

DIFFERENTIAL CONSTRAINTS FOR THE PROBABILITY DENSITY FUNCTION OF STOCHASTIC SOLUTIONS TO THE WAVE EQUATION

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By using functional integral methods we determine new types of differential constraints satisfied by the joint probability density function of stochastic solutions to the wave equation subject to uncertain boundary and initial conditions. These differential constraints involve unusual limit partial differential operators and, in general, they can be grouped into two main classes: the first one depends on the specific field equation under consideration, i.e., on the stochastic wave equation; the second class includes a set of intrinsic relations determined by the structure of the joint probability density function of the wave and its derivatives. Preliminary results we have obtained for stochastic dynamical systems and first-order nonlinear stochastic PDEs suggest that the set of differential constraints is complete and therefore it allows to determine uniquely the probability density function of the solution to the stochastic problem. The proposed new approach can be extended to arbitrary nonlinear stochastic PDEs and it could be an effective way to overcome the curse of dimensionality for random boundary and initial conditions. An application of theory developed is presented and discussed for a simple random wave in one spatial dimension.

KEY WORDS: *Stochastic partial differential equations, high-dimensional methods, random fields.*

1. INTRODUCTION

Many physical phenomena such as sound propagation, electromagnetic scattering at random surfaces and random vibrations in solid mechanics can be described in terms of random waves satisfying the standard wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = U^2 \nabla^2 \psi \quad (1)$$

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1 for random boundary conditions, random initial conditions or random group velocity. The purpose of this paper is
2 to introduce a new method to compute the statistical properties of the stochastic solutions to Eq. (1). This method
3 is based on a set of differential constraints for the joint probability density function of the wave ψ and its deriva-
4 tives and it may present an advantage with respect to more classical stochastic approaches such as polynomial chaos
5 [1–4], probabilistic collocation [5, 6] and generalized spectral decompositions [7–11]. In fact, it seems that it does
6 not suffer from the curse of dimensionality problem, at least when randomness comes from boundary or initial con-
7 ditions. Indeed, since we are solving for probability density function of the system, we can actually prescribe these
8 conditions in terms of probability distributions and this is obviously not dependent on the number of random variables
9 characterizing the underlying probability space.

10 This observation immediately lead us to the question: *is it possible to determine a closed evolution equation*
11 *for the probability density function of the solution to Eq. (1) at a specific space-time location?* Unfortunately, the
12 answer is negative. In fact, the self-interacting nature of the wave equation is associated with nonlocal solutions in
13 space and time which, in turn, yield the impossibility to determine a pointwise equation for the probability density
14 function. However, an equation for the probability density *functional* of the wave can always be obtained. This very
15 general approach relies on the use of functional integral techniques [12–16], in particular those ones involving the
16 Hopf characteristic functional (see appendix APPENDIX B). These functional methods aim to cope with the global
17 probabilistic structure of the solution and they have been extensively studied in the past as a possible tool to tackle
18 many fundamental problems in physics such as turbulence [17, 18]. Their usage grew very rapidly in the 70s, when
19 it became clear that diagrammatic functional techniques [12] could be applied, at least formally, to many different
20 problems in classical statistical physics. However, functional differential equations involving the Hopf characteristic
21 functional are unfortunately not amenable to numerical simulation. In addition, the amount of statistical information
22 carried on by the Hopf characteristic functional is often far beyond the needs of practical uncertainty quantification,
23 which usually reduces only to the computation of a few statistical moments of the solution at specific space-time
24 locations.

25 Thus, we are led to look for alternative ways to determine the evolution of the probability density function asso-
26 ciated with the solution to Eq. (1). To this end, by using a functional integral technique we have recently introduced
27 in the context of stochastic dynamical systems, in this paper we will show that it is possible to formulate a set of *dif-*
28 *ferential constraints* that have to be satisfied locally by the probability density function of *every* random wave process
29 governed by Eq. (1). These constraints involve, in general, unusual limit partial differential operators and they can
30 be grouped into two main classes: the first one depends on the specific field equation under consideration, i.e., on

1 the stochastic wave equation; the second class is represented by a set of intrinsic relations arising from the structure
2 of the joint probability density function of the wave and its derivatives. A fundamental question we address in this
3 paper is whether the set of these differential constraints allows to determine uniquely the probability density function
4 associated with the stochastic solution to the wave equation.

5 This paper is organized as follows. In section 2 we consider random waves in one spatial dimension and we show
6 how to determine a closed and exact differential constraint that has to be satisfied by the probability density function
7 of every random wave process for random boundary conditions and random initial conditions. All the calculations
8 are based on a functional integral technique which is presented in detail in appendix APPENDIX A. Section 3
9 deals with the formulation of additional differential constraints depending on the structure of the joint probability
10 density function of the wave and its derivatives. The completeness of the set of differential constraints is discussed
11 in section 4 for a prototype problem involving a first-order nonlinear stochastic PDE. In section 5 we generalize the
12 theory developed to two- and three-dimensional random wave processes. An example of application of the theory
13 is presented in section 6. Finally, the main findings and their implications are summarized in section 7. We also
14 include two additional appendices. The first one, i.e. appendix APPENDIX B, deals with the application of the Hopf
15 characteristic functional approach to one-dimensional random waves. The second one, i.e. appendix APPENDIX C,
16 deals with the perturbation expansion of differential constraints in the neighborhood of a selected space-time location.

