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probabilistic collocation (ME-PCM) methods. We have found that the response-excitation approach yields accurate representations of the statistical properties associated with the stochastic solution. The question of high-dimensionality for evolution equations involving multidimensional joint response-excitation probability densities is also addressed.

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Dear Editor:

Please consider for publication in the Journal of Computational Physics the attached copy of the manuscript entitled ``New evolution equations for the joint response-excitation probability density function of stochastic solutions to first-order nonlinear PDEs".

In this paper we determine new types of evolution equations satisfied by the joint response excitation probability density function associated with the stochastic solution to first-order nonlinear scalar PDEs subject to random boundary conditions, random initial conditions or random forcing terms. This extends recent work on the subject, e.g., by J.-B. Chen and J. Li [Prob. Eng. Mech 24 (2009), pp. 51-59], that holds for ODEs with random parameters, to nonlinear PDEs. The new equations we obtain are of interest in many areas of mathematical physics since they can model, e.g., ocean waves in an Eulerian framework linear and nonlinear advection problems, advection-reaction systems and, more generally, scalar conservation laws.

By using a Fourier-Galerkin spectral method we obtain numerical solutions of the PDEs governing the response-excitation probability density function and we compare the numerical results against those obtained from probabilistic collocation (PCM) and multi-element probabilistic collocation (ME-PCM) methods. The numerical results suggest that the response excitation approach yields accurate representations of the statistical properties associated with the stochastic solution. Therefore, it can be effectively employed as a new computational method for uncertainty quantification.

Thank you for your time and consideration.

George EM Karniadakis

# New evolution equations for the joint response-excitation probability density function of stochastic solutions to first-order nonlinear PDEs

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**Abstract**

By using functional integral methods we determine new types of differential equations satisfied by the *joint* response-excitation probability density function associated with the stochastic solution to first-order nonlinear scalar PDEs. The theory is developed for arbitrary fully nonlinear and for quasilinear first-order stochastic PDEs subject to random boundary conditions, random initial conditions or random forcing terms. Particular applications are presented for a nonlinear advection problem with an additional quadratic nonlinearity, for the classical linear and nonlinear advection equations and for the advection-reaction equation. By using a Fourier-Galerkin spectral method we obtain numerical solutions of the PDEs governing the response-excitation probability density function and we compare the numerical results against those obtained from probabilistic collocation (PCM) and multi-element probabilistic collocation (ME-PCM) methods. We have found that the response-excitation approach yields accurate representations of the statistical properties associated with the stochastic solution. The question of high-dimensionality for evolution equations involving multidimensional joint response-excitation probability densities is also addressed.

*Key words:* PDF methods, multi-element probabilistic collocation, Hopf characteristic functional, uncertainty quantification.

## 1. Introduction

The purpose of this paper is to introduce a new probability density function approach for computing the statistical properties of the stochastic solution to first-order (in space and time) nonlinear scalar PDEs subject to uncertain initial conditions, boundary conditions or external random forces. Specifically, we will consider two different classes of model problems. The first one is a fully nonlinear PDE with random excitation[1]

$$\frac{\partial u}{\partial t} + \mathcal{N}(u, u_x, x, t; \boldsymbol{\xi}) = 0\,, \tag{1}$$

where $\mathcal{N}$ is a continuously differentiable function, $u_x = \partial u/\partial x$ and $\boldsymbol{\xi}$ is a finite-dimensional vector of random variables with known joint probability density function. The second one is a multidimensional quasilinear PDE with random excitation

$$\frac{\partial u}{\partial t} + \mathcal{P}\left(u, t, \boldsymbol{x}; \boldsymbol{\xi}\right) \cdot \nabla_{x} u = \mathcal{Q}\left(u, t, \boldsymbol{x}; \boldsymbol{\eta}\right)\,, \tag{2}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are continuously differentiable functions, $\boldsymbol{x}$ denotes a set of independent variables while $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are two vectors of random variables with known joint probability density function.

As is well known, Eqs. (1) and (2) can model many physically interesting phenomena such as ocean waves in an Eulerian framework [1, 2], linear and nonlinear advection problems, advection-reaction systems [3] and, more generally, scalar conservation laws. The key aspect in order to determine the statistical properties of the solution to Eq. (1) or Eq. (2) relies in an efficient representation of the functional relation between the input uncertainty and the solution

[1] In this paper we will often refer to PDEs with random coefficients as "stochastic PDEs".

field. This topic has received great attention in recent years. Well known approaches are generalized polynomial chaos [4, 5, 6], multi-element generalized polynomial chaos [7, 8], multi-element and sparse grid adaptive probabilistic collocation [9, 10, 11], high-dimensional model representations [12, 13], stochastic biorthogonal expansions [14, 15, 16] and generalized spectral decompositions [17, 18]. However, first-order nonlinear or quasilinear PDEs with stochastic excitation admit an exact reformulation in terms of joint *response-excitation* [19] probability density functions. This formulation has a great advantage with respect to more classical stochastic approaches since it does not suffer from the curse of dimensionality problem, at least when randomness comes only from boundary or initial conditions. In fact, we can prescribe these conditions in terms of probability distributions and this is obviously not dependent on the number of random variables characterizing the underlying probability space. Therefore, a probability density function approach seems more appropriate to tackle several open problems such as curse of dimensionality, discontinuities in random space [8], and long-term integration [20, 21]. However, if an external random forcing term appears within the equations of motion, e.g., represented in the form of a finite-dimensional Karhunen-Loève expansion, then the dimensionality of the corresponding problem in probability space could increase significantly. This happens because the exact stochastic dynamics in this case develops over a high-dimensional manifold and therefore the exact probabilistic description of the system necessarily involves a multidimensional probability density function, or even a probability density functional. The evolution equation for the joint response-excitation probability density function associated with the solution to Eq. (1) or Eq. (2) can be determined by using different stochastic approaches. In this paper we will employ a functional integral technique we have recently introduced in the context of nonlinear stochastic dynamical systems. This allows for very efficient mathematical derivations compared to those ones based on the more rigorous Hopf characteristic functional approach [22, 23, 24].

This paper is organized as follows. In section 2 we present a general functional integral technique for the representation of the joint response-excitation probability density function associated with the stochastic solution to an arbitrary stochastic PDE. In sections 3 and 4 we apply this technique to Eq. (1) and Eq. (2), respectively. In these sections we also present several applications: to a nonlinear advection problem with an additional quadratic nonlinearity, to the classical linear and nonlinear advection equations, and to the advection-reaction equation [3]. In section 5 we raise some questions related to the numerical integration of problems involving a high-dimensional joint response-excitation probability density function. By using an accurate Fourier-Galerkin spectral method [25, 26], in section 6 we obtain numerical solutions of the PDE governing the response-excitation probability density function corresponding to the randomly forced nonlinear advection equation. The numerical results are compared against those obtained from probabilistic collocation (PCM) and multi-element probabilistic collocation (ME-PCM) approaches [9, 10]. Finally, the main findings and their implications are summarized in section 7. We also include two appendices. The first one, i.e. appendix A, discusses the application of the Hopf characteristic functional approach to some of the equations considered in the paper. In Appendix B we present the Fourier-Galerkin systems we have employed for the numerical simulations.

## 2. Functional representation of the probability density function of the solution to stochastic PDEs

Let us consider a physical system described in terms of partial differential equations subject to uncertain initial conditions, boundary conditions, physical parameters or external forcing terms. The solution to these types of boundary value problems is a random field whose regularity properties in space and time are strongly related to the type of nonlinearities appearing in the equations as well as on the statistical properties of the random input processes. In this paper we assume that the probability density function of the solution field exists. In order to fix ideas, let us consider the advection-diffusion equation

$$\frac{\partial u}{\partial t} + \xi(\omega)\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{3}$$

with deterministic boundary and initial conditions. The parameter $\xi(\omega)$ is assumed to be a random variable with known probability density function. The solution to Eq. (3) is a random field depending on the random variable $\xi(\omega)$ in a possibly nonlinear way. We shall denote such a functional dependence as $u(x, t; [\xi])$. The joint probability density of $u(x, t; [\xi])$ and $\xi$, i.e. the solution field at the space-time location $(x, t)$ and the random variable $\xi$, admits the following integral representation

$$p^{(a,b)}_{u(x,t)\xi} = \langle\delta(a - u(x, t; [\xi]))\delta(b - \xi)\rangle, \quad a, b \in \mathbb{R}. \tag{4}$$

The average operator $\langle\cdot\rangle$, in this particular case, is defined as a simple integral with respect to the probability density of $\xi(\omega)$, i.e.

$$p^{(a,b)}_{u(x,t)\xi} = \int_{-\infty}^{\infty} \delta(a - u(x,t;[z]))\delta(b - z) p^{(z)}_{\xi} dz\,, \tag{5}$$

where $p^{(z)}_{\xi}$ denotes the probability density of $\xi$ which could be compactly supported (e.g., a uniform distribution). The support of the probability density function $p^{(a,b)}_{u(x,t)\xi}$ is actually determined by the nonlinear transformation $\xi \to u(x,t;[\xi])$ appearing within the delta function $\delta(a - u(x,t;[\xi]))$ (see, e.g., Ch. 3 in [27]). The simple representation (5) can be easily generalized to infinite dimensional random input processes. To this end, let us examine the case where the scalar field $u$ is advected by a random velocity field $U$ according to the equation

$$\frac{\partial u}{\partial t} + U(x,t;\omega)\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}\,, \tag{6}$$

for some deterministic initial condition and boundary conditions. Disregarding the particular form of the random field $U(x,t;\omega)$, let us consider its collocation representation for a given discretization of the space-time domain. This gives us a certain number of random variables $\{U(x_i,t_j;\omega)\}$ ($i = 1,..,N$, $j = 1,...,M$). The random field $u$ solving Eq. (6) at each one of these locations is, in general, a nonlinear function of *all* the variables $\{U(x_i,t_j;\omega)\}$. In order to see this, it is sufficient to write an explicit finite difference numerical scheme of Eq. (6). The joint probability density of the solution field $u$ at $(x_i,t_j)$ and the forcing field $U$ at a different space-time location, say $(x_n,t_m)$, admits the following integral representation

$$p^{(a,b)}_{u(x_i,t_j)U(x_n,t_m)} = \langle \delta(a - u(x_i,t_j;[U(x_1,t_1),...,U(x_N,t_M)]))\delta(b - U(x_n,t_m))\rangle\,, \tag{7}$$

where the average is with respect to the joint probability of all the random variables $\{U(x_n,t_m;\omega)\}$ ($n = 1,..,N$, $m = 1,...,M$). The notation $u(x_i,t_j;[U(x_1,t_1),...,U(x_N,t_M)])$ emphasizes that the solution field $u(x_i,t_j)$ is, in general, a nonlinear function of all the random variables $\{U(x_n,t_m;\omega)\}$. If we send the number of these variables to infinity, i.e. we refine the space-time mesh to the continuum level, we obtain a functional integral representation of the joint probability density

$$\begin{aligned} p^{(a,b)}_{u(x,t)U(x',t')} &= \langle \delta(a - u(x,t;[U]))\delta(b - U(x',t'))\rangle \\ &= \int \mathcal{D}[U] W[U] \delta(a - u(x,t;[U]))\delta(b - U(x',t'))\,, \end{aligned} \tag{8}$$

where $W[U]$ is the probability density functional of random field $U(x,t;\omega)$ and $\mathcal{D}[U]$ is the usual functional integral measure [28, 29, 30]. Depending on the stochastic PDE, we will need to consider different joint probability density functions, e.g., the joint probability of a field and its derivatives with respect to space variables. The functional representation described above allows us to deal with these different situations in a very practical way. For instance, the joint probability density of a field $u(x,y,t)$ and its first-order spatial derivatives at different space-time locations can be represented as

$$p^{(a,b,c)}_{u(x,y,t)u_x(x',y',t')u_y(x'',y'',t'')} = \langle \delta(a - u(x,y,t))\delta(b - u_x(x',y',t'))\delta(c - u_y(x'',y'',t''))\rangle\,, \tag{9}$$

where, for notational convenience, we have denoted by $u_x \overset{\text{def}}{=} \partial u/\partial x$, $u_y \overset{\text{def}}{=} \partial u/\partial y$ and we have omitted the functional dependence on the random input variables (i.e. the variables in the brackets $[\cdot]$) within each field. Similarly, the joint probability of $u(x,y,t)$ at two different spatial locations is

$$p^{(a,b)}_{u(x,y,t)u(x',y',t)} = \langle \delta(a - u(x,y,t))\delta(b - u(x',y',t))\rangle\,. \tag{10}$$

In the sequel we will be mostly concerned with joint probability densities of different fields at the *same* space-time location. Therefore, in order to lighten the notation further, sometimes we will drop the subscripts indicating the space-time variables and write, for instance

$$p^{(a,b)}_{uu_x} = \langle \delta(a - u(x,y,t))\delta(b - u_x(x,y,t))\rangle\,, \tag{11}$$

or even more compactly

$$p^{(a,b)}_{uu_x} = \langle \delta(a - u)\delta(b - u_x)\rangle\,. \tag{12}$$

### 2.1. Representation of derivatives

We will often need to differentiate the probability density function, e.g. (4) or (8), with respect to space or time variables. This operation involves generalized derivatives of the Dirac delta function and it can be carried out in a systematic manner. To this end, let us consider Eq. (4) and define the following linear functional

$$\int_{-\infty}^{\infty} p_{u(x,t)}^{(a)} \rho(a) da = \langle \int_{-\infty}^{\infty} \delta(a - u(x,t))\rho(a)da \rangle = \langle \rho(u) \rangle , \tag{13}$$

where $\rho(a)$ is a continuously differentiable and compactly supported function. A differentiation of Eq. (13) with respect to $t$ gives

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\partial p_{u(x,t)}^{(a)}}{\partial t} \rho(a) da &= \langle u_t \frac{\partial \rho}{\partial u} \rangle \\ &= \langle u_t \int_{-\infty}^{\infty} \frac{\partial \rho}{\partial a} \delta(a - u(x,t)) da \rangle \\ &= \int_{-\infty}^{\infty} -\frac{\partial}{\partial a} \langle u_t \delta(a - u(x,t)) \rangle \rho(a) da \,. \end{aligned} \tag{14}$$

This equation holds for an arbitrary $\rho(a)$ and therefore we have the identity

$$\frac{\partial p_{u(x,t)}^{(a)}}{\partial t} = -\frac{\partial}{\partial a} \langle \delta\left(a - u(x,t)\right) u_t(x,t) \rangle \,. \tag{15}$$

Similarly,

$$\frac{\partial p_{u(x,t)}^{(a)}}{\partial x} = -\frac{\partial}{\partial a} \langle \delta\left(a - u(x,t)\right) u_x(x,t) \rangle \,. \tag{16}$$

Straightforward extensions of these results allow us to compute derivatives of joint probability density functions involving more fields, e.g., $u(x,t)$ and its first-order spatial derivative $u_x(x,t)$. For instance, we have

$$\frac{\partial p_{uu_x}^{(a,b)}}{\partial t} = -\frac{\partial}{\partial a} \langle \delta\left(a - u(x,t)\right) \delta(b - u_x(x,t)) u_t(x,t) \rangle - \frac{\partial}{\partial b} \langle \delta\left(a - u(x,t)\right) \delta(b - u_x(x,t)) u_{tx}(x,t) \rangle \,, \tag{17}$$

where $u_{tx} \stackrel{\text{def}}{=} \partial^2 u / \partial t \partial x$.

