  \nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} + \left( \frac{\delta_{me}}{c_0^2} + \frac{\delta_{th}}{c_0^2} \right) \frac{\partial^\alpha}{\partial t^\alpha} \nabla^2 u = -\frac{\beta}{\rho_0 c_0^4} \frac{\partial^2 p^2}{\partial t^2}
  \]

- Standard Westervelt equation for \( \gamma=\alpha=1 \)
Conclusion

- The constitutive equation, a material property, is what determines wave behavior.
- Both the low and high-frequency regions are important.
- Which one is relevant depends on application.
Fractional viscoelastic wave equations

• Fractional mechanical models:
  – Fract. Kelvin-Voigt: \[ \nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} + \tau_\sigma \frac{\partial^\alpha}{\partial t^\alpha} \nabla^2 u = 0 \]
  – Fract. Zener: \[ \nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} + \tau_\sigma \frac{\partial^\alpha}{\partial t^\alpha} \nabla^2 u - \frac{\tau_\epsilon^\beta}{c_0^2} \frac{\partial^{\beta+2} u}{\partial t^{\beta+2}} = 0 \]
  – Attenuation \( \propto \omega^y \)
    • Low frequencies: \( y = 1+\alpha \)
    • High/Intermediate frequencies: \( y = 1-\alpha/2 \)
    • High frequencies (for Zener): \( y = 1-\alpha \)