17 2. ONE-DIMENSIONAL RANDOM WAVES

18 In one spatial dimension Eq. (1) can be written as

$$\frac{\partial^2 \psi}{\partial t^2} = U^2 \frac{\partial^2 \psi}{\partial x^2}. \quad (2)$$

19 We supplement this equation with appropriate random boundary conditions and with a random initial condition in a
20 suitable space-time domain. The solution to this boundary value problem is clearly a *random wave*, which we assume
21 to admit a probability density function. At this point a fundamental question is whether we can actually determine an
22 evolution equation for such a probability density based on Eq. (2). To this end, let us first look for a representation
23 of the joint probability of the wave and its first-order derivatives with respect to space and time calculated at *different*
24 space-time locations. The reason why we need such joint probability density will become clear in a while. For

1 notational convenience, let us set

$$\psi \stackrel{\text{def}}{=} \psi(x, t; \omega), \quad \psi'_t \stackrel{\text{def}}{=} \frac{\partial \psi}{\partial t}(x', t'; \omega), \quad \psi''_x \stackrel{\text{def}}{=} \frac{\partial \psi}{\partial x}(x'', t''; \omega). \quad (3)$$

2 This allows us to write the joint probability density of ψ , ψ'_t and ψ''_x as (see appendix APPENDIX A for further
3 details)

$$p_{\psi \psi'_t \psi''_x}^{(a,b,c)} \stackrel{\text{def}}{=} \langle \underbrace{\delta(a - \psi)}_{\text{constructor field}} \underbrace{\delta(b - \psi'_t) \delta(c - \psi''_x)}_{\text{absorbing fields}} \rangle, \quad (4)$$

4 where δ denotes the Dirac delta function and the average operator $\langle \cdot \rangle$ is defined as a functional integral with respect
5 to the joint measure of the random initial condition and the random boundary conditions. The definition we give of
6 “*constructor field*” relies on the fact that we will extract from that Dirac delta function, called also “*indicator function*”
7 by Klyatskin [19] (p. 42), all the derivatives we need to build up the wave equation. As we will see, this procedure
8 generates also additional terms that can be treated in a limit process by employing the “*absorbing fields*”. By using
9 the differentiation rules for the probability density function discussed in appendix APPENDIX A we obtain

$$\frac{\partial p_{\psi \psi'_t \psi''_x}^{(a,b,c)}}{\partial t} = -\frac{\partial}{\partial a} \langle \delta(a - \psi) \psi_t \delta(b - \psi'_t) \delta(c - \psi''_x) \rangle \quad (5)$$

10 and

$$\frac{\partial^2 p_{\psi \psi'_t \psi''_x}^{(a,b,c)}}{\partial t^2} = \frac{\partial^2}{\partial a^2} \langle \delta(a - \psi) \psi_t^2 \delta(b - \psi'_t) \delta(c - \psi''_x) \rangle - \frac{\partial}{\partial a} \langle \delta(a - \psi) \psi_{tt} \delta(b - \psi'_t) \delta(c - \psi''_x) \rangle. \quad (6)$$

11 Similarly,

$$\frac{\partial^2 p_{\psi \psi'_t \psi''_x}^{(a,b,c)}}{\partial x^2} = \frac{\partial^2}{\partial a^2} \langle \delta(a - \psi) \psi_x^2 \delta(b - \psi'_t) \delta(c - \psi''_x) \rangle - \frac{\partial}{\partial a} \langle \delta(a - \psi) \psi_{xx} \delta(b - \psi'_t) \delta(c - \psi''_x) \rangle. \quad (7)$$

12 If we subtract Eq. (7) from Eq. (6) and we take Eq. (2) into account we obtain

$$\frac{\partial^2 p_{\psi \psi'_t \psi''_x}^{(a,b,c)}}{\partial t^2} - U^2 \frac{\partial^2 p_{\psi \psi'_t \psi''_x}^{(a,b,c)}}{\partial x^2} = \frac{\partial^2}{\partial a^2} \langle \delta(a - \psi) (\psi_t^2 - U^2 \psi_x^2) \delta(b - \psi'_t) \delta(c - \psi''_x) \rangle. \quad (8)$$

The average appearing in Eq. (8) can be easily represented if we take the limit for (x', x'') and (t', t'') going to x and t , respectively. In fact, by using the “absorbing fields” appearing in Eq. (4) we see that

$$(b^2 - U^2 c^2) p_{\psi\psi_t\psi_x}^{(a,b,c)} = \lim_{\substack{t', t'' \rightarrow t \\ x', x'' \rightarrow x}} \langle \delta(a - \psi) (\psi_t^2 - U^2 \psi_x^2) \delta(b - \psi_t') \delta(c - \psi_x'') \rangle. \quad (9)$$

1 This yields the final result

$$\lim_{\substack{t', t'' \rightarrow t \\ x', x'' \rightarrow x}} \frac{\partial^2 p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial t^2} = U^2 \lim_{\substack{t', t'' \rightarrow t \\ x', x'' \rightarrow x}} \frac{\partial^2 p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial x^2} + (b^2 - U^2 c^2) \frac{\partial^2 p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial a^2}. \quad (10)$$