### 2.2. Representation of averages

In this section we determine important formulae that allow us to compute the average of a product between Dirac delta functions and various fields[2]. To this end, let us first consider the average $\langle \delta(a-u)u \rangle$. By applying well-known properties of Dirac delta functions it can be shown that

$$\langle \delta(a-u)u \rangle = a p_{u(x,t)}^{(a)} \,. \tag{18}$$

This result is a multidimensional extension of the following trivial identity that holds for only one random variable $\xi$ (with probability density $p_\xi^{(z)}$) and a nonlinear function $g(\xi)$ (see, e.g., Ch. 3 of [27] or [31])

$$\int_{-\infty}^{\infty} \delta(a - g(z)) g(z) p_\xi^{(z)} dz = \sum_n \frac{1}{|g'(\hat{z}_n)|} \int_{-\infty}^{\infty} \delta(z - \hat{z}_n) g(z) p_\xi^{(z)} dz = \sum_n \frac{g(\hat{z}_n) p_\xi^{(\hat{z}_n)}}{|g'(\hat{z}_n)|} \,, \tag{19}$$

[2] These formulae are actually the key results that allow us to obtain a closure for the evolution equation involving the probability density function of the solution to first-order SPDEs.

where $\hat{z}_n = g^{-1}(z)$ are roots of $g(z) = a$. Since $g(\hat{z}_n) = g(g^{-1}(a)) = a$, from Eq. (19) it follows that

$$\int_{-\infty}^{\infty} \delta(a - g(z))g(z)p_\xi^{(z)}dz = a \sum_n \frac{p_\xi^{(\hat{z}_n)}}{|g'(\hat{z}_n)|} = a \int_{-\infty}^{\infty} \delta(a - g(z))p_\xi^{(z)}dz\,, \tag{20}$$

which is equivalent to Eq. (18). Similarly, one can show that

$$\langle \delta(a - u)\delta(b - u_x)u \rangle = ap_{uu_x}^{(a,b)}\,, \tag{21a}$$

$$\langle \delta(a - u)\delta(b - u_x)u_x \rangle = bp_{uu_x}^{(a,b)}\,, \tag{21b}$$

and, more generally, that

$$\langle \delta(a - u)\delta(b - u_x)h(x, t, u, u_x) \rangle = h(x, t, a, b)p_{uu_x}^{(a,b)}\,, \tag{22}$$

where, for the purposes of the present paper, $h(u, u_x, x, t)$ is any continuous function of $u$, $u_x$, $x$ and $t$. The result (22) can be generalized even further to averages involving a product of functions in the form

$$\langle \delta(a - u)\delta(b - u_x)h(x, t, u, u_x)g(x, t, u_{xx}, u_{xt}, u_{tt}, ...) \rangle = h(x, t, a, b)\langle \delta(a - u)\delta(b - u_x)g(x, t, u_{xx}, u_{xt}, u_{tt}, ...) \rangle\,. \tag{23}$$

In short, the general rule is: *we are allowed to take out of the average all those functions involving fields for which we have available a Dirac delta.* As an example, if $u$ is a time-dependent field in a two-dimensional spatial domain we have

$$\langle \delta(a - u)\delta(b - u_x)\delta(c - u_y)e^{-(x^2+y^2)} \sin(u)u_x u_y^2 u_{xx} \rangle = e^{-(x^2+y^2)} \sin(a)bc^2 \langle \delta(a - u)\delta(b - u_x)\delta(c - u_y)u_{xx} \rangle\,. \tag{24}$$

### *2.3. Intrinsic relations depending on the structure of the joint probability density function*

The fields appearing in the joint probability density function are usually not independent from each other and therefore, as a consequence, we expect that there exist a certain number of relations between the probability density function and itself. These relations are independent of the particular stochastic PDE describing the physical system. In order to determine them, let us consider the following joint density

$$p_{u(x,t)u_x(x',t')}^{(a,b)} = \langle \delta(a - u(x, t))\delta(b - u_x(x', t')) \rangle \tag{25}$$

involving a random field and its first-order spatial derivative at *different* space-time locations. There is clearly a deterministic relation between the random field $u$ and its derivative at the *same* space-time location. This relation reduces to defining the partial derivative as the limit of the increment ratio

$$u_x(x, t) = \lim_{x'\to x} \frac{u(x', t) - u(x, t)}{(x' - x)}\,. \tag{26}$$

The condition (26) can be actually translated into a relation involving the probability density function (25). In fact,

$$\frac{\partial p_{u(x,t)u_x(x',t')}^{(a,b)}}{\partial x} = -\frac{\partial}{\partial a}\langle \delta(a - u(x, t))u_x(x, t)\delta(b - u_x(x', t')) \rangle\,. \tag{27}$$

Taking the limit for $x' \to x$ and $t' \to t$ and using the results of section 2.2 yields

$$\lim_{x'\to x} \frac{\partial p_{u(x,t)u_x(x',t)}^{(a,b)}}{\partial x} = -bp_{u(x,t)u_x(x,t)}^{(a,b)}\,. \tag{28}$$

This is a *differential constraint* that arises only because $u_x$ is *locally related* to $u$ by Eq. (26). As already pointed out, condition (28) is independent of the specific PDE describing the physical phenomenon. Additional regularity properties of $u$, e.g. the existence of the second-order spatial derivative, yield additional intrinsic relations. For instance,

$$\frac{\partial p_{u(x,t)u_x(x',t')}^{(a,b)}}{\partial x'} = -\frac{\partial}{\partial a}\langle \delta(a - u(x, t))u_{xx}(x', t')\delta(b - u_x(x', t')) \rangle\,. \tag{29}$$

At the same time,

$$\frac{\partial^2 p^{(a,b)}_{u(x,t)u_x(x',t')}}{\partial x^2} = \frac{\partial^2}{\partial a^2}\langle \delta(a-u(x,t))u_x(x,t)^2\delta(b-u_x(x',t'))\rangle - \frac{\partial}{\partial a}\langle \delta(a-u(x,t))u_{xx}(x,t)\delta(b-u_x(x',t'))\rangle\,. \tag{30}$$

By taking the limit for $x' \to x$ and $t' \to t$ of both Eqs. (29) and (30), we obtain the differential constraint

$$\lim_{x'\to x}\frac{\partial^2 p^{(a,b)}_{u(x,t)u_x(x',t)}}{\partial x^2} = b^2\frac{\partial p^{(a,b)}_{u(x,t)u_x(x',t)}}{\partial a^2} + \frac{\partial}{\partial a}\int_{-\infty}^{b}\lim_{x\to x'}\frac{\partial p^{(a,b')}_{u(x,t)u_x(x',t)}}{\partial x'}db'\,. \tag{31}$$

The existence of the third-order spatial derivative yields another relation and so forth. In principle, if the field $u$ is analytic we can construct an infinite set of differential constraints to be satisfied at every space-time location. In other words, the local regularity properties of the field $u$ can be translated into a set differential constraints involving the joint probability density function (25). We conclude this section by noting that the correlation between the two Dirac delta functions arising in (25) can be also formally expanded in a functional power series [32, 33]. This yields a series expansion of the joint probability density in terms of cumulants of the random input variables.

## 3. Kinetic equations for the probability density function of the solution to first-order nonlinear stochastic PDEs

In this section we address the question of how to obtain a kinetic equation for the probability density of the solution to the nonlinear evolution equation

$$\frac{\partial u}{\partial t} + \mathcal{N}(u, u_x, x, t) = 0\,, \tag{32}$$

where $\mathcal{N}$ is a continuously differentiable function. For the moment, we shall consider only one spatial dimension and assume that the field $u(x,t;\omega)$ is random as a consequence of the fact that the initial condition or the boundary condition associated with Eq. (32) are set to be random. A more general case involving a random forcing term will be discussed later in this section. As is well known, the full statistical information of the solution to Eq. (32) can be always encoded in the Hopf characteristic functional of the system (see appendix A). In some very special cases, however, the functional differential equation satisfied by the Hopf functional can be reduced to a standard partial differential equation for the characteristic function or, equivalently, for the probability density function of the system. First-order nonlinear stochastic PDEs belong to this class and, in general, they admit a reformulation in terms of the joint density of $u$ and its first order spatial derivative $u_x$ at the *same* space-time location, i.e.,

$$p^{(a,b)}_{uu_x} = \langle \delta(a-u(x,t))\delta(b-u_x(x,t))\rangle\,. \tag{33}$$

The average operator $\langle\cdot\rangle$ here is defined as an integral with respect to the joint probability density functional of the random initial condition and the random boundary condition. A differentiation of Eq. (33) with respect to time yields

$$\frac{\partial p^{(a,b)}_{uu_x}}{\partial t} = -\frac{\partial}{\partial a}\langle \delta(a-u)u_t\delta(b-u_x)\rangle - \frac{\partial}{\partial b}\langle \delta(a-u)\delta(b-u_x)u_{xt}\rangle\,. \tag{34}$$

If we substitute Eq. (32) and its derivative with respect to $x$ into Eq. (34) we obtain

$$\frac{\partial p^{(a,b)}_{uu_x}}{\partial t} = \frac{\partial}{\partial a}\left(\mathcal{N}p^{(a,b)}_{uu_x}\right) + \frac{\partial}{\partial b}\langle\left(\frac{\partial \mathcal{N}}{\partial u}u_x + \frac{\partial \mathcal{N}}{\partial u_x}u_{xx} + \frac{\partial \mathcal{N}}{\partial x}\right)\delta(a-u)\delta(b-u_x)\rangle\,. \tag{35}$$

Next, let us recall that $\mathcal{N}$ and its derivatives are at least continuous functions (by assumption) and therefore by using Eq. (23) they can be taken out of the averages. Thus, the only item that is missing in order to close Eq. (35) is an expression for the average of $u_{xx}$ in terms of the probability density function. Such an expression can be easily obtained by integrating the identity

$$\frac{\partial p^{(a,b)}_{uu_x}}{\partial x} = -b\frac{\partial p^{(a,b)}_{uu_x}}{\partial a} - \frac{\partial}{\partial b}\langle \delta(a-u)\delta(b-u_x)u_{xx}\rangle \tag{36}$$

with respect to $b$ from $-\infty$ to $b$ and taking into account the fact that the average of any field vanishes when $b \to \pm\infty$ due to the properties of the underlying probability density functional. Therefore, Eq. (36) can be equivalently written as

$$\langle \delta(a-u)\delta(b-u_x)u_{xx}\rangle = -\int_{-\infty}^{b} \frac{\partial p_{uu_x}^{(a,b')}}{\partial x} db' - \int_{-\infty}^{b} b' \frac{\partial p_{uu_x}^{(a,b')}}{\partial a} db' \, . \tag{37}$$

A substitution of this relation into Eq. (35) yields the final result

$$\frac{\partial p_{uu_x}^{(a,b)}}{\partial t} = \frac{\partial}{\partial a}\left(\mathcal{N} p_{uu_x}^{(a,b)}\right) + \frac{\partial}{\partial b}\left[\left(b\frac{\partial \mathcal{N}}{\partial a} + \frac{\partial \mathcal{N}}{\partial x}\right) p_{uu_x}^{(a,b)} - \frac{\partial \mathcal{N}}{\partial b}\left(\int_{-\infty}^{b} \frac{\partial p_{uu_x}^{(a,b')}}{\partial x} db' + \int_{-\infty}^{b} b' \frac{\partial p_{uu_x}^{(a,b')}}{\partial a} db'\right)\right], \tag{38}$$

where $\mathcal{N}$ here is a function of $a$, $b$, $x$ and $t$, respectively. Equation (38) is the correct evolution equation for the joint probability density associated with the solution to an arbitrary nonlinear evolution problem in the form (32). This equation made its first appearance in [34], although the original published version has many typos and a rather doubtful derivation[3]. A natural generalization of Eq. (32) includes an external driving force in the form

$$\frac{\partial u}{\partial t} + \mathcal{N}(u, u_x, x, t) = f(x, t; \omega) \, . \tag{39}$$

Depending on the type of the random field $f$ and on its correlation structure, different stochastic methods can be employed. For instance, if the characteristic variation of $f$ is much shorter than the characteristic variation of the solution $u$ then we can use small correlation space-time expansions. In particular, if the field $f$ is Gaussian then we can use the Furutsu-Novikov-Donsker [35, 36, 37] formula (see also [24]). Alternatively, if we have available a Karhunen-Loève expansion

$$f(x, t; \omega) = \sum_{k=1}^{m} \lambda_k \xi_k(\omega) \psi_k(x, t) \, , \tag{40}$$

then we can obtain a closed and exact equation for the joint probability of $u$, $u_x$ and all the (uncorrelated) random variables $\{\xi_k(\omega)\}$ appearing in the series (40), i.e.,

$$p_{u(x,t)u_x(x,t)\{\xi_k\}}^{(a,b,\{c_k\})} = \langle \delta(a - u(x,t))\delta(b - u_x(x,t)) \prod_{k=1}^{m} \delta(c_k - \xi_k)\rangle \, . \tag{41}$$

For the specific case of Eq. (39) we obtain the final result

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial a}(\mathcal{N}P) + \frac{\partial}{\partial b}\left[\left(b\frac{\partial \mathcal{N}}{\partial a} + \frac{\partial \mathcal{N}}{\partial x}\right)P - \frac{\partial \mathcal{N}}{\partial b}\left(\int_{-\infty}^{b} \frac{\partial P}{\partial x} db' + \int_{-\infty}^{b} b' \frac{\partial P}{\partial a} db'\right)\right] - \left[\sum_{k=1}^{m} \lambda_k c_k \psi_k\right]\frac{\partial P}{\partial a}, \tag{42}$$

where we have used the shorthand notation

$$P \stackrel{\text{def}}{=} p_{u(x,t)u_x(x,t)\{\xi_k\}}^{(a,b,\{c_k\})} \, . \tag{43}$$

Note that Eq. (42) is linear and exact but it involves 4 variables ($t$, $x$, $a$ and $b$) and $m$ parameters ($\{c_1, ..., c_m\}$). In any case, once the solution is available[4] we can integrate out the variables $(b, \{c_k\})$ and obtain the *response probability of the system*, i.e. the probability density of the solution $u$ at every space-time point as

$$p_{u(x,t)}^{(a)} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{u(x,t)u_x(x,t)\{\xi_k\}}^{(a,b,\{c_k\})} db dc_1 \cdots dc_m \, . \tag{44}$$

[3](Equation numbering hereafter corresponds to Ref. [34]). First of all, we notice a typo in Eq. (1.5), i.e. two brackets are missing. Secondly, according to Eq. (1.1) $f$ is a multivarible function that includes also $x$ and therefore one term is missing in Eq. (1.6). Also, despite the rather doubtful derivation, the final result (1.8) seems to have three typos, i.e., the variable $v$ is missing in the last integral within the brackets (this typo was corrected in the subsequent Eq. (1.9)) and there are two signs that are wrong. These sign errors are still present in Eq. (1.9).

[4]Later on we will discuss in more detail numerical algorithms and techniques that can be employed to compute the numerical solution to a multidimensional linear PDE like (42).