Note that this identity involves unusual partial differential operators which we shall call *limit partial derivatives*. For subsequent mathematical developments it is convenient to reserve a special symbol for these operators, i.e. we shall define

$$\frac{\partial^2 p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial t^2} \stackrel{\text{def}}{=} \lim_{\substack{t', t'' \rightarrow t \\ x', x'' \rightarrow x}} \frac{\partial^2 p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial t^2}, \quad \frac{\partial^2 p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial x^2} \stackrel{\text{def}}{=} \lim_{\substack{t', t'' \rightarrow t \\ x', x'' \rightarrow x}} \frac{\partial^2 p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial x^2}. \quad (11)$$

2 This allows us to write Eq. (10) in a compact form as

$$\frac{\partial^2 p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial t^2} = U^2 \frac{\partial^2 p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial x^2} + (b^2 - U^2 c^2) \frac{\partial^2 p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial a^2}. \quad (12)$$

3 Equation (12) is a *differential constraint* that must be satisfied by the probability density function of *every* one-
 4 dimensional random wave governed by Eq. (2) for random initial condition or random boundary conditions. These
 5 type of differential constraints made their first appearance in [20] in the context of stochastic dynamical systems.
 6 Note that Eq. (12) always involves only three variables (x, t, a) and two parameters $(b$ and $c)$ independently of
 7 the dimensionality of the random boundary conditions and the random initial condition. This very attractive feature
 8 immediately lead us to the question: is it possible to solve a boundary value problem for Eq. (12) and therefore obtain
 9 the joint probability density function of the random wave and its derivatives at a specific space-time location uniquely?
 10 Certainly, if we would have this possibility then we would be able to overcome the curse of dimensionality problem.
 11 As we will see in section 6, however, there exist an infinite number of probability densities satisfying the differential
 12 constraint (12) for the same boundary and initial conditions. This suggests that a boundary value problem for Eq. (12)
 13 is, in general, *ill-posed*.

1 A possible way to get rid of the limit partial derivatives is to integrate Eq. (12) with respect to the variables b and
 2 c . This procedure yields the following equation for the *response probability*, i.e. for the probability density of the
 3 wave ψ

$$\frac{\partial^2 p_{\psi}^{(a)}}{\partial t^2} = U^2 \frac{\partial^2 p_{\psi}^{(a)}}{\partial x^2} + \frac{\partial^2}{\partial a^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (b^2 - U^2 c^2) p_{\psi\psi_t\psi_x}^{(a,b,c)} dbdc, \quad (13)$$

4 where

$$p_{\psi(x,t)}^{(a)} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{\psi\psi_t\psi_x}^{(a,b,c)} dbdc \quad (14)$$

denotes the marginal density of $p_{\psi\psi_t\psi_x}^{(a,b,c)}$. The integrals appearing in Eq. (14) are formally written from $-\infty$ to ∞ although the probability density function we integrate out may be compactly supported. Note that Eq. (13) is not closed. In fact, the last term at the right hand side can be written as

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (b^2 - U^2 c^2) p_{\psi\psi_t\psi_x}^{(a,b,c)} dbdc &= \langle \delta(a - \psi) \int_{-\infty}^{\infty} b^2 \delta(b - \psi_t) db \int_{-\infty}^{\infty} \delta(c - \psi_x) dc \rangle \\ &\quad - U^2 \langle \delta(a - \psi) \int_{-\infty}^{\infty} \delta(b - \psi_t) db \int_{-\infty}^{\infty} c^2 \delta(c - \psi_x) dc \rangle \\ &= \langle \delta(a - \psi) \psi_t^2 \rangle - U^2 \langle \delta(a - \psi) \psi_x^2 \rangle. \end{aligned} \quad (15)$$

5 The last two averages of this equation involve the calculation of the correlations between two functionals of the random
 6 initial conditions and the random boundary conditions, e.g., $\delta(a - \psi)$ and ψ_t^2 . Such a correlation structure can be
 7 disentangled, for example, by using a functional power series [21, 22] (see also [23], p. 311, and [19], p. 70). In other
 8 words, the correlations $\langle \delta(a - \psi) \psi_t^2 \rangle$ and $\langle \delta(a - \psi) \psi_x^2 \rangle$ can be represented in terms of cumulants of the random initial
 9 conditions and the random boundary conditions. Alternatively, other types of closures can be considered [24–27].

10 We conclude this section by emphasizing that Eq. (13) can be obtained more directly from the representation of
 11 the response probability of the wave, i.e.

$$p_{\psi(x,t)}^{(a)} = \langle \delta(a - \psi(x, t)) \rangle. \quad (16)$$

In fact, if we differentiate Eq. (16) twice with respect to t and x

$$\frac{\partial^2 p_{\psi(x,t)}^{(a)}}{\partial t^2} = -\frac{\partial}{\partial a} \langle \delta(a - \psi) \psi_{tt} \rangle + \frac{\partial^2}{\partial a^2} \langle \delta(a - \psi) \psi_t^2 \rangle, \quad (17)$$

$$\frac{\partial^2 p_{\psi(x,t)}^{(a)}}{\partial x^2} = -\frac{\partial}{\partial a} \langle \delta(a - \psi) \psi_{xx} \rangle + \frac{\partial^2}{\partial a^2} \langle \delta(a - \psi) \psi_x^2 \rangle \quad (18)$$

1 and sum up the derivatives, we immediately obtain

$$\frac{\partial^2 p_{\Psi(x,t)}^{(a)}}{\partial t^2} = U^2 \frac{\partial^2 p_{\Psi(x,t)}^{(a)}}{\partial x^2} + \frac{\partial^2}{\partial a^2} \langle \delta(a - \Psi) \Psi_t^2 \rangle - U^2 \frac{\partial^2}{\partial a^2} \langle \delta(a - \Psi) \Psi_x^2 \rangle. \quad (19)$$

2 This shows that the differential constraint (12) includes the evolution equation of the response probability of the wave
3 as a subcase.

4 *Some remarks on evolution equations involving limit partial differential operators*

5 Equation (12) involves unusual limit partial differential operators in space and time. These operators arise very
6 often when the functional integral method or, equivalently, the Hopf characteristic functional approach (see appendix
7 APPENDIX B) are applied to a stochastic PDE in order to determine a pointwise equation for the probability density
8 function of its solution. In this section we would like to provide some heuristic justification of why this happens. To
9 this end, let us first rewrite the one-dimensional wave equation (2) as a first-order system

$$\frac{\partial \Psi}{\partial t} = U^2 \frac{\partial \eta}{\partial x}, \quad \frac{\partial \eta}{\partial t} = \frac{\partial \Psi}{\partial x}. \quad (20)$$

10 This formulation clearly shows that the temporal evolution of the wave Ψ is driven by the random field $\partial \eta / \partial x$ while
11 the temporal evolution of η is governed by $\partial \Psi / \partial x$. In other words, the stochastic dynamics of random waves is,
12 in general, *self-interacting*. A similar situation arises in the simpler context of time-dependent stochastic dynamical
13 systems of order greater than one, where the dynamics of the stochastic solution is influenced by the dynamics of
14 higher-order time derivatives. A self-interacting system can yield to *nonlocal solutions* in space and time. Perhaps,
15 the simplest example in this context is the diffusion equation. In this cases, a pointwise equation for the probability
16 density function cannot be derived in a closed form. There are, however, a few exceptions where we can actually obtain
17 a closed equation for the probability density function. Among them we recall first-order nonlinear and quasilinear
18 stochastic PDEs. In these cases the limit partial derivatives do not appear and the evolution problem for the probability
19 density function is well defined in a classical sense. A prime example is the inviscid Burgers equation

$$\frac{\partial \Psi}{\partial t} + \Psi \frac{\partial \Psi}{\partial x} = 0 \quad (21)$$

1 with random boundary or initial conditions. In this case it can be shown that the probability density function of the
2 solution satisfies

$$\frac{\partial p_{\psi(x,t)}^{(a)}}{\partial t} + a \frac{\partial p_{\psi(x,t)}^{(a)}}{\partial x} + \int_{-\infty}^a \frac{\partial p_{\psi(x,t)}^{(a')}}{\partial x} da' = 0. \quad (22)$$

3 INTRINSIC RELATIONS DEPENDING ON THE STRUCTURE OF THE JOINT PROBABILITY DEN- 4 SITY FUNCTION

5 The fields appearing in the joint probability density (4) are locally related to each other and therefore we expect that
6 there exist a certain number of relations between the limit partial derivatives of the joint probability density with
7 respect to different arguments. These relations are independent of the particular stochastic field equation describing
8 the physical phenomenon, e.g. Eq. (2), but they are *intrinsically* defined by the structure of the joint probability
9 density function. For instance, there is clearly a local deterministic relation between the random field ψ and its spatial
10 derivative

$$\psi_x(x, t; \omega) \stackrel{\text{def}}{=} \lim_{x' \rightarrow x} \frac{\psi(x', t; \omega) - \psi(x, t; \omega)}{x' - x}. \quad (23)$$

This relation, and similar ones for other derivatives, can be translated into an intrinsic relation involving the joint probability density function (4). To this end, we first observe that the limit derivatives of Eq. (4) with respect to different arguments, e.g. x , x' and x'' are different. Indeed,

$$\frac{\partial p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial x} = -\frac{\partial}{\partial a} \langle \delta(a - \psi) \psi_x \delta(b - \psi_t) (c - \psi_x) \rangle = -c \frac{\partial p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial a}, \quad (24)$$

$$\frac{\partial p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial x'} = -\frac{\partial}{\partial b} \langle \delta(a - \psi) \delta(b - \psi_t) \psi_{tx} \delta(c - \psi_x) \rangle, \quad (25)$$

$$\frac{\partial p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial x''} = -\frac{\partial}{\partial c} \langle \delta(a - \psi) \delta(b - \psi_t) \delta(c - \psi_x) \psi_{xx} \rangle. \quad (26)$$