The integrals above are formally written from $-\infty$ to $\infty$ although the probability density we are integrating out may be compactly supported. We conclude this section by observing that the knowledge of the probability density function of the solution to a stochastic PDE at a specific location does *not* provide all the statistical information of the system. For instance, the calculation of the two-point correlation function $\langle u(x,t)u(x',t')\rangle$ requires the knowledge of the joint probability density of the solution $u$ at two different locations, i.e. $p^{(a,b)}_{u(x,t)u(x',t')}$. We will go back to this point in section 4.

*3.1. An example: nonlinear advection problem with an additional quadratic nonlinearity*

Let us consider the following quadratic prototype problem (see, e.g., [38], p. 358)

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + \nu\left(\dfrac{\partial u}{\partial x}\right)^2 = 0\,, \quad \nu \geq 0\,, \quad x \in [0, 2\pi]\,, \quad t \geq t_0 \\ u(x, t_0; \omega) = A\sin(x) + \eta(\omega)\,, \quad A > 0 \\ \text{Periodic B.C.} \end{cases} \tag{45}$$

where $\eta(\omega)$ is a random variable with known probability density function. In this introductory example we would like to provide full details on the calculation of the equation for the probability density function. To this end, let us substitute Eq. (45) and its derivative with respect to $x$ into Eq. (34) to obtain

$$\frac{\partial p^{(a,b)}_{uu_x}}{\partial t} = \left(ab + \nu b^2\right)\frac{\partial p^{(a,b)}_{uu_x}}{\partial a} + b p^{(a,b)}_{uu_x} + \frac{\partial}{\partial b}\langle\delta(a-u)\delta(b-u_x)\left(u_x^2 + uu_{xx} + 2\nu u_x u_{xx}\right)\rangle\,. \tag{46}$$

At this point we need an explicit expression for the last average at the right hand side of Eq. (46) in terms of the probability density function (33). Such an expression can be easily obtained by using the averaging rule (23) and identity (37). We finally get

$$\frac{\partial p^{(a,b)}_{uu_x}}{\partial t} = -a\frac{\partial p^{(a,b)}_{uu_x}}{\partial x} + b p^{(a,b)}_{uu_x} + \frac{\partial}{\partial b}\left(b^2 p^{(a,b)}_{uu_x}\right) - \nu b^2 \frac{\partial p^{(a,b)}_{uu_x}}{\partial a} - 2\nu\left(b\frac{\partial p^{(a,b)}_{uu_x}}{\partial x} + \int_{-\infty}^{b} b'\frac{\partial p^{(a,b')}_{uu_x}}{\partial a}db' + \int_{-\infty}^{b}\frac{\partial p^{(a,b')}_{uu_x}}{\partial x}db'\right). \tag{47}$$

This equation is obviously consistent with the general law (38) with $\mathcal{N}(a,b,x,t) = ab + \nu b^2$. An alternative derivation of Eq. (47) is also provided in Appendix A.3 by employing the Hopf characteristic functional approach. Note that Eq. (47) is a linear partial differential equation in 4 variables ($a$, $b$, $x$, $t$) that can be integrated for $t \geq t_0$ once the joint probability of $u$ and $u_x$ is provided at some initial time $t_0$. In the present example, such an initial condition can be obtained by observing that the spatial derivative of the random initial state $u(x,t_0;\omega) = A\sin(x) + \eta(\omega)$ is the *deterministic* function

$$u_x(x, t_0; \omega) = A\cos(x)\,. \tag{48}$$

Therefore, by applying the Dirac delta formalism, we see that the initial condition for the joint probability density of $u$ and $u_x$ is

$$\begin{aligned} p^{(a,b)}_{u(x,t_0)u_x(x,t_0)} &= \langle\delta(a - A\sin(x) - \eta)\delta(b - A\cos(x))\rangle \\ &= \delta(b - A\cos(x))\langle\delta(a - A\sin(x) - \eta)\rangle \\ &= \delta(b - A\cos(x))\frac{1}{\sqrt{2\pi}}e^{-(a-A\sin(x))^2/2}\,, \end{aligned} \tag{49}$$

provided $\eta(\omega)$ is a Gaussian random variable. At this point it is clear that Eq. (47) has to be interpreted in a weak sense in order for the initial condition (49) to be meaningful. From a numerical viewpoint the presence of the Dirac delta function within the initial condition introduces significant difficulties. In fact, if we adopt a Fourier-Galerkin framework then we need a very high (theoretically infinite) resolution in the $b$ direction in order to resolve such initial condition and, consequently, the proper temporal dynamics of the probability function. In addition, the Fourier-Galerkin system associated with Eq. (47) is fully coupled and therefore inaccurate representations of the Dirac delta appearing in the initial condition rapidly propagate within the Galerkin system, leading to numerical errors. However, we can always apply a Fourier transformation with respect to $a$ and $b$ to Eqs. (47) and (49), before performing the numerical discretization. This is actually equivalent to look for a solution in terms of the joint characteristic function instead of the joint probability density function. The corresponding evolution equation is obtained in appendix A.3 and it is rewritten hereafter for convenience ($\phi^{(a,b)}_{uu_x}$ denotes the joint characteristic function of $u$ and $u_x$, while $i$ is the imaginary unit)

$$\frac{\partial \phi^{(a,b)}_{uu_x}}{\partial t} = ib\frac{\partial^2 \phi^{(a,b)}_{uu_x}}{\partial b^2} - i\frac{\partial \phi^{(a,b)}_{uu_x}}{\partial b} + i\frac{\partial^2 \phi^{(a,b)}_{uu_x}}{\partial a \partial x} - i\nu a\frac{\partial^2 \phi^{(a,b)}_{uu_x}}{\partial b^2} - 2\frac{i\nu}{b}\left(\frac{\partial \phi^{(a,b)}_{uu_x}}{\partial x} - a\frac{\partial \phi^{(a,b)}_{uu_x}}{\partial b} - b\frac{\partial^2 \phi^{(a,b)}_{uu_x}}{\partial b \partial x}\right). \tag{50}$$

The initial condition for this equation is obtained by Fourier transformation of Eq. (49). This yields the well-defined complex function

$$\phi^{(a,b)}_{u(x,t_0)u_x(x,t_0)} = \frac{1}{\sqrt{2\pi}} e^{ibA\cos(x)} \int_{-\infty}^{\infty} e^{ia\alpha-(\alpha - A\sin(x))^2/2} d\alpha \,. \tag{51}$$

We do not address here the computation of the numerical solution to the problem defined by Eqs. (47) and (49). Instead, we aim to prove the consistency of such problem with the nonlinear advection problem discussed in the forthcoming section 4.2. To this end, let us first observe that Eq. (36) can be rewritten as

$$bp^{(a,b)}_{uu_x} = -\int_{-\infty}^{a} \frac{\partial p^{(a',b)}_{uu_x}}{\partial x} da' - \frac{\partial}{\partial b}\int_{-\infty}^{a} \langle \delta(a'-u)\delta(b-u_x)u_{xx}\rangle da' \,. \tag{52}$$

Thus, if we set $\nu = 0$ in Eq. (47) and integrate it with respect to $b$ from $-\infty$ to $\infty$, together with the initial condition (49), we obtain

$$\begin{cases} \dfrac{\partial p^{(a)}_{u(x,t)}}{\partial t} + \dfrac{\partial p^{(a)}_{u(x,t)}}{\partial x} + \displaystyle\int_{-\infty}^{a} \dfrac{\partial p^{(a')}_{u(x,t)}}{\partial x} da' = 0 \\ p^{(a)}_{u(x,t_0)} = \dfrac{1}{\sqrt{2\pi}} e^{-(a-A\sin(x))^2/2} \end{cases} \tag{53}$$

The problem (53) coincides with (73) with $\psi = 0$[5] and therefore we have shown that Eq. (47) is consistent with the standard nonlinear advection equation when $\nu = 0$. We note that, the equations derived in [34] are not consistent in this sense due to the aforementioned typos.

## 4. Kinetic equations for the probability density of the solution to first-order quasilinear stochastic PDEs

In this section we discuss the possibility to obtain a kinetic equation for the probability density function of stochastic solutions to multidimensional quasilinear stochastic PDE in the form

$$\frac{\partial u}{\partial t} + \boldsymbol{\mathcal{P}}(u, t, \boldsymbol{x}; \boldsymbol{\xi}) \cdot \nabla_x u = Q(u, t, \boldsymbol{x}; \boldsymbol{\eta}) \,, \tag{54}$$

where $\boldsymbol{\mathcal{P}}$ and $Q$ are assumed to be continuously differentiable, $\boldsymbol{x}$ denotes a set of independent variables[6] while $\boldsymbol{\xi} = [\xi_1, ..., \xi_m]$ and $\boldsymbol{\eta} = [\eta_1, ..., \eta_n]$ are two vectors of random variables with known joint probability density function. We remark that Eq. (54) models many physically interesting phenomena such as ocean waves [1], linear and nonlinear advection problems, advection-reaction equations [3] and, more generally, scalar conservation laws. We first consider the case where the stochastic solution $u(\boldsymbol{x}, t; \omega)$ is random as consequence of the fact that the initial condition or the boundary conditions are random. In other words, we temporarily remove the dependence on $\{\xi_k\}$ and $\{\eta_k\}$ in $\boldsymbol{\mathcal{P}}$ and $Q$, respectively. In this case an exact evolution equation for the probability density function

$$p^{(a)}_{u(\boldsymbol{x},t)} = \langle \delta(a - u(\boldsymbol{x}, t))\rangle \tag{55}$$

[5]Simply set $\psi = 0$ in (73) and integrate both the equation and the initial condition with respect to $b$ from $-\infty$ to $\infty$.

[6]In many applications $\boldsymbol{x}$ is just a vector of spatial coordinates, e.g., $\boldsymbol{x} = (x, y, z)$. More generally, $\boldsymbol{x}$ is a vector of independent variables including, e.g., spatial coordinates and parameters. For example, the two-dimensional action balance equation for ocean waves in the Eulerian framework [1, 2] is defined in terms of the following variables $\boldsymbol{x} = (x, y, \theta, \sigma)$ where $\theta$ and $\sigma$ denote wave direction and wavelength, respectively.

can be determined. The average here is with respect to the joint probability density functional of the random initial condition and the random boundary conditions. Differentiation of (55) with respect to $t$ yields

$$\frac{\partial p^{(a)}_{u(\boldsymbol{x},t)}}{\partial t} = -\frac{\partial}{\partial a}\langle \delta(a-u(\boldsymbol{x},t))\left[-\boldsymbol{\mathcal{P}}(u,t,\boldsymbol{x})\cdot\nabla_x u + \boldsymbol{Q}(u,t,\boldsymbol{x})\right]\rangle\,. \tag{56}$$

By using the results of the previous sections it is easy to show that this equation can be equivalently written as

$$\frac{\partial p^{(a)}_{u(\boldsymbol{x},t)}}{\partial t} + \frac{\partial}{\partial a}\left(\boldsymbol{\mathcal{P}}(a,t,\boldsymbol{x})\cdot\int_{-\infty}^{a}\nabla_x p^{(a')}_{u(\boldsymbol{x},t)}da'\right) = -\frac{\partial}{\partial a}\left(\boldsymbol{Q}(a,t,\boldsymbol{x})\,p^{(a)}_{u(\boldsymbol{x},t)}\right)\,. \tag{57}$$

If we denote by $D$ the number of independent variables appearing in the vector $\boldsymbol{x}$, then we see that Eq. (57) is a linear partial differential equation in $D+2$ variables. Note that Eq. (57) is independent of the number of random variables describing the boundary conditions or the initial conditions.

As we have previously pointed out, the knowledge of the probability density function of the solution to a stochastic PDE does not provide all the statistical information about the stochastic system. For instance, the computation of the two-point correlation function requires the knowledge of the joint probability of the solution field at two different locations. In order to determine such equation let us consider the joint density

$$p^{(a,b)}_{u(\boldsymbol{x},t)u(\boldsymbol{x}',t)} = \langle \delta(a-u(\boldsymbol{x},t))\delta(b-u(\boldsymbol{x}',t))\rangle\,. \tag{58}$$

Differentiation of Eq. (55) with respect to time yields

$$\begin{aligned}\frac{\partial}{\partial t}p^{(a,b)}_{u(\boldsymbol{x},t)u(\boldsymbol{x}',t)} =& -\frac{\partial}{\partial a}\langle \delta(a-u(\boldsymbol{x},t))\delta(b-u(\boldsymbol{x}',t))\left[-\boldsymbol{\mathcal{P}}(u,t,\boldsymbol{x})\cdot\nabla)xu + \boldsymbol{Q}(u,t,\boldsymbol{x})\right]\rangle\\ &-\frac{\partial}{\partial b}\langle \delta(a-u(\boldsymbol{x},t))\delta(b-u(\boldsymbol{x}',t))\left[-\boldsymbol{\mathcal{P}}(u,t,\boldsymbol{x}')\cdot\nabla_{x'} u + \boldsymbol{Q}(u,t,\boldsymbol{x}')\right]\rangle\,,\end{aligned} \tag{59}$$

and therefore

$$\begin{aligned}\frac{\partial}{\partial t}p^{(a,b)}_{u(\boldsymbol{x},t)u(\boldsymbol{x}',t)} + \frac{\partial}{\partial a}\left(\boldsymbol{\mathcal{P}}(a,t,\boldsymbol{x})\cdot\int_{-\infty}^{a}\nabla_x p^{(a',b)}_{u(\boldsymbol{x},t)u(\boldsymbol{x}',t)}da'\right) + \frac{\partial}{\partial b}\left(\boldsymbol{\mathcal{P}}(b,t,\boldsymbol{x})\cdot\int_{-\infty}^{b}\nabla_{x'} p^{(a,b')}_{u(\boldsymbol{x},t)u(\boldsymbol{x}',t)}db\right) =\\ -\frac{\partial}{\partial a}\left(\boldsymbol{Q}(a,t,\boldsymbol{x})\,p^{(a,b)}_{u(\boldsymbol{x},t)u(\boldsymbol{x}',t)}\right) - \frac{\partial}{\partial b}\left(\boldsymbol{Q}(b,t,\boldsymbol{x}')\,p^{(a,b)}_{u(\boldsymbol{x},t)u(\boldsymbol{x}',t)}\right)\,.\end{aligned} \tag{60}$$

Next, we consider Eq. (54) and we look for a kinetic equation involving the joint probability density of $u$ and all the random variables $\{\xi_i\}$ and $\{\eta_j\}$

$$p^{(a,\{b_i\},\{c_j\})}_{u(\boldsymbol{x},t)\{\xi_i\}\{\eta_j\}} \stackrel{\text{def}}{=} \langle \delta(a-u(\boldsymbol{x},t))\prod_{k=1}^{m}\delta(b_k-\xi_k)\prod_{j=1}^{n}\delta(c_k-\eta_k)\rangle\,. \tag{61}$$

The average here is with respect to the joint probability density functional of the random initial condition, the random boundary conditions and all the random variables $\{\xi_i\}$ and $\{\eta_j\}$. By following exactly the same steps that led us to Eq. (57) we obtain

$$\frac{\partial}{\partial t}p^{(a,\{b_i\},\{c_j\})}_{u(\boldsymbol{x},t)\{\xi_i\}\{\eta_j\}} + \frac{\partial}{\partial a}\left(\boldsymbol{\mathcal{P}}(a,t,\boldsymbol{x},\boldsymbol{b})\cdot\int_{-\infty}^{a}\nabla p^{(a',\{b_i\},\{c_j\})}_{u(\boldsymbol{x},t)\{\xi_i\}\{\eta_j\}}da'\right) = -\frac{\partial}{\partial a}\left(\boldsymbol{Q}(a,t,\boldsymbol{x},\boldsymbol{c})\,p^{(a,\{b_i\},\{c_j\})}_{u(\boldsymbol{x},t)\{\xi_i\}\{\eta_j\}}\right)\,. \tag{62}$$

Thus, if $\boldsymbol{x}$ is a vector of $D$ variables then Eq. (62) involves $D+2$ variables and $n+m$ parameters, i.e. $\boldsymbol{b}=(b_1,...,b_m)$, $\boldsymbol{c}=(c_1,...,c_n)$. Therefore, the numerical solution to Eq. (62) necessarily involves the use of computational schemes specifically designed for high-dimensional problems such as sparse grid or separated representations [39, 40]. However, let us remark that if $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{Q}$ are easily integrable then we can apply the method of characteristics directly to Eq. (54) (or Eq. (62)) and obtain even an analytical solution to the problem. Unfortunately, this is not always possible and therefore the use of numerical approaches is often unavoidable.