Similarly,

$$\frac{\partial p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial t} = -\frac{\partial}{\partial a} \langle \delta(a - \psi) \psi_t \delta(b - \psi_t) (c - \psi_x) \rangle = -b \frac{\partial p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial a}, \quad (27)$$

$$\frac{\partial p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial t'} = -\frac{\partial}{\partial b} \langle \delta(a - \psi) \delta(b - \psi_t) \psi_{tt} \delta(c - \psi_x) \rangle, \quad (28)$$

$$\frac{\partial p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial t''} = -\frac{\partial}{\partial c} \langle \delta(a - \psi) \delta(b - \psi_t) \delta(c - \psi_x) \psi_{xt} \rangle. \quad (29)$$

1 From Eqs. (25) and (29) it follows that

$$\int_{-\infty}^b \frac{\partial p_{\psi\psi_t\psi_x}^{(a,b',c)}}{\partial x'} db' = \int_{-\infty}^c \frac{\partial p_{\psi\psi_t\psi_x}^{(a,b,c')}}{\partial t''} dc'. \quad (30)$$

2 Next, let us consider the average of the second-order spatial derivative of ψ . A comparison between Eq. (26) and Eq.
 3 (7) (in the limit for $x', x'' \rightarrow x$ and $t', t'' \rightarrow t$) immediately gives

$$\int_{-\infty}^a \frac{\partial^2 p_{\psi\psi_t\psi_x}^{(a',b,c)}}{\partial x^2} da' = c^2 \frac{\partial p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial a} + \int_{-\infty}^c \frac{\partial p_{\psi\psi_t\psi_x}^{(a,b,c')}}{\partial x''} dc'. \quad (31)$$

4 Similarly, a comparison between Eqs. (28) and (6) yields

$$\int_{-\infty}^a \frac{\partial^2 p_{\psi\psi_t\psi_x}^{(a',b,c)}}{\partial t^2} da' = b^2 \frac{\partial p_{\psi\psi_t\psi_x}^{(a,b,c)}}{\partial a} + \int_{-\infty}^b \frac{\partial p_{\psi\psi_t\psi_x}^{(a,b',c)}}{\partial t'} db'. \quad (32)$$

5 Equations (24), (27), (30), (31) and (32) are simple examples of *intrinsic relations* involving the probability density
 6 function. These relations are in the form of differential constraints and they have to be satisfied independently of
 7 the particular stochastic problem under consideration, e.g. Eq. (2). In principle, if the wave field ψ is analytic, by
 8 repeatedly applying the arguments above we can construct an infinite set of differential constraints to be satisfied at
 9 every space-time location by the joint probability density function of the wave and its derivatives. Note that from Eqs.
 10 (28), (26) and (2) we easily obtain that

$$\int_{-\infty}^b \frac{\partial p_{\psi\psi_t\psi_x}^{(a,b',c)}}{\partial t'} db' = U^2 \int_{-\infty}^c \frac{\partial p_{\psi\psi_t\psi_x}^{(a,b,c')}}{\partial x''} dc', \quad (33)$$

11 which is another, more compact, way to write the differential constraint for the probability density function associated
 12 with the wave equation (2). Indeed, a substitution of Eqs. (31) and (32) in Eq. (33) consistently gives Eq. (12).

13 4. ON THE COMPLETENESS OF THE SET OF DIFFERENTIAL CONSTRAINTS

14 In some cases, it can be proved that the set of differential constraints involving the first-order limit partial derivatives is
 15 equivalent to a standard evolution equation for the joint probability density function of the system. This circumstance
 16 implies that the set of differential constraints is complete, i.e., that it allows for the computation of the joint density.
 17 This happens, for example, in the context of high-order stochastic dynamical systems subject to a random initial state.
 18 The set of differential constraints, in this case, is equivalent to the well-known Liouville equation. In this section we

1 would like to prove a similar result for first-order nonlinear stochastic PDEs. To this end, let us consider the equation

$$\frac{\partial \psi}{\partial t} + \left(\frac{\partial \psi}{\partial x} \right)^2 = 0 \quad (34)$$

2 subject to random boundary and initial conditions of arbitrary dimensionality. The solution to this problem can be
3 represented in probability space by using the joint density

$$p_{\psi\psi'_x}^{(a,b)} = \langle \delta(a - \psi) \delta(b - \psi'_x) \rangle. \quad (35)$$

The full set of first-order differential constraints associated with Eqs. (34) and (35) is easily obtained as

$$\frac{\partial p_{\psi\psi'_x}^{(a,b)}}{\partial t} = b^2 \frac{\partial p_{\psi\psi'_x}^{(a,b)}}{\partial a}, \quad (36)$$

$$\frac{\partial p_{\psi\psi'_x}^{(a,b)}}{\partial t'} = \frac{\partial}{\partial b} (2b \langle \delta(a - \psi) \delta(b - \psi_x) \psi_{xx} \rangle), \quad (37)$$

$$\frac{\partial p_{\psi\psi'_x}^{(a,b)}}{\partial x} = -b \frac{\partial p_{\psi\psi'_x}^{(a,b)}}{\partial a}, \quad (38)$$

$$\frac{\partial p_{\psi\psi'_x}^{(a,b)}}{\partial x'} = -\frac{\partial}{\partial b} \langle \delta(a - \psi) \delta(b - \psi_x) \psi_{xx} \rangle. \quad (39)$$

4 From Eqs. (37) and (39) it follows that

$$\frac{\partial p_{\psi\psi'_x}^{(a,b)}}{\partial t'} = -\frac{\partial}{\partial b} \left(2b \int_{-\infty}^b \frac{\partial p_{\psi\psi'_x}^{(a,b')}}{\partial x'} db' \right). \quad (40)$$

At this point, let us recall the following identities between standard partial derivatives and limit partial derivatives

$$\frac{\partial p_{\psi\psi'_x}^{(a,b)}}{\partial t} = \frac{\partial p_{\psi\psi'_x}^{(a,b)}}{\partial t} + \frac{\partial p_{\psi\psi'_x}^{(a,b)}}{\partial t'}, \quad (41)$$

$$\frac{\partial p_{\psi\psi'_x}^{(a,b)}}{\partial x} = \frac{\partial p_{\psi\psi'_x}^{(a,b)}}{\partial x} + \frac{\partial p_{\psi\psi'_x}^{(a,b)}}{\partial x'}. \quad (42)$$

Note that at the left hand side we first set $x = x'$ and $t = t'$ and then we differentiate while at the right hand side we first differentiate and then we set $x = x'$ and $t = t'$. By substituting Eqs. (36) and (40) into Eq. (41) and using Eqs.

(42) and (38), we obtain

$$\frac{\partial p_{\psi\psi_x}^{(a,b)}}{\partial t} = b^2 \frac{\partial p_{\psi\psi_x}^{(a,b)}}{\partial a} - \frac{\partial}{\partial b} \left[2b \left(\int_{-\infty}^b \frac{\partial p_{\psi\psi_x}^{(a,b')}}{\partial x} db' + \int_{-\infty}^b b' \frac{\partial p_{\psi\psi_x}^{(a,b')}}{\partial a} db' \right) \right]. \quad (43)$$

1 This is the correct evolution equation for the joint probability density function associated with the stochastic solution
 2 to Eq. (34). We can show this by considering Eq. (35), with both ψ and ψ_x set at the same space-time location. A
 3 differentiation with respect to time yields, in this case,

$$\frac{\partial p_{\psi\psi_x}^{(a,b)}}{\partial t} = b^2 \frac{\partial p_{\psi\psi_x}^{(a,b)}}{\partial a} + \frac{\partial}{\partial b} (2b \langle \delta(a - \psi) \delta(b - \psi_x) \psi_{xx} \rangle), \quad (44)$$

4 where the last term can be written in terms of the probability density function by inverting the relation

$$\frac{\partial p_{\psi\psi_x}^{(a,b)}}{\partial x} = -b \frac{\partial p_{\psi\psi_x}^{(a,b)}}{\partial a} - \frac{\partial}{\partial b} \langle \delta(a - \psi) \delta(b - \psi_x) \psi_{xx} \rangle, \quad (45)$$

5 i.e.,

$$\langle \delta(a - \psi) \delta(b - \psi_x) \psi_{xx} \rangle = - \int_{-\infty}^b \frac{\partial p_{\psi\psi_x}^{(a,b')}}{\partial x} db' - \int_{-\infty}^b b' \frac{\partial p_{\psi\psi_x}^{(a,b')}}{\partial a} db'. \quad (46)$$

6 A substitution of Eq. (46) into Eq. (44) gives exactly Eq. (43). Thus, we have shown that the set of first-order
 7 differential constraints (36)-(39) is equivalent to a standard PDE for the joint probability density function (35) on the
 8 hyperplane $x' = x, t' = t$. This implies that the set of differential constraints, in this case, is complete. It is still
 9 an open question, however, if such set is complete for more general stochastic problems, e.g. for the stochastic wave
 10 equation (2). The dynamics of the joint probability density function in these cases does not satisfy a standard PDE.
 11 This is due to the fact that such dynamics, in general, arises from the projection of a functional differential equation
 12 on the hyperplane $x = x' = x'' = \dots, t = t' = t'' = \dots$. In some sense, the set of differential constraints we have
 13 obtained represents the components of such projection.

14 5. TWO- AND THREE-DIMENSIONAL RANDOM WAVES

15 The theoretical apparatus developed in the previous sections can be easily extended to two- and three-dimensional
 16 wave equations. To this end, let us first consider two spatial dimensions and the equation

$$\frac{\partial^2 \psi}{\partial t^2} = U^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right). \quad (47)$$

- 1 The joint probability density function of the random wave and its derivatives has the functional integral representation

$$p_{\Psi\Psi'_t\Psi''_x\Psi'''_y}^{(a,b,c,d)} = \langle \underbrace{\delta(a-\Psi)}_{\text{constructor field}} \underbrace{\delta(b-\Psi'_t)\delta(c-\Psi''_x)\delta(d-\Psi'''_y)}_{\text{absorbing fields}} \rangle. \quad (48)$$

As before, we now differentiate the constructor field with respect to time and space variables and then we use the absorbing fields in a limit procedure in order to get rid of the additional terms arising from the differentiation process.