### 4.1. Example 1: Linear advection

Let us consider the simple linear advection problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = \sigma\xi(\omega)\psi(x,t)\,, \quad \sigma \geq 0\,, \quad x \in [0, 2\pi]\,, \quad t \geq t_0 \\ u(x, t_0; \omega) = u_0(x; \omega) \\ \text{Periodic B.C.} \end{cases} \tag{63}$$

where $u_0(x;\omega)$ is a random initial condition of arbitrary dimensionality, $\xi(\omega)$ is a random variable and $\psi$ is a prescribed deterministic function. We look for an equation involving the joint response-excitation probability density function

$$p^{(a,b)}_{u(x,t)\xi} = \langle \delta(a - u(x,t))\delta(b - \xi)\rangle\,. \tag{64}$$

The average here is with respect to the joint probability measure of $u_0(x;\omega)$ and $\xi(\omega)$. Differentiation of (64) with respect to $t$ and $x$ yields, respectively

$$\frac{\partial p^{(a,b)}_{u(x,t)\xi}}{\partial t} = -\frac{\partial}{\partial a}\langle \delta(a-u)u_t\delta(b-\xi)\rangle\,, \tag{65a}$$

$$\frac{\partial p^{(a,b)}_{u(x,t)\xi}}{\partial x} = -\frac{\partial}{\partial a}\langle \delta(a-u)u_x\delta(b-\xi)\rangle\,. \tag{65b}$$

A summation of Eqs. (65a) and (65b) gives the final result

$$\begin{aligned} \frac{\partial p^{(a,b)}_{u(x,t)\xi}}{\partial t} + \frac{\partial p^{(a,b)}_{u(x,t)\xi}}{\partial x} &= -\frac{\partial}{\partial a}\langle \delta(a-u)\left[u_t + u_x\right]\delta(b-\xi)\rangle \\ &= -\frac{\partial}{\partial a}\langle \delta(a-u)\sigma\xi\psi\delta(b-\xi)\rangle \\ &= -\sigma b\psi(x,t)\frac{\partial p^{(a,b)}_{u(x,t)\xi}}{\partial a}\,. \end{aligned} \tag{66}$$

Thus, the problem corresponding to Eq. (63) can be formulated in probability space as

$$\begin{cases} \dfrac{\partial p^{(a,b)}_{u(x,t)\xi}}{\partial t} + \dfrac{\partial p^{(a,b)}_{u(x,t)\xi}}{\partial x} = -\sigma\dfrac{\partial p^{(a,b)}_{u(x,t)\xi}}{\partial a}b\psi(x,y)\,, \quad \sigma \geq 0\,, \quad x \in [0,2\pi]\,, \quad t \geq t_0 \\ p^{(a,b)}_{u(x,t_0)\xi} = p^{(a)}_{u_0(x)}p^{(b)}_{\xi} \\ \text{Periodic B.C.} \end{cases} \tag{67}$$

where we have assumed that the process $u_0(x;\omega)$ is independent of $\xi$ and we have denoted by $p^{(a)}_{u_0(x)}$ and $p^{(b)}_{\xi}$ the probability densities of the initial condition and $\xi(\omega)$, respectively. Equation (67) is derived also in appendix (A.1) by employing a Hopf characteristic functional approach. Once the solution to Eq. (67) is available, we can compute the *response probability* of the system as

$$p^{(a)}_{u(x,t)} = \int_{-\infty}^{\infty} p^{(a,b)}_{u(x,t)\xi}db \tag{68}$$

and then extract all the statistical moments we are interested in, e.g.,

$$\langle u^m(x,t;\omega)\rangle = \int_{-\infty}^{\infty} a^m p^{(a)}_{u(x,t)}da\,. \tag{69}$$

### 4.2. Example 2: Nonlinear advection

A more interesting problem concerns the computation of the statistical properties of the solution to the randomly forced inviscid Burgers equation

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = \sigma\xi(\omega)\psi(x,t)\,, \quad \sigma \geq 0\,, \quad x \in [0, 2\pi]\,, \quad t \geq t_0 \\
u(x, t_0; \omega) = A\sin(x) + \eta(\omega) \quad A \in \mathbb{R} \\
\text{Periodic B.C.}
\end{cases}
\tag{70}
$$

where, as before, $\xi$ and $\eta$ are assumed as independent Gaussian random variables and $\psi$ is a prescribed deterministic function. Note that the amplitude of the initial condition controls the initial speed of the wave. We look for an equation involving the probability density (64). To this end, we differentiate it with respect to $t$ and $x$

$$
\frac{\partial p^{(a,b)}_{u(x,t)\xi}}{\partial t} = -\frac{\partial}{\partial a}\langle \delta(a-u)u_t\delta(b-\xi)\rangle\,, \tag{71a}
$$

$$
\frac{\partial p^{(a,b)}_{u(x,t)\xi}}{\partial x} = -\frac{\partial}{\partial a}\langle \delta(a-u)u_x\delta(b-\xi)\rangle\,, \tag{71b}
$$

$$
a\frac{\partial p^{(a,b)}_{u(x,t)\xi}}{\partial x} = -\frac{\partial}{\partial a}\langle \delta(a-u)uu_x\delta(b-\xi)\rangle + \langle \delta(a-u)u_x\delta(b-\xi)\rangle\,. \tag{71c}
$$

By using Eq. (71b) we obtain

$$
\langle \delta(a-u)u_x\delta(b-\xi)\rangle = -\int_{-\infty}^{a} \frac{\partial p^{(a',b)}_{u(x,t)\xi}}{\partial x} da'\,. \tag{72}
$$

Finally, a summation of Eq. (71a) and Eq. (71c) (with the last term given by Eq. (72)) gives

$$
\begin{cases}
\dfrac{\partial p^{(a,b)}_{u(x,t)\xi}}{\partial t} + a\dfrac{\partial p^{(a,b)}_{u(x,t)\xi}}{\partial x} + \displaystyle\int_{-\infty}^{a} \dfrac{\partial p^{(a',b)}_{u(x,t)\xi}}{\partial x} da' = -\sigma b\psi(x,t)\dfrac{\partial p^{(a,b)}_{u(x,t)\xi}}{\partial a}\,, \quad x \in [0, 2\pi]\,, \quad t \geq t_0 \\
p^{(a,b)}_{u(x,t_0)\xi} = p_\eta(a,x)p_\xi(b) \\
\text{Periodic B.C.}
\end{cases}
\tag{73}
$$

where, as before

$$
p_\xi(b) = \frac{1}{\sqrt{2\pi}}e^{-b^2/2}\,, \qquad p_\eta(a,x) = \frac{1}{\sqrt{2\pi}}e^{-(a-A\sin(x))^2/2}\,. \tag{74}
$$

Equation (73) is derived also in appendix (A.2) by employing a Hopf characteristic functional approach.

### 4.3. Example 3: Nonlinear advection with high-dimensional random forcing

A natural generalization of the problem considered in §4.2 involves a high-dimensional random forcing term

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = \sigma f(t,x;\omega)\,, \quad x \in [0, 2\pi]\,, \quad t \geq 0\,, \quad \sigma \geq 0 \\
u(x, 0; \omega) = A\sin(x) + \eta(\omega) \\
\text{Periodic B.C.}
\end{cases}
\tag{75}
$$

where $f(t,x;\omega)$ is a random field with stipulated statistical properties, e.g. a Gaussian random field with prescribed correlation function. For convenience, let us assume that we have available a Karhunen-Loève representation of $f$, i.e.

$$
f(x,t;\omega) = \sum_{k=1}^{m} \lambda_k\xi_k(\omega)\psi_k(x,t)\,, \tag{76}
$$

where $\{\xi_k(\omega)\}$ is a set of uncorrelated Gaussian random variables. With these assumptions the space-time autocorrelation of the random field (76) can be rather arbitrary. Indeed, the functions $\psi_k$ can be constructed based on a prescribed correlation structure [15]. Now, let us look for an evolution equation involving the joint probability density function of the solution $u$ and all the random variables $\{\xi_1, \cdots, \xi_m\}$

$$p^{(a,\{b_j\})}_{u(x,t)\{\xi_j\}} = \langle \delta(a-u) \prod_{k=1}^{m} \delta(b_k - \xi_k) \rangle \, . \tag{77}$$

By following the same steps that led us to Eq. (73) it can be shown that

$$\frac{\partial}{\partial t} p^{(a,\{b_j\})}_{u(x,t)\{\xi_j\}} + a \frac{\partial}{\partial x} p^{(a,\{b_j\})}_{u(x,t)\{\xi_j\}} + \int_{-\infty}^{a} \frac{\partial}{\partial x} p^{(a',\{b_j\})}_{u(x,t)\{\xi_j\}} da' = -\sigma \sum_{k=1}^{m} \lambda_k b_k \psi_k(x,t) \frac{\partial}{\partial a} p^{(a,\{b_j\})}_{u(x,t)\{\xi_j\}} \, . \tag{78}$$

This is a linear equation that involves 3 variables ($a$, $t$ and $x$) and $m$ parameters ($b_1$, ...,$b_m$).

### 4.4. Example 4: Three-dimensional advection-reaction equation

As a last example, let us consider a multidimensional advection-reaction system governed by the stochastic PDE

$$\frac{\partial u}{\partial t} + \boldsymbol{U}(\boldsymbol{x}, t; \omega) \cdot \nabla u = \mathcal{H}(u) \, , \tag{79}$$

where $\boldsymbol{x} = (x, y, z)$ are spatial coordinates, $\boldsymbol{U}(\boldsymbol{x}, t; \omega)$ is a vectorial random field with known statistics and $\mathcal{H}$ is a nonlinear function of $u$. Equation (79) has been recently investigated by Tartakovsky and Broyda [3] in the context of transport phenomena in heterogeneous porous media with uncertain properties[7]. In the sequel, we shall assume that we have available a representation of the random field $\boldsymbol{U}(\boldsymbol{x}, t; \omega)$, e.g., a Karhunen-Loève series of each component in the form

$$U^{(x)}(\boldsymbol{x}, t; \omega) = \sum_{i=1}^{m_x} \lambda_i^{(x)} \xi_i(\omega) \Psi_i^{(x)}(\boldsymbol{x}, t) \, , \tag{80a}$$

$$U^{(y)}(\boldsymbol{x}, t; \omega) = \sum_{j=1}^{m_y} \lambda_j^{(y)} \eta_j(\omega) \Psi_j^{(y)}(\boldsymbol{x}, t) \, , \tag{80b}$$

$$U^{(z)}(\boldsymbol{x}, t; \omega) = \sum_{k=1}^{m_z} \lambda_k^{(z)} \zeta_k(\omega) \Psi_k^{(z)}(\boldsymbol{x}, t) \, . \tag{80c}$$

Note that each set of random variables $\{\xi_i\}$, $\{\eta_j\}$ and $\{\zeta_k\}$ is uncorrelated, but we can have a correlation between different sets. This gives us the possibility to prescribe a correlation structure between different velocity components at the same space-time location. Given this, let us look for an equation satisfied by the joint probability density function

$$p^{(a,\{b_i\},\{c_j\},\{d_k\})}_{u(\boldsymbol{x},t)\{\xi_i\}\{\eta_j\}\{\zeta_k\}} = \langle \delta(a - u(\boldsymbol{x}, t)) \prod_{i=1}^{m_x} \delta(b_i - \xi_i) \prod_{j=1}^{m_y} \delta(b_j - \eta_j) \prod_{k=1}^{m} \delta(d_k - \zeta_k) \rangle \, , \tag{81}$$

where the average is with respect to the joint probability density functional of the initial conditions, boundary conditions and random variables $\{\{\xi_i\}, \{\eta_j\}, \{\zeta_k\}\}$. By following the same steps that led us to Eq. (57), we obtain

$$\frac{\partial P}{\partial t} + \left( \sum_{i=1}^{m_x} \lambda_i^{(x)} b_i \Psi_i^{(x)} \right) \frac{\partial P}{\partial x} + \left( \sum_{i=1}^{m_y} \lambda_i^{(y)} c_i \Psi_i^{(y)} \right) \frac{\partial P}{\partial y} + \left( \sum_{i=1}^{m_z} \lambda_i^{(z)} d_i \Psi_i^{(z)} \right) \frac{\partial P}{\partial z} = -\frac{\partial}{\partial a} \left( \mathcal{H} P \right) \, , \tag{82}$$

where $P$ is a shorthand notation for Eq. (81). Equation (82) involves 5 variables ($a$, $x$, $y$, $z$, $t$) and $m_x + m_y + m_z$ parameters. Thus, the exact stochastic dynamics of this advection-reaction system develops over a high-dimensional manifold. In order to overcome such a dimensionality issue, Tartakovsky & Broyda [3] have focused in obtaining a *closure approximation* of the response probability associated with the solution to Eq. (79) based on a large eddy

[7] In [3] it is assumed that $\mathcal{H}$ is random as a consequence of an uncertain reaction rate constant $\kappa(\boldsymbol{x}; \omega)$.

diffusivity scheme. Note that a marginalization of Eq. (82) with respect to all the parameters $\{\{b_i\}, \{c_j\}, \{d_k\}\}$ yields an exact evolution for $p^{(a)}_{u(x,t)}$ involving various integrals of (81). All these integrals, however, are in the form of first-order moments of one-dimensional marginals. Under this perspective, a closure model for the probability density of the solution to Eq. (82) aims at quantifying the relation between such probability density and the aforementioned moments involving one-dimensional marginals.