We obtain,

$$\begin{aligned} \frac{\partial^2 p_{\Psi\Psi'_t\Psi''_x\Psi'''_y}^{(a,b,c,d)}}{\partial t^2} &= \frac{\partial^2}{\partial a^2} \langle \delta(a-\Psi) \Psi_t^2 \delta(b-\Psi'_t) \delta(c-\Psi''_x) \delta(d-\Psi'''_y) \rangle - \\ &\quad - \frac{\partial}{\partial a} \langle \delta(a-\Psi) \Psi_{tt} \delta(b-\Psi'_t) \delta(c-\Psi''_x) \delta(d-\Psi'''_y) \rangle, \end{aligned} \quad (49)$$

$$\begin{aligned} \frac{\partial^2 p_{\Psi\Psi'_t\Psi''_x\Psi'''_y}^{(a,b,c,d)}}{\partial x^2} &= \frac{\partial^2}{\partial a^2} \langle \delta(a-\Psi) \Psi_x^2 \delta(b-\Psi'_t) \delta(c-\Psi''_x) \delta(d-\Psi'''_y) \rangle - \\ &\quad - \frac{\partial}{\partial a} \langle \delta(a-\Psi) \Psi_{xx} \delta(b-\Psi'_t) \delta(c-\Psi''_x) \delta(d-\Psi'''_y) \rangle, \end{aligned} \quad (50)$$

$$\begin{aligned} \frac{\partial^2 p_{\Psi\Psi'_t\Psi''_x\Psi'''_y}^{(a,b,c,d)}}{\partial y^2} &= \frac{\partial^2}{\partial a^2} \langle \delta(a-\Psi) \Psi_y^2 \delta(b-\Psi'_t) \delta(c-\Psi''_x) \delta(d-\Psi'''_y) \rangle - \\ &\quad - \frac{\partial}{\partial a} \langle \delta(a-\Psi) \Psi_{yy} \delta(b-\Psi'_t) \delta(c-\Psi''_x) \delta(d-\Psi'''_y) \rangle. \end{aligned} \quad (51)$$

- 2 A summation of Eqs. (49)-(51) and a following limit procedure for (x', x'', x''') and (t', t'', t''') going to x and t ,
3 respectively, yields

$$\frac{\partial^2 p_{\Psi\Psi_t\Psi_x\Psi_y}^{(a,b,c,d)}}{\partial t^2} = U^2 \left(\frac{\partial^2 p_{\Psi\Psi_t\Psi_x\Psi_y}^{(a,b,c,d)}}{\partial x^2} + \frac{\partial^2 p_{\Psi\Psi_t\Psi_x\Psi_y}^{(a,b,c,d)}}{\partial y^2} \right) + (b^2 - U^2(c^2 + d^2)) \frac{\partial^2 p_{\Psi\Psi_t\Psi_x\Psi_y}^{(a,b,c,d)}}{\partial a^2}. \quad (52)$$

that is the differential constraint for the joint probability density function of *every* two-dimensional random wave satisfying the wave equation (47) for random boundary or random initial conditions. Straightforward extensions of the arguments above yield the following differential constraint for the joint probability density function of three-

dimensional random waves

$$\frac{\partial^2 p_{\Psi\Psi_t\Psi_x\Psi_y\Psi_z}^{(a,b,c,d,e)}}{\partial t^2} = U^2 \left(\frac{\partial^2 p_{\Psi\Psi_t\Psi_x\Psi_y\Psi_z}^{(a,b,c,d,e)}}{\partial x^2} + \frac{\partial^2 p_{\Psi\Psi_t\Psi_x\Psi_y\Psi_z}^{(a,b,c,d,e)}}{\partial y^2} + \frac{\partial^2 p_{\Psi\Psi_t\Psi_x\Psi_y\Psi_z}^{(a,b,c,d,e)}}{\partial z^2} \right) + (b^2 - U^2 (c^2 + d^2 + e^2)) \frac{\partial^2 p_{\Psi\Psi_t\Psi_x\Psi_y}^{(a,b,c,d)}}{\partial a^2}. \tag{53}$$

1 The differential constraints (52) and (53) can be supplemented with additional intrinsic relations involving the joint
 2 probability density function. These relations can be constructed by following the same mathematical steps presented
 3 in section 3.

4 6. AN EXAMPLE

5 Let us consider a random wave in one spatial dimension satisfying the following boundary value problem

$$\begin{cases} \frac{\partial^2 \Psi}{\partial t^2} = U^2 \frac{\partial^2 \Psi}{\partial x^2}, & -\infty < x < \infty, \quad t \geq t_0 \\ \Psi(x, t_0; \omega) = \sum_{k=1}^3 \xi_k(\omega) h_k(x) \\ \Psi_t(x, t_0; \omega) = 0 \end{cases} \tag{54}$$

6 where $\xi_k(\omega)$ ($k = 1, \dots, 3$) are random variables with known joint probability density function and $h_k(x)$ are
 7 prescribed deterministic functions on the real line. The analytical solution to the problem (54) is the well known
 8 d'Alambert wave ([28], p. 41)

$$\Psi(x, t; \omega) = \frac{1}{2} \sum_{k=1}^3 \xi_k(\omega) [h_k(x + Ut) + h_k(x - Ut)]. \tag{55}$$