## 5. Some remarks on kinetic equations involving high-dimensional joint response-excitation probability density functions

All the equations presented in this paper for the probability density function are linear evolution equations that can be formally written as [41, 42]

$$\frac{\partial p}{\partial t} = Hp\,. \tag{83}$$

where $H$ is, in general, a linear operator depending on space, time as well as on many parameters, here denoted by $\{b_i\}$. The parametric dependence of $H$ on the set $\{b_i\}$ is usually linear. For instance, the kinetic equation (78) can be written in the general form (83) provided we define

$$H \stackrel{\text{def}}{=} L + \sigma B\,, \tag{84a}$$

$$L \stackrel{\text{def}}{=} -a\frac{\partial}{\partial x}(\cdot) - \int_{-\infty}^{a} \frac{\partial}{\partial x}(\cdot)\, da'\,, \tag{84b}$$

$$B \stackrel{\text{def}}{=} -\sum_{k=1}^{d} b_k \psi_k(x,t) \frac{\partial}{\partial a}(\cdot)\,. \tag{84c}$$

A discretization of Eq. (83) with respect to the variables of the system, e.g. $a$ and $x$ in case of Eq. (78), yields a linear system of ordinary differential equations for the Fourier coefficients[8] $\hat{p}(t; \{b_i\})$

$$\frac{d\hat{p}(t;\{b_i\})}{dt} = \hat{H}(t;\{b_i\})\, \hat{p}(t;\{b_i\})\,. \tag{85}$$

The matrix $\hat{H}(t;\{b_i\})$ is the finite-dimensional version of the linear operator $H$. The *short-time propagator* associated with the system (85) can be expressed analytically in terms of a Magnus series [43, 44]

$$\hat{p}(t;\{b_i\}) = \exp\left[\sum_{k=1}^{\infty} \Omega_k(t, t_0;\{b_i\})\right] \hat{p}(t_0;\{b_i\})\,, \tag{86}$$

where the matrices $\Omega_k(t;\{b_i\})$ are

$$\Omega_1(t,t_0;\{b_i\}) = \int_{t_0}^{t} \hat{H}(t_1;\{b_i\})\, dt_1 \tag{87a}$$

$$\Omega_2(t,t_0;\{b_i\}) = \frac{1}{2}\int_{t_0}^{t}\int_{t_0}^{t_1} [\hat{H}(t_1,;\{b_i\}), \hat{H}(t_2;\{b_i\})] dt_1 dt_2\,, \tag{87b}$$

$$\begin{aligned}\Omega_3(t,t_0;\{b_i\}) = \frac{1}{6}\int_{t_0}^{t}\int_{t_0}^{t_1}\int_{t_0}^{t_2} \Big( & [\hat{H}(t_1;\{b_i\}), [\hat{H}(t_2;\{b_i\}), \hat{H}(t_3;\{b_i\})]] + \\ & + [\hat{H}(t_3;\{b_i\}), [\hat{H}(t_2;\{b_i\}), \hat{H}(t_1;\{b_i\})]] \Big) dt_1 dt_2 dt_3\,, \end{aligned} \tag{87c}$$

$$\ldots$$

[8] In Eq. (85) $\hat{p}(t;\{b_i\})$ denotes a vector of Fourier coefficients.

$[\,,]$ being the matrix commutator (Lie bracket), i.e. $[A, B] \stackrel{\text{def}}{=} AB - BA$. As is well known, the convergence radius of the representation (86), i.e. the break-down time $T_b$ where we have no guarantee that the solution is in the form (86), depends on the norm of the operator $H$ (see §2.7 in [43]). Thus, we cannot expect that the representation (86) in general holds for all $t \geq 0$. However, in some particular cases, e.g. when the matrix $\hat{H}(t; \{b_i\})$ has the structure

$$\hat{H}(t; \{b_i\}) = \sum_{l=1}^{d} z_k(t) Q_k(\{b_i\}), \qquad z_k(t) \text{ are scalar functions, } Q_k \text{ are matrices} \tag{88}$$

then there exist a global representation of the solution in terms of a product of $d$ matrix exponentials [45]. A global representation can also be constructed based on the general the formula (90). To this end, let us partition the integration period $[0, T]$ into $n$ small chunks $[t_{i+1}, t_i]$ ($t_{n+1} = T$, $t_0$) and consider a Magnus expansion within each time interval. This allows us to represent the time evolution of the solution through a product of exponential operators in the form

$$\mathcal{G}(t, t_0; \{b_i\}) = \prod_{j=0}^{n} \exp\left[\sum_{k=1}^{p} \Omega_k\left(t_{j+1}, t_j; \{b_i\}\right)\right], \tag{89}$$

where, e.g., $\Omega_1\left(t_{j+1}, t_j; \{b_i\}\right)$ is defined by Eq. (87a) but with integration limits $t_j$ and $t_{j+1}$. In order to achieve a time integration method order $2p$, only terms up to $\Omega_{2p-2}$ have to be retained in the exponents appearing in (89). We also remark that these exponential operators can be can be decomposed further by using the Suzuki's formulae [46]. The response probability of the system at a particular time then is obtained by averaging the unitary evolution $\mathcal{G}(t, t_0; \{b\})\hat{p}(t_0; \{b_i\})$ with respect to all the parameters $\{b_j\}$, i.e. over *all the possible histories* of the system. In other words, once the transformation (89) has been constructed then we can, in principle, integrate out all the variables $\{b_i\}$ and obtain the following expression for the Fourier coefficients of the *response probability*

$$\hat{p}(t) = \langle \mathcal{G}(t, t_0; \{b_i\}) \hat{p}(t_0; \{b_i\}) \rangle_b, \quad t \in [0, T], \tag{90}$$

where the average $\langle \cdot \rangle_b$ denotes a multidimensional integral with respect to the parameters $\{b_j\}$. It is clear at this point that the computation of the multidimensional integral appearing in Eq. (90) is the key aspect for the calculation of *the response probability* associated with the random wave. Remarkably, this averaging operation is very similar to the computation of the *effective Lagrangian* in the functional integral formalism of classical statistical physics ([28], p. 196). Unfortunately, there is still no explicit analytical formula for the integral of the exponential operator (89) with respect to the parameters $b_k$ (see, e.g., [47, 48, 49, 50]) and therefore it seems that an analytical calculation of (90) cannot be performed explicitly. From a numerical viewpoint, however, recent multidimensional extensions [47] of the Van Loan [48] algorithm could lead to effective techniques for the calculation of the average appearing in Eq. (90). Let us conclude this section by observing that, in general, the numerical solution of equations involving a high-dimensional probability density function is a challenging problem. However, recent advances in numerical methods for high-dimensional partial differential equations [40, 51, 39] allow us to perform the simulation of these high-dimensional systems by exploiting the power of separated respresentation. Indeed, the computational complexity of separated schemes scales linearly with the dimension of the space in which the model is defined, instead of the exponential growing characteristic of mesh-based discretization strategies.

## 6. Numerical results

The main purpose of this section is to provide evidence that the evolution equations for the probability density function obtained in this paper are correct. To this end we will consider two specific prototype problems discussed in section §4.1 and §4.2, i.e., the randomly forced linear and nonlinear advection equations. For the linear advection problem we will provide an analytical solution while for the nonlinear advection equation we will resort to a Fourier-Galerkin spectral method [25, 26] combined with high-order probabilistic collocation [9, 10].

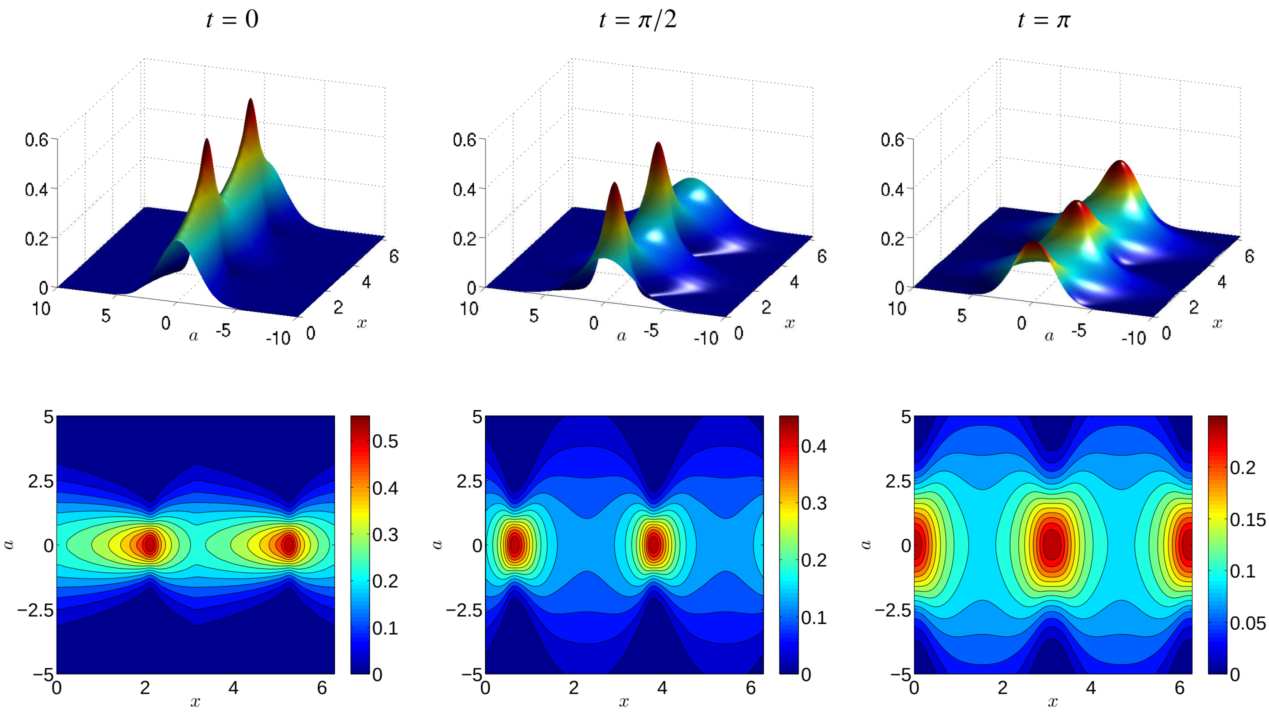


Figure 1: Response probability $p^{(a)}_{u(x,t)}$ of the solution to the linear advection problem (63) as computed form Eq. (67). Shown are snapshots of the probability density function at different times for $\sigma = 3$.

### 6.1. Linear Advection

By using the method of characteristics ([52] p. 97, [38] p. 66) it can be proved that the analytical solution to (63) assuming

$$\psi(x,t) = \sin(x)\sin(2t)\,, \tag{91a}$$

$$u_0(x;\omega) = \sum_{k=1}^{10} \eta_k(\omega)\frac{1}{k}\sin(kx+k) + \sum_{k=1}^{10} \zeta_k(\omega)\frac{1}{k}\cos(kx)\,, \tag{91b}$$

is

$$u(x,t;\omega) = u_0(x-t;\omega) + \sigma\xi(\omega)Q(x,t)\,, \tag{92}$$

where

$$Q(x,t) \overset{\text{def}}{=} \frac{2}{3}\sin(x-t) - \frac{1}{2}\sin(x-2t) - \frac{1}{6}\sin(x+2t)\,. \tag{93}$$

Thus, if $\eta_k(\omega)$, $\zeta_k(\omega)$ and $\xi(\omega)$ are independent Gaussian random variables then we obtain the following statistical moments

$$\langle u(x,t;\omega)^n\rangle = \begin{cases} 0 & n \text{ odd} \\ (n-1)(n-3)\cdots\left[Z(x-t)^2 + \sigma^2 Q(x,t)^2\right]^{n/2} & n \text{ even} \end{cases} \tag{94}$$

where

$$Z(x)^2 \overset{\text{def}}{=} \sum_{k=1}^{10} \frac{1}{k^2}\left[\sin(kx+k)^2 + \cos(kx)^2\right]. \tag{95}$$

Similarly, the analytical solution to the problem (67) for the initial condition

$$p^{(a,b)}_{u(x,t_0)\xi} = \frac{1}{2\pi\,|Z(x)|}\exp\left[-\frac{a^2}{2Z(x)^2} - \frac{b^2}{2}\right] \tag{96}$$

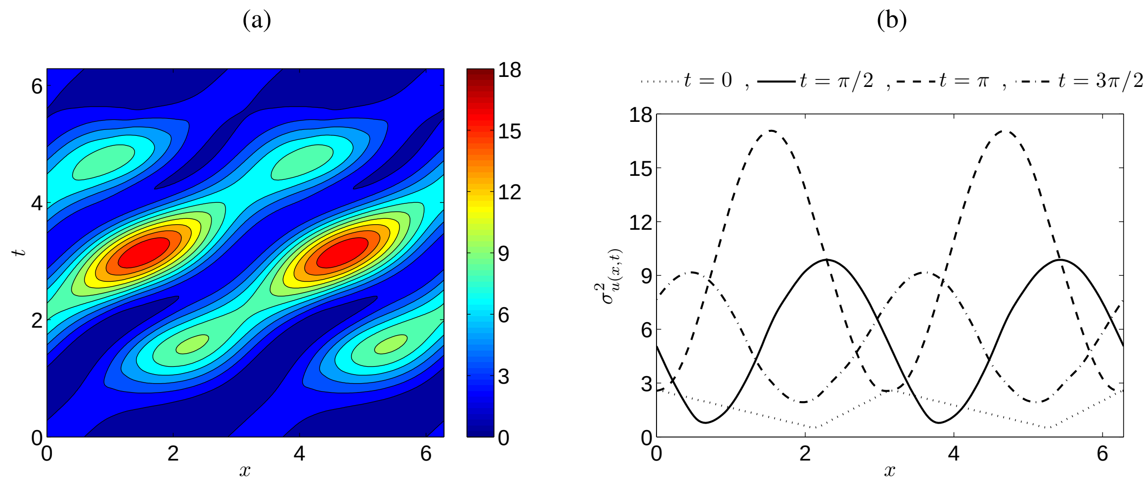


Figure 2: Linear advection equation. (a) Variance of the solution field $u(x, t; \omega)$ in space-time for $\sigma = 3$ and (b) time-slices of the variance in spatial domain.

is easily obtained through the method of characteristics as

$$p^{(a,b)}_{u(x,t)\xi} = \frac{1}{2\pi\,|Z(x-t)|} \exp\left[-\frac{a^2 + b^2\left(Z(x-t)^2 + \sigma^2 Q(x,t)^2\right) - 2ab\sigma Q(x,t)}{2Z(x-t)^2}\right]. \tag{97}$$

An integration of Eq. (97) with respect to $b$ form $-\infty$ to $\infty$, gives the following expression for the response probability density function (i.e. the probability of $u(x, t; \omega)$)

$$p^{(a)}_{u(x,t)} = \frac{1}{\sqrt{2\pi}\,|W(x,t)|} \exp\left[-\frac{a^2}{2W(x,t)^2}\right], \tag{98}$$

where

$$W(x,t)^2 \stackrel{\text{def}}{=} Z(x-t)^2 + \sigma^2 Q(x,t)^2 . \tag{99}$$

Note that the statistical moments (94) are in perfect agreement with the moments of the probability density function (98) and therefore the solutions to Eqs. (67) and (63) are fully consistent. The response probability density (68) is shown in figure 1 at different time instants. Similarly, in figure 2 we plot the variance of the solution field at different times (note that the mean field is identically zero).

### *6.2. Nonlinear advection*

An analytical solution to the problem (70) is not available in an explicit form and therefore we need to resort to numerical approaches. To this end, we employ a Fourier-Galerkin method combined with high-order probabilistic collocation for the simulation of both Eqs. (70) and (73) (see appendix B). For smooth initial conditions, the spectral convergence rate of the Fourier-Galerkin method [26] allows us to simulate the time evolution of the probability density very accurately. In addition, the integral appearing in Eq. (73) can be computed analytically. However, let us remark that the use of a global Fourier series for the representation of the probability density function has also some drawbacks. First of all, the positivity and the normalization condition are not automatically guaranteed. The normalization condition, however can be enforced by using suitable geometric time integrators (see section 5). Secondly, if the initial condition for the probability density is compactly supported, e.g. a uniform distribution, then the convergence rate of the Fourier series significantly deteriorates [25] (although the representation theoretically converges to any distribution in $L^2$). Indeed, the problem of representing the evolution of a possibly discontinuous field, i.e. the joint probability density function, within a multidimensional hypercube with periodic (zero) boundary conditions is challenging.

Given these preliminary remarks, let us now recall that the nonlinear advection equation easily develops shock singularities as the time goes on ([38], p. 276). Therefore, a careful selection of the simulation parameters and the type of random forcing is necessary in order to avoid these situations. In particular, we shall consider the following choices in Eqs. (70) and (73)

$$\psi(x,t) = \sin(x)\sin(20t)\,, \quad \sigma = \frac{1}{2}\,, \quad A = \frac{1}{2}\,. \tag{100}$$

The Fourier resolution in space is set to $N = 100$ modes for both fields $u$ and $p^{(a,b)}_{u(x,t)\xi}$ while the resolution in the $a$ direction for the probability density is set to $Q = 250$ modes (see appendix B for further details). The Gaussian random variables $\xi$ and $\eta$ appearing in Eq. (70) are first sampled on a $50 \times 50$ Gauss-Hermite collocation grid (PCM) and then on a multi-element collocation grid of 10 elements each element being of order 10 (ME-PCM) [9] for higher accuracy. Similarly, the dependence on the parameter $b$ in Eq. (73) is handled through a multi-element Gauss-Lobatto-Legendre quadrature. Specifically, we have employed 10 equally-spaced finite elements, each element being of order 10. Once the solution to Eq. (73) is available, we can compute the *response probability* of the system and the relevant statistical moments exactly as before (see Eqs. (68) and (69)). The response probability density is shown in figure 3 at different time instants. The corresponding mean and variance are shown in figure 4. We remark that these statistical

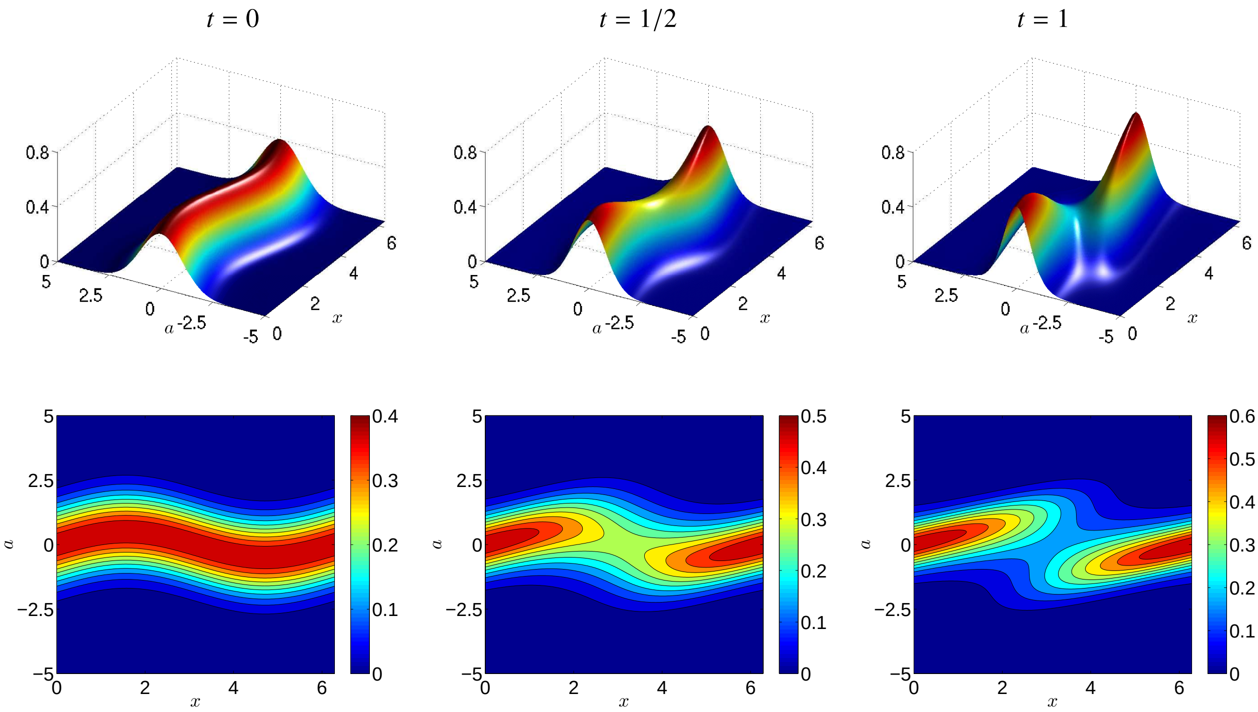


Figure 3: Response probability $p^{(a)}_{u(x,t)}$ of the solution to the nonlinear advection problem (70) as computed from Eq. (73). Shown are snapshots of the probability density function at different times.

moments are in very good agreement with the ones obtained from high-order collocation approaches applied to Eq. (70). In order to show this, in figure 5 we report the time-dependent relative errors between the mean and the variance of the field $u$ as computed from Eqs. (70) and (73). These relative errors are defined as

$$e_2[\langle u\rangle](t) \stackrel{\text{def}}{=} \frac{\|\langle\tilde{u}\rangle - \langle u\rangle\|_{L_2([0,2\pi])}}{\|\langle u\rangle\|_{L_2([0,2\pi])}}\,, \tag{101a}$$

$$e_2[\sigma_u^2](t) \stackrel{\text{def}}{=} \frac{\|\tilde{\sigma}_u^2 - \sigma_u^2\|_{L_2([0,2\pi])}}{\|\sigma_u^2\|_{L_2([0,2\pi])}}\,, \tag{101b}$$

where the quantities with a tilde are obtained from probabilistic collocation of Eq. (70) (either Gauss-Hermite or

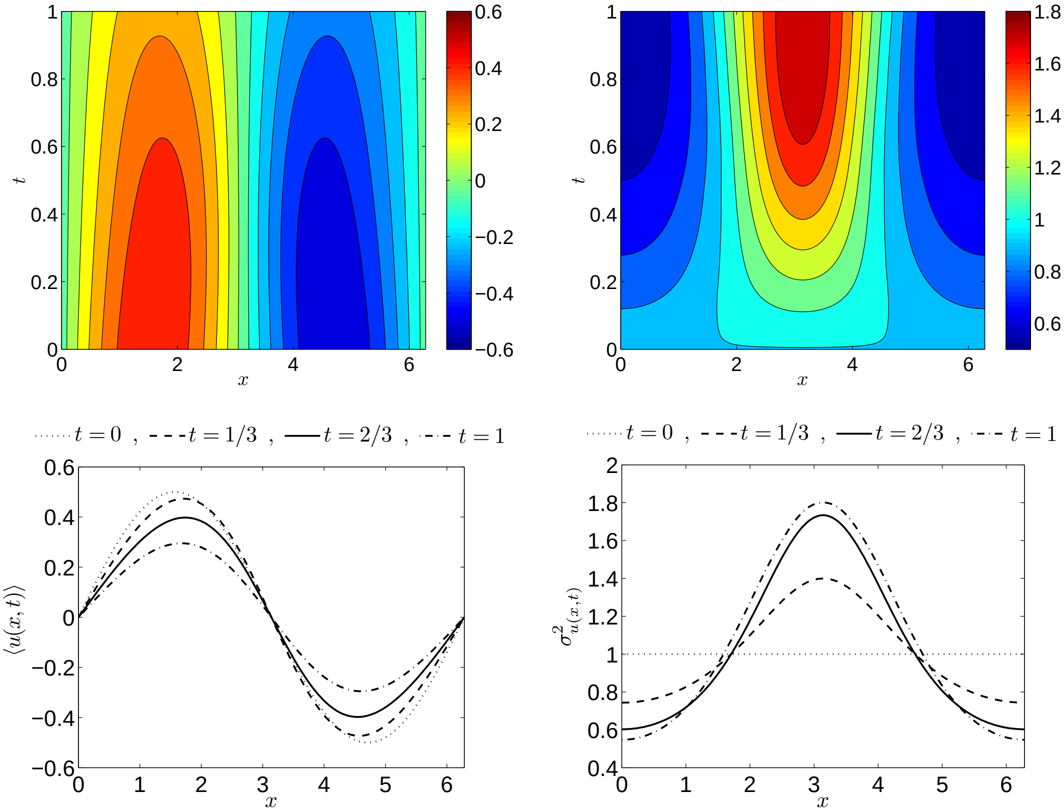


Figure 4: Nonlinear advection equation. Mean (Left column) and variance (Right column) the solution field $u(x, t; \omega)$ in space and time. The second row reports on specific time slices of the contour plots shown in the first row.

ME-PCM), while $\|\cdot\|_{L_2([0,2\pi])}$ denotes the standard $L_2$ norm in $[0, 2\pi]$, i.e.

$$\|g(x)\|_{L_2([0,2\pi])} \stackrel{\text{def}}{=} \left( \int_0^{2\pi} g(x)^2 dx \right)^{1/2} . \tag{102}$$

The error growth in time observed in figure 5(a) is not due to a random frequency problem [20, 7], but rather to the fact that the response probability of the system tends to split into two distinct parts after time $t = 1/2$ (see figure 3). This yields to accuracy problems when a global Gauss-Hermite collocation scheme is used. A similar issue has been discussed in [8], in the context of stochastic Rayleigh-Bénard convection subject to random initial states. In figure 5(b) we compare the numerical results obtained from the simulation simulation of Eq. (73) against multi-element probabilistic collocation (ME-PCM) of Eq. (70). It is seen that the error growth is stabilized in time. This suggests that the probability density function approach has accuracy which is comparable with ME-PCM. Indeed, the maximum pointwise errors between the mean and the variance fields as computed by the two methods are

$$\max_{\substack{t\in[0,1]\\x\in[0,2\pi]}} |\langle \tilde{u} \rangle - \langle u \rangle| = 3.0 \times 10^{-7} , \qquad \max_{\substack{t\in[0,1]\\x\in[0,2\pi]}} \left| \tilde{\sigma}_u^2 - \sigma_u^2 \right| = 1.7 \times 10^{-6} . \tag{103}$$

## 7. Summary and discussion

We have obtained and discussed new evolution equations for the joint response-excitation probability density function of the stochastic solution to first-order nonlinear scalar PDEs subject to uncertain initial conditions, boundary conditions or random forcing terms. The theoretical predictions are confirmed well by numerical simulations

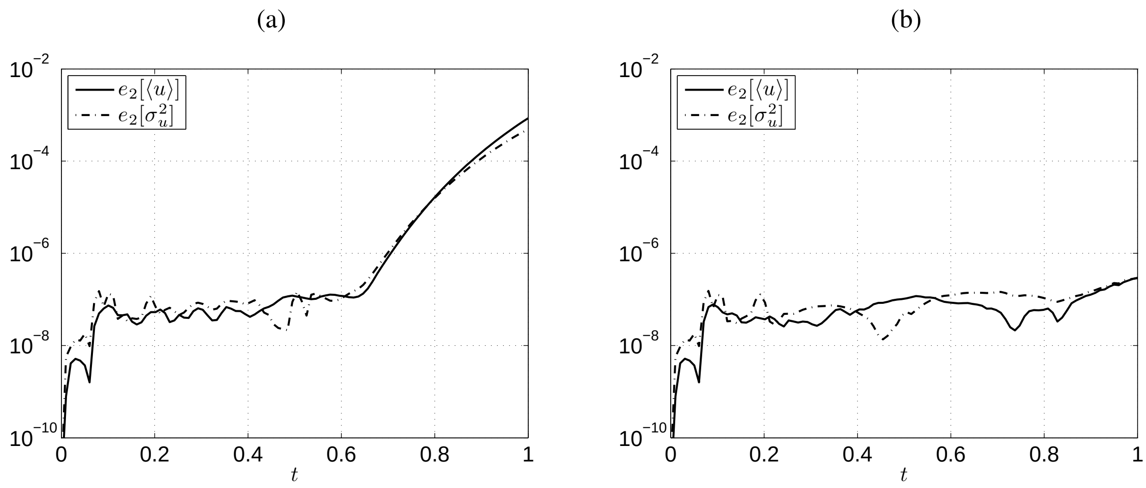


Figure 5: Nonlinear Advection. Relative errors (101a) and (101b) between the mean and the variance of the field $u$ as computed from probabilistic collocation of Eq. (70) and integration of the probability density function satisfying Eq. (73). We show two different results: (a) Gauss-Hermite collocation of Eq. (70) on a $50\times50$ grid; (b) ME-PCM of Eq. (70) with 10 finite elements of order 10 for both $\xi(\omega)$ and $\eta(\omega)$ ($100\times100$ collocation points).

based on an accurate Fourier-Galerkin spectral method. A reformulation of a stochastic problem in terms of the probability density function has an advantage with respect to more classical stochastic approaches in that it does not suffer from the curse of dimensionality if randomness comes from boundary or initial conditions. In fact, we can prescribe these conditions in terms of probability distributions and this is obviously not dependent on the number of random variables characterizing the underlying probability space. However, if an external random forcing appears within the scalar PDE modeling the physical system, then the dimensionality of the corresponding problem in probability space could increase significantly (see §3, §4 and §5) . This happens because the exact stochastic dynamics in this case develops over a high-dimensional manifold and therefore the exact probabilistic description necessarily involves a multidimensional joint response-excitation probability density function, or even a probability density functional. In order to overcome this issue a *closure approximation* [3, 53, 54, 55] of the evolution equation involving the probability density function can be constructed. This usually yields a system which is amenable to numerical simulation. However, the computation of the numerical solution to an equation involving a probability density function is itself a challenging problem [56, 57]. In fact, in addition to the question of dimensionality, which may be handled by using closures, we have different constraints that have to be satisfied, e.g., the positivity and the normalization condition. Moreover, the probability density function could be compactly supported over *disjoint domains* and this obviously requires the use of suitable numerical techniques such as the discontinuous Galerkin method. Finally, if the boundary conditions or the initial condition associated with the problem in physical space are set to be deterministic then the corresponding conditions in probability space will be defined in terms of Dirac delta functions (see §3.1).

A fundamental question is whether the statistical approaches developed in this paper for first-order nonlinear stochastic PDEs can be extended to more general equations involving higher order derivatives in space and time, such as the second-order wave equation, the diffusion equation or the Klein-Gordon equation [58, 59, 60, 61]. Unfortunately, the self-interacting nature of these higher-order problems is often associated with *nonlocal solutions* which, in turn, make it impossible to obtain a closed evolution equation governing the probability density function a specific space-time location. Even in these nonlocal cases, however, it is possible to formulate a set of differential constraints satisfied locally by the probability density function of the stochastic solution [62]. These differential constraints involve, in general, unusual partial differential operators, i.e. limit partial derivatives, and it is still not clear if the are sufficient to determine uniquely the probability density function associated with the solution to the underlying stochastic PDE. An alternative and very general approach relies on the use of functional integral techniques [63, 29, 28, 30, 64], in particular those ones involving the Hopf characteristic functional (see also appendix A). These methods aim to cope with the global probabilistic structure of the solution to a stochastic PDE and they have been extensively studied in the past as a possible tool to tackle many fundamental problems in physics such as turbulence [22]. Their usage grew very rapidly around the 70s, when it became clear that the diagrammatic functional techniques [63] could be applied, at least formally, to many different problems in classical statistical physics. However, functional differential equations involving the Hopf characteristic functional are unfortunately not amenable to numerical simulation.


### Acknowledgements

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## A. Hopf characteristic functional approach

In this appendix we derive some of the equations for the joint response-excitation probability density function we have considered in the paper by employing a Hopf characteristic functional approach [23, 22, 65, 66]. This allow us to show how these equations can be obtained from general principles.

### *A.1. Linear advection*

Let us consider the boundary value problem (63) and introduce the joint characteristic functional of the solution field $u(x,t;\omega)$ and the random variable $\xi(\omega)$

$$F[\alpha(X,\tau),b] \stackrel{\text{def}}{=} \langle e^{i\int_X\int_T u(X,\tau;\omega)\alpha(X,\tau)dXd\tau+ib\xi(\omega)}\rangle\,, \tag{104}$$

where $\alpha(X,\tau)$ is a test function and the average $\langle\cdot\rangle$ is with respect to the joint probability measure of $\xi$ and $\eta$, namely the amplitude of the forcing and the amplitude of the spatially uniform initial condition. The Volterra functional derivative of (104) with respect to $u(x,t)$, i.e. the Gâteaux differential [67] of the functional $F[\alpha,b]$ with respect to $\alpha$ evaluated at $z(X,\tau)=\delta(t-\tau)\delta(x-X)$, is

$$\frac{\delta F[\alpha,b]}{\delta u(x,t)} = i\langle u(x,t;\omega)e^{i\int_X\int_T u(X,\tau;\omega)\alpha(X,\tau)dXd\tau+ib\xi(\omega)}\rangle\,. \tag{105}$$

A differentiation of Eq. (105) with respect to $x$ and $t$ yields the identity

$$\frac{\partial}{\partial t}\left(\frac{\delta F[\alpha,b]}{\delta u(x,t)}\right)+\frac{\partial}{\partial x}\left(\frac{\delta F[\alpha,b]}{\delta u(x,t)}\right) = i\sigma\psi(x,t)\langle\xi(\omega)e^{i\int_X\int_T u(X,\tau;\omega)\alpha(X,\tau)dXd\tau+ib\xi(\omega)}\rangle\,, \tag{106}$$

where we have used Eq. (63). Equation (106) is a *functional differential equation* that holds for arbitrary test functions $\alpha(X,\tau)$. In particular, it holds for

$$\alpha(X,\tau)^+ \stackrel{\text{def}}{=} a\delta(t-\tau)\delta(x-X)\,, \qquad a\in\mathbb{R}\,. \tag{107}$$

Thus, if we evaluate Eq. (106) for $\alpha=\alpha^+$ we obtain

$$i\langle\left[u_t(x,t;\omega)+u_x(x,t;\omega)-\sigma\psi(x,t)\xi(\omega)\right]e^{iau(x,t;\omega)+ib\xi(\omega)}\rangle = 0\,. \tag{108}$$

This condition is equivalent to a partial differential equation involving the *joint characteristic function* of the random variables $u(x,t;\omega)$ and $\xi(\omega)$

$$\phi^{(a,b)}_{u(x,t)\xi} \stackrel{\text{def}}{=} \langle e^{iau(x,t;\omega)+ib\xi(\omega)}\rangle\,. \tag{109}$$

In order to see this, let us notice that

$$\frac{\partial\phi^{(a,b)}_{u(x,t)\xi}}{\partial t} = ia\langle u_t(x,t;\omega)e^{iau(x,t;\omega)+ib\xi(\omega)}\rangle\,, \tag{110a}$$

$$\frac{\partial\phi^{(a,b)}_{u(x,t)\xi}}{\partial x} = ia\langle u_x(x,t;\omega)e^{iau(x,t;\omega)+ib\xi(\omega)}\rangle\,, \tag{110b}$$

$$\frac{\partial\phi^{(a,b)}_{u(x,t)\xi}}{\partial b} = i\langle\xi(\omega)e^{iau(x,t;\omega)+ib\xi(\omega)}\rangle\,. \tag{110c}$$

A substitution of Eqs. (110a)-(110c) into Eq. (108) immediately gives

$$\frac{\partial \phi^{(a,b)}_{u(x,t)\xi}}{\partial t} + \frac{\partial \phi^{(a,b)}_{u(x,t)\xi}}{\partial x} = \sigma\psi(x,t) a \frac{\partial \phi^{(a,b)}_{u(x,t)\xi}}{\partial b} \,, \tag{111}$$

which is the result we were looking for. An inverse Fourier transformation of Eq. (111) with respect to $a$ and $b$ gives exactly Eq. (67). In order to see this, let us simply recall the definition of $p^{(a,b)}_{u(x,t)\xi}$ as the inverse Fourier transform of the characteristic function $\phi^{(a,b)}_{u(x,t)\xi}$

$$p^{(a,b)}_{u(x,t)\xi} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ial-ibq} \phi^{(l,q)}_{u(x,t)\xi} dl dq \,,$$

and two simple relations arising from Fourier transformation theory of a one-dimensional function $g(x)$

$$\int_{-\infty}^{\infty} e^{-iax} \frac{d^n g(x)}{dx^n} dx = (ia)^n \int_{-\infty}^{\infty} e^{-iax} g(x) dx \,,$$
$$\int_{-\infty}^{\infty} e^{-iax} x^n g(x) dx = i^n \frac{d^n}{da^n} \int_{-\infty}^{\infty} e^{-iax} g(x) dx \,.$$

*A.2. Nonlinear advection*

Let us consider the nonlinear advection problem (70) where, for simplicity, we neglect the additive random forcing term at the right hand side. This simplification does not alter in any way the main aspects of the proof presented hereafter[9]. The Hopf characteristic functional of the solution field is

$$F[\alpha] = \langle e^{i \int_X \int_T u(X,\tau;\omega)\alpha(X,\tau) dX d\tau} \rangle \,. \tag{112}$$

As before, the Volterra functional derivative of (112) with respect to $u(x,t)$ is

$$\frac{\delta F[\alpha]}{\delta u(x,t)} = i \langle u(x,t;\omega) e^{i \int_X \int_T u(X,\tau;\omega)\alpha(X,\tau) dX d\tau} \rangle \,. \tag{113}$$

A differentiation of Eq. (113) with respect to $x$ and $t$ gives, respectively

$$\frac{\partial}{\partial t}\left(\frac{\delta F[\alpha]}{\delta u(x,t)}\right) = i \langle u_t(x,t;\omega) e^{i \int_X \int_T u(X,\tau;\omega)\alpha(X,\tau) dX d\tau} \rangle \,, \tag{114a}$$

$$\frac{\partial}{\partial x}\left(\frac{\delta F[\alpha]}{\delta u(x,t)}\right) = i \langle u_x(x,t;\omega) e^{i \int_X \int_T u(X,\tau;\omega)\alpha(X,\tau) dX d\tau} \rangle \,. \tag{114b}$$

Now we perform an additional functional differentiation of Eq. (114b) with respect to $u(x',t')$

$$\frac{\delta}{\partial u(x',t')}\left(\frac{\partial}{\partial x}\frac{\delta F[\alpha]}{\delta u(x,t)}\right) = -\langle u_x(x,t;\omega) u(x',t';\omega) e^{i \int_X \int_T u(X,\tau;\omega)\alpha(X,\tau) dX d\tau} \rangle \tag{115}$$

and then we take the limits for $x' \to x$ and $t' \to t$ to obtain

$$\frac{\delta}{\partial u(x,t)}\left(\frac{\partial}{\partial x}\frac{\delta F[\alpha]}{\delta u(x,t)}\right) = -\langle u(x,t;\omega) u_x(x,t;\omega) e^{i \int_X \int_T u(X,\tau;\omega)\alpha(X,\tau) dX d\tau} \rangle \,. \tag{116}$$

A summation Eq. (114a) and Eq. (116) gives (taking Eq. (70) into account)

$$\frac{\partial}{\partial t}\frac{\delta F[\alpha]}{\delta u(x,t)} - i \frac{\delta}{\partial u(x,t)}\left(\frac{\partial}{\partial x}\frac{\delta F[\alpha]}{\delta u(x,t)}\right) = 0 \,. \tag{117}$$

[9]Indeed, the random forcing function can be included in the Hopf characteristic functional exactly as we have done in Eq. (104).

This functional differential equation holds for arbitrary test functions $\alpha(X,\tau)$. In particular it holds for

$$\alpha^+(X,\tau) = a\delta(t-\tau)\delta(x-X)\,.$$

Evaluating Eq. (117) for $\alpha = \alpha^+$ yields

$$\langle (u_t + uu_x)\, e^{iu(x,t;\omega)a}\rangle = 0\,. \tag{118}$$

Let us define the *characteristic function* of the random variable $u(x,t;\omega)$.

$$\phi^{(a)}_{u(x,t)} \stackrel{\text{def}}{=} \langle e^{iu(x,t;\omega)a}\rangle \tag{119}$$

From the definition (119) it easily follows that

$$\frac{\partial \phi^{(a)}_{u(x,t)}}{\partial t} = ai\langle u_t e^{iu(x,t;\omega)a}\rangle \tag{120a}$$

$$\frac{\partial \phi^{(a)}_{u(x,t)}}{\partial a} = i\langle u e^{iu(x,t;\omega)a}\rangle \tag{120b}$$

$$\frac{\partial^2 \phi^{(a)}_{u(x,t)a}}{\partial a \partial x} = -a\langle uu_x e^{iu(x,t;\omega)a}\rangle + \frac{1}{a}\frac{\partial \phi^{(a)}_{u(x,t)}}{\partial x}\,. \tag{120c}$$

Substituting Eq. (120c) and Eq. (120a) into Eq. (118) yields the following equation for the characteristic function $\phi^{(a)}_{u(x,t)}$

$$a\frac{\partial \phi^{(a)}_{u(x,t)}}{\partial t} - ia\frac{\partial^2 \phi^{(a)}_{u(x,t)}}{\partial a\partial x} + i\frac{\partial \phi^{(a)}_{u(x,t)}}{\partial x} = 0\,. \tag{121}$$

The inverse Fourier transformation of Eq. (121) with respect to $a$ gives

$$\frac{\partial^2 p^{(a)}_{u(x,t)}}{\partial a\partial t} + a\frac{\partial^2 p^{(a)}_{u(x,t)}}{\partial a\partial x} + \frac{\partial p^{(a)}_{u(x,t)}}{\partial x} = 0\,. \tag{122}$$

Taking into account the fact that $p^{(a)}_{u(x,t)}$ vanishes at infinity, together with all its derivatives, we easily see that Eq. (122) is equivalent to

$$\frac{\partial p^{(a)}_{u(x,t)}}{\partial t} + a\frac{\partial p^{(a)}_{u(x,t)}}{\partial x} + \int_{-\infty}^{a}\frac{\partial p^{(a')}_{u(x,t)}}{\partial x}da' = 0\,. \tag{123}$$

This coincides with Eq. (73) with $\sigma = 0$.

### *A.3. Nonlinear advection with an additional quadratic nonlinearity*

In this section we discuss the application of the Hopf characteristic functional approach for the derivation of an equation involving the probability density function of the solution to the problem (45). To this end, let us consider the following joint Hopf characteristic functional

$$F[\alpha(X,\tau),\beta(X,\tau)] = \langle e^{i\int_X\int_T u(X,\tau;\omega)\alpha(X,\tau)dXd\tau + i\int_X\int_T u_x(X,\tau;\omega)\beta(X,\tau)dXd\tau}\rangle\,, \tag{124}$$

where $\alpha(X,\tau)$ and $\beta(X,\tau)$ are two test fields. Functional differentiation with respect to $u$ and $u_x$ yields

$$\frac{\delta F[\alpha,\beta]}{\delta u(x,t)} = \langle iu(x,t;\omega)e^{i\int_X\int_T u(X,\tau;\omega)\alpha(X,\tau)dXd\tau + i\int_X\int_T u_x(X,\tau;\omega)\beta(X,\tau)dXd\tau}\rangle\,,$$

$$\frac{\delta F[\alpha,\beta]}{\delta u_x(x,t)} = \langle iu_x(x,t;\omega)e^{i\int_X\int_T u(X,\tau;\omega)\alpha(X,\tau)dXd\tau + i\int_X\int_T u_x(X,\tau;\omega)\beta(X,\tau)dXd\tau}\rangle\,,$$

$$\frac{\delta^2 F[\alpha,\beta]}{\delta u(x,t)\delta u_x(x,t)} = -\langle u(x,t;\omega)u_x(x,t;\omega)e^{i\int_X\int_T u(X,\tau;\omega)\alpha(X,\tau)dXd\tau + i\int_X\int_T u_x(X,\tau;\omega)\beta(X,\tau)dXd\tau}\rangle\,.$$

By combining different functional derivatives of $F[\alpha,\beta]$ with respect to $u$ and $u_x$ it is straightforward to obtain the following functional differential equation corresponding to Eq. (45)

$$\frac{\partial}{\partial t}\frac{\delta F[\alpha,\beta]}{\delta u(x,t)} - i\frac{\delta^2 F[\alpha,\beta]}{\delta u(x,t)\delta u_x(x,t)} - i\nu\frac{\delta^2 F[\alpha,\beta]}{\delta u_x(x,t)^2} = 0\,. \tag{125}$$

This equation holds for arbitrary test functions $\alpha$ and $\beta$. In particular it holds for

$$\alpha(X,\tau)^+ = a\delta(t-\tau)\delta(x-X)\,,$$
$$\beta(X,\tau)^+ = b\delta(t-\tau)\delta(x-X)\,.$$

An evaluation of Eq. (125) for $\alpha = \alpha^+$ and $\beta = \beta^+$ gives us the condition

$$\left\langle\left[u_t(x,t;\omega) + u(x,t;\omega)u_x(x,t;\omega) + \nu u_x(x,t;\omega)^2\right] e^{iau(x,t;\omega)+ibu_x(x,t;\omega)}\right\rangle = 0\,. \tag{126}$$

Next we show that the integral equation (126) is equivalent to a partial differential equation for the joint characteristic function of the random variables $u(x,t;\omega)$ and $u_x(x,t;\omega)$, i.e. the characteristic function of the solution field and its first-order spatial derivative at the same space-time location

$$\phi^{(a,b)}_{uu_x} \stackrel{\text{def}}{=} \langle e^{iau(x,t;\omega)+ibu_x(x,t;\omega)}\rangle\,. \tag{127}$$

To this end, let us first notice that

$$\begin{aligned}\frac{\partial\phi^{(a,b)}_{uu_x}}{\partial t} &= \langle [iau_t + ibu_{xt}]\, e^{iau(x,t;\omega)+ibu_x(x,t;\omega)}\rangle\\ &\stackrel{(45)}{=} \left\langle\left[iau_t - ib\left(u_x^2 + uu_{xx} + 2\nu u_x u_{xx}\right)\right] e^{iau(x,t;\omega)+ibu_x(x,t;\omega)}\right\rangle\,.\end{aligned} \tag{128}$$

At this point we need an expression for the average appearing in Eq. (128) in terms of the characteristic function. This expression is obtained by observing that

$$\frac{\partial\phi^{(a,b)}_{uu_x}}{\partial x} = \langle [iau_x + ibu_{xx}]\, e^{iau(x,t;\omega)+ibu_x(x,t;\omega)}\rangle\,, \tag{129a}$$

$$\begin{aligned}\frac{\partial^2\phi^{(a,b)}_{uu_x}}{\partial a\partial x} &= \langle iu_x e^{iau(x,t;\omega)+ibu_x(x,t;\omega)}\rangle - \langle [auu_x + buu_{xx}]\, e^{iau(x,t;\omega)+ibu_x(x,t;\omega)}\rangle\\ &= \frac{\partial\phi^{(a,b)}_{uu_x}}{\partial b} + a\frac{\partial^2\phi^{(a,b)}_{uu_x}}{\partial a\partial b} - b\langle uu_{xx}e^{iau(x,t;\omega)+ibu_x(x,t;\omega)}\rangle\,,\end{aligned} \tag{129b}$$

$$\begin{aligned}\frac{\partial^2\phi^{(a,b)}_{uu_x}}{\partial b\partial x} &= \langle iu_{xx} e^{iau(x,t;\omega)+ibu_x(x,t;\omega)}\rangle - \left\langle\left[au_x^2 + bu_xu_{xx}\right] e^{iau(x,t;\omega)+ibu_x(x,t;\omega)}\right\rangle\\ &= \frac{1}{b}\frac{\partial\phi^{(a,b)}_{uu_x}}{\partial x} - \frac{a}{b}\frac{\partial\phi^{(a,b)}_{uu_x}}{\partial b} + a\frac{\partial^2\phi^{(a,b)}_{uu_x}}{\partial b^2} - b\langle u_xu_{xx}e^{iau(x,t;\omega)+ibu_x(x,t;\omega)}\rangle\,.\end{aligned} \tag{129c}$$

Therefore, by using Eqs. (128), (129b) and (129c) we obtain the following explicit representation for the time derivative appearing in Eq. (126)

$$\begin{aligned}ia\langle u_t e^{iau(x,t;\omega)+ibu_x(x,t;\omega)}\rangle =& \frac{\partial\phi^{(a,b)}_{uu_x}}{\partial t} - ib\frac{\partial^2\phi^{(a,b)}_{uu_x}}{\partial b^2} + i\frac{\partial\phi^{(a,b)}_{uu_x}}{\partial b} + ia\frac{\partial^2\phi^{(a,b)}_{uu_x}}{\partial a\partial b} - i\frac{\partial^2\phi^{(a,b)}_{uu_x}}{\partial a\partial x}\\ &+ 2i\nu\left(\frac{1}{b}\frac{\phi^{(a,b)}_{uu_x}}{\partial x} - \frac{a}{b}\frac{\phi^{(a,b)}_{uu_x}}{\partial b} + a\frac{\partial^2\phi^{(a,b)}_{uu_x}}{\partial b^2} - \frac{\partial^2\phi^{(a,b)}_{uu_x}}{\partial b\partial x}\right)\,.\end{aligned} \tag{130}$$

The other terms in Eq. (126) are

$$\langle uu_x e^{iau(x,t;\omega)+ibu_x(x,t;\omega)}\rangle = -\frac{\partial^2\phi^{(a,b)}_{uu_x}}{\partial a\partial b}\,, \tag{131a}$$

$$\langle u_x^2 e^{iau(x,t;\omega)+ibu_x(x,t;\omega)}\rangle = -\frac{\partial^2\phi^{(a,b)}_{uu_x}}{\partial b^2}\,. \tag{131b}$$

Finally, a substitution of Eqs. (130)-(131b) into Eq. (126) gives

$$b\frac{\partial \phi_{uu_x}^{(a,b)}}{\partial t} = ib^2\frac{\partial^2 \phi_{uu_x}^{(a,b)}}{\partial b^2} - ib\frac{\partial \phi_{uu_x}^{(a,b)}}{\partial b} + ib\frac{\partial^2 \phi_{uu_x}^{(a,b)}}{\partial a \partial x} - i\nu ab\frac{\partial^2 \phi_{uu_x}^{(a,b)}}{\partial b^2} - 2i\nu\left(\frac{\partial \phi_{uu_x}^{(a,b)}}{\partial x} - a\frac{\partial \phi_{uu_x}^{(a,b)}}{\partial b} - b\frac{\partial^2 \phi_{uu_x}^{(a,b)}}{\partial b \partial x}\right). \tag{132}$$

An inverse Fourier transform of Eq. (132) with respect to $a$ and $b$ yields

$$\frac{\partial^2 p_{uu_x}^{(a,b)}}{\partial b \partial t} = \frac{\partial^2}{\partial b^2}\left(b^2 p_{uu_x}^{(a,b)}\right) + \frac{\partial}{\partial b}\left(b p_{uu_x}^{(a,b)}\right) - \frac{\partial}{\partial b}\left(b\frac{\partial p_{uu_x}^{(a,b)}}{\partial x}\right) - \nu\frac{\partial}{\partial b}\left(b^2\frac{\partial^2 p_{uu_x}^{(a,b)}}{\partial a}\right) - 2\nu\left[\frac{\partial p_{uu_x}^{(a,b)}}{\partial x} + b\frac{\partial p_{uu_x}^{(a,b)}}{\partial a} + \frac{\partial}{\partial b}\left(b\frac{\partial p_{uu_x}^{(a,b)}}{\partial x}\right)\right], \tag{133}$$

which, upon integration with respect to $b$ from $-\infty$ to $b$, gives exactly Eq. (47).

## B. Fourier-Galerkin systems for the nonlinear advection equation in physical and probability spaces

In this appendix we obtain the Fourier-Galerkin system corresponding to the nonlinear advection problem considered in section 4.2 This allows the interested reader to perform numerical simulations and easily reproduce our numerical results.

*Fourier-Galerkin system in physical domain.* Let us consider a Fourier series representation of the solution to the problem (70) (periodic in $x \in [0, 2\pi]$)

$$u(x,t;\omega) = \sum_{n=-N}^{N} \hat{u}_n(t;\omega)e^{inx} \tag{134}$$

A substitution into Eq. (70) and subsequent projection onto the space

$$\mathfrak{B}_{2N+1} = \text{span}\left\{e^{-ipx}\right\}_{p=-N,..,N} \tag{135}$$

yields

$$2\pi\frac{d\hat{u}_p}{dt} + \sum_{n=-N}^{N}\sum_{m=-N}^{N} im\hat{u}_n\hat{u}_m \int_0^{2\pi} e^{i(n+m-p)x}dx = \sigma\xi(\omega)\int_0^{2\pi}\psi(x,t)e^{-ipx}dx\,. \tag{136}$$

At this point, let us set $\psi(x,t) = \sin(kx)\sin(jt)$. The integral in Eq. (136) then is easily obtained as

$$\int_0^{2\pi}\psi(x,t)e^{-ipx}dx = -i\sin(jt)\int_0^{2\pi}\sin(kx)\sin(px)dx = -i\pi\sin(jt)\delta_{pk} + i\pi\sin(jt)\delta_{(-p)k}\,. \tag{137}$$

Therefore, we obtain the following Galerkin system

$$\begin{cases} \dfrac{d\hat{u}_p}{dt} + i\displaystyle\sum_{m=-N}^{N} m\hat{u}_{p-m}\hat{u}_m = i\frac{\sigma}{2}\xi(\omega)\sin(jt)\,, & p=-k \\ \dfrac{d\hat{u}_p}{dt} + i\displaystyle\sum_{m=-N}^{N} m\hat{u}_{p-m}\hat{u}_m = -i\frac{\sigma}{2}\xi(\omega)\sin(jt)\,, & p=k \\ \dfrac{d\hat{u}_p}{dt} + i\displaystyle\sum_{m=-N}^{N} m\hat{u}_{p-m}\hat{u}_m = 0\,, & \text{otherwise} \end{cases} \tag{138}$$

The initial condition for $\hat{u}_p(t;\omega)$ is obtained by projection as

$$\hat{u}_p(t_0,\omega) = \delta_{p0}\eta(\omega) + \frac{iA}{2}\left(\delta_{-1p} - \delta_{1p}\right). \tag{139}$$

The solution strategy is as follows:

1. we sample $\xi(\omega)$ and $\eta(\omega)$ at suitable quadrature points, e.g., Gauss-Hermite or ME-PCM points [9];
2. for each realization of $\xi(\omega)$ and $\eta(\omega)$ solve the system (138) with the initial condition (139).

When the solutions corresponding to all these realizations are available, we compute the mean and the second order moment of the solution as

$$\langle u(x,t;\omega)\rangle = \sum_{n=-N}^{N} \langle \hat{u}_n(t;\omega)\rangle e^{inx}\,, \tag{140a}$$

$$\langle u(x,t;\omega)^2\rangle = \sum_{n,p=-N}^{N} \langle \hat{u}_n(t;\omega)\hat{u}_p(t;\omega)\rangle e^{i(n+p)x}\,. \tag{140b}$$

In a collocation representation the averaging operation can be explicitly written as

$$\langle \hat{u}_n(t;\omega)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{u}_n(t;\eta,\xi)e^{-(\eta^2+\xi^2)/2}d\eta d\xi \simeq \sum_{i=1}^{K_\eta}\sum_{j=1}^{K_\xi} w_i^\xi w_j^\eta \hat{u}_n(t;\eta_i,\xi_j)\,, \tag{141}$$

where $\{\eta_i\}$ and $\{\xi_j\}$ are quadrature points while $w_i^\xi$ and $w_i^\eta$ are the corresponding integration weights.

*Fourier-Galerkin system in probability space.* Let us consider the following Fourier series representation of the solution to the problem (73) (periodic in $[0,2\pi]\times[-L,L]$)

$$p_{u(x,t)\xi}^{(a,b)} = \sum_{n=-N}^{N}\sum_{m=-Q}^{Q} \hat{p}_{nm}(t,b)e^{inx+ima\pi/L}\,, \tag{142}$$

where $[-L,L]$ is large enough in order to include the support of the response probability function[10]. For subsequent mathematical developments it is convenient to set

$$A_{mq}^{(0)} \overset{\text{def}}{=} \int_{-L}^{L} e^{i(m-q)a\pi/L}da = \begin{cases} 2L & m=q \\ 0 & m\neq q \end{cases} \tag{143a}$$

$$A_{mq}^{(1)} \overset{\text{def}}{=} \int_{-L}^{L} a e^{i(m-q)a\pi/L}da = \begin{cases} 0 & m=q \\ (-1)^{(m-q)}\dfrac{2L^2}{(m-q)i\pi} & m\neq q \end{cases} \tag{143b}$$

$$A_{mq}^{(2)} \overset{\text{def}}{=} \int_{-L}^{L} a^2 e^{i(m-q)a\pi/L}da = \begin{cases} \dfrac{2}{3}L^3 & m=q \\ (-1)^{(m-q)}\dfrac{4L^3}{((m-q)\pi)^2} & m\neq q \end{cases} \tag{143c}$$

Now, the projection of Eq. (73) onto the Fourier space

$$\mathfrak{B}_{2N+2Q+2} = \operatorname{span}\left\{e^{-ihx-iqa\pi/L}\right\}_{\substack{h=-N,..,N\\ q=-Q,..,Q}} \tag{144}$$

yields the Fourier-Galerkin system

$$\frac{d\hat{p}_{hq}}{dt} = -\frac{ih}{2L}\sum_{m=-Q}^{Q}\hat{p}_{hm}\left(A_{mq}^{(1)}+B_{mq}\right) + \sigma\frac{bq\pi}{2L}\left(\hat{p}_{(h-k)q}-\hat{p}_{(h+k)q}\right)\sin(jt)\,. \tag{145}$$

[10] We will set $L=20$ in the present numerical study (see the comments before Eq. (148)).

where

$$B_{mq} \stackrel{\text{def}}{=} \int_{-L}^{L} \left( \int_{-L}^{a} e^{ima'\pi/L} da' \right) e^{-iqa\pi/L} da = \begin{cases} A_{0q}^{(1)} + L A_{0q}^{(0)} & m = 0 \\ \dfrac{L}{i\pi m} \left( A_{mq}^{(0)} - e^{-il\pi} A_{0q}^{(0)} \right) & m \neq 0 \end{cases} \tag{146}$$

In order to set the initial condition for $\hat{p}_{hq}$ we need to calculate the projection of $p_\eta(a, x)$ onto the space (144). This is given by

$$\int_0^{2\pi} \int_{-L}^{L} p_\eta(a, x) \, e^{-ihx - iqa\pi/L} da dx = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-ihx} \int_{-L}^{L} e^{-(a - A\sin(x))^2/2} e^{-iqa\pi/L} da dx \,. \tag{147}$$

The last integral can be manipulated further

$$\begin{aligned} \int_{-L}^{L} e^{-(a-A\sin(x))^2/2} e^{-iqa\pi/L} da &= e^{-A^2 \sin(x)^2/2} \int_{-L}^{L} e^{-a^2/2 + (A\sin(x) - iq\pi/L)a} da \\ &\simeq \sqrt{2\pi} e^{-q^2\pi^2/(2L^2) - iq\pi A \sin(x)/L} \,, \end{aligned}$$

The error in this approximation is far below the machine precision ($10^{-14}$) for all $q \in \mathbb{Z}$ if $L \geq 20$, i.e., we can consider it as *numerically exact* for all $a \in [-20, 20]$. Therefore the initial condition for the Fourier Galerkin system (145) is

$$\hat{p}_{hq}(t_0, b) = \frac{p_\xi^{(b)}}{4\pi L} e^{-q^2\pi^2/(2L^2)} I_{hq} \,. \tag{148}$$

where $p_\xi^{(b)}$ is Gaussian and we have defined

$$I_{hq} \stackrel{\text{def}}{=} \int_0^{2\pi} e^{-ihx - iq\pi A \sin(x)/L} dx \,. \tag{149}$$

These integrals can be evaluated numerically to the desired accuracy. At this point the solution strategy is as follows:

1. We sample the variable $b$ at quadrature points in $[-20, 20]$, e.g., at multi-element Gauss-Legendre-Lobatto points.
2. For each realization $b = b_k$ we set the initial condition (148) and we integrate the system (145) in time.

The response probability, i.e. the marginal of Eq. (142) with respect to $b$ then can obtained through simple Gauss quadrature. Once the response probability is available, we can compute analytically the mean and the second order moment of $u$ as

$$\begin{aligned} \langle u(x,t) \rangle &= \int_{-L}^{L} a p_{u(x,t)}^{(a)} da = \sum_{n=-N}^{N} \sum_{m=-Q}^{Q} \hat{p}_{nm}(t) A_{m0}^{(1)} e^{inx} \,, \\ \langle u(x,t)^2 \rangle &= \int_{-L}^{L} a^2 p_{u(x,t)}^{(a)} da = \sum_{n=-N}^{N} \sum_{m=-Q}^{Q} \hat{p}_{nm}(t) A_{m0}^{(2)} e^{inx} \,. \end{aligned}$$

where the matrices $A_{nm}^{(1)}$ and $A_{nm}^{(2)}$ are defined in Eqs. (143b) and (143c) while

$$\hat{p}_{nm}(t) \stackrel{\text{def}}{=} \sum_{i=1}^{K_b} w_i^b \hat{p}_{nm}(t; b_i) \tag{150}$$

are Fourier coefficients of the response probability ($w_i^b$ are quadrature weights).

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