For subsequent mathematical developments, it is convenient to define

$$G_{1k} \stackrel{\text{def}}{=} \frac{1}{2} [h_k(x + Ut) + h_k(x - Ut)], \tag{56}$$

$$G'_{2k} \stackrel{\text{def}}{=} \frac{\partial G_{1k}}{\partial t}(x', t'), \tag{57}$$

$$G''_{3k} \stackrel{\text{def}}{=} \frac{\partial G_{1k}}{\partial x}(x'', t''). \tag{58}$$

This allows us to write the solution (55) and its derivatives with respect to x and t in a compact form as

$$\psi = \sum_{k=1}^3 \xi_k G_{1k}, \quad \psi'_t = \sum_{k=1}^3 \xi_k G'_{2k}, \quad \psi''_x = \sum_{k=1}^3 \xi_k G''_{3k}. \quad (59)$$

1 The joint probability density of ψ , ψ'_t and ψ''_x , i.e. the joint probability of the wave and its first-order derivatives at
 2 different space-time locations, can be obtained by using the classical mapping approach. To this end, let us notice that
 3 Eqs. (59) define a three-dimensional *linear* transformation from $\{\xi_1, \xi_2, \xi_3\}$ to $\{\psi, \psi'_x, \psi''_t\}$. In a matrix-vector form
 4 this transformation can be represented as

$$\begin{pmatrix} \psi \\ \psi'_t \\ \psi''_x \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G'_{21} & G'_{22} & G'_{23} \\ G''_{31} & G''_{32} & G''_{33} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}. \quad (60)$$

5 The determinant of the system's matrix is

$$J \stackrel{\text{def}}{=} G_{11} (G'_{22} G''_{33} - G''_{32} G'_{23}) + G_{12} (G''_{31} G'_{23} - G'_{21} G''_{33}) + G_{13} (G'_{21} G''_{32} - G''_{31} G'_{22}), \quad (61)$$

6 while the transpose of its algebraic complement has entries

$$\begin{aligned} H_{11} &= G'_{22} G''_{33} - G''_{32} G'_{23}, & H_{12} &= G_{13} G''_{32} - G_{12} G''_{33}, & H_{13} &= G_{12} G'_{23} - G'_{22} G_{13}, \\ H_{21} &= G''_{31} G'_{23} - G'_{21} G''_{33}, & H_{22} &= G_{11} G''_{33} - G_{13} G''_{31}, & H_{23} &= G_{13} G'_{21} - G_{11} G'_{23}, \\ H_{31} &= G'_{21} G''_{32} - G''_{31} G'_{22}, & H_{32} &= G''_{31} G_{12} - G_{11} G''_{32}, & H_{33} &= G_{11} G'_{22} - G'_{21} G_{12}. \end{aligned} \quad (62)$$

7 A classical result of probability theory ([29], p. 183) states that if $p_{\xi_1 \xi_2 \xi_3}^{(\alpha_1, \alpha_2, \alpha_3)}$ is the joint probability density of ξ_1 , ξ_2
 8 and ξ_3 then the joint probability of ψ , ψ'_t and ψ''_x is given by

$$p_{\psi \psi'_t \psi''_x}^{(a,b,c)} = \frac{1}{|J|} p_{\xi_1 \xi_2 \xi_3}^{(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)}, \quad (63)$$

9 where

$$\hat{\alpha}_i(a, b, c, x, x', x'', t, t', t'') = \frac{1}{J} (H_{i1}a + H_{i2}b + H_{i3}c). \quad (64)$$

Note that every $\hat{\alpha}_i$ is also function of all the space-time variables because of the matrix components H_{ij} . Therefore,
 (63) is a rather complex function of nine variables. Under the assumption that $\xi_i(\omega)$ are mutually independent normal

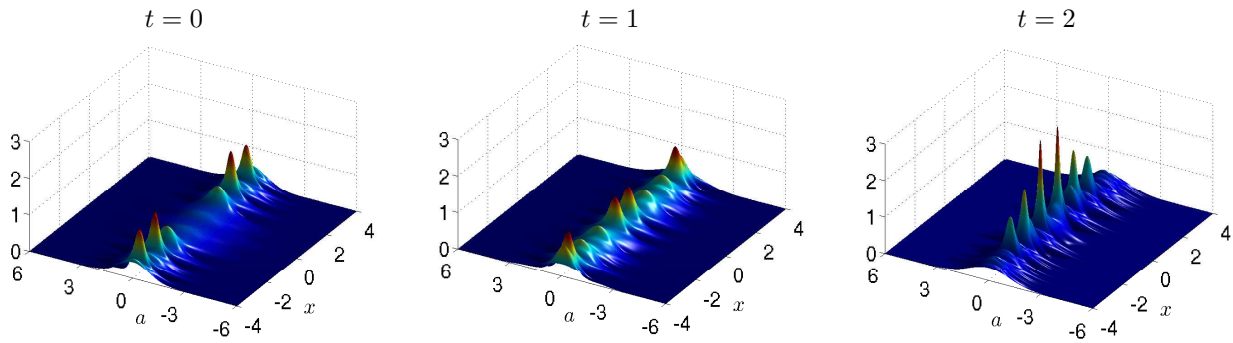


FIG. 1: Probability density function $p_{\Psi(x,t)}^{(a)}$ of the random wave at different time instants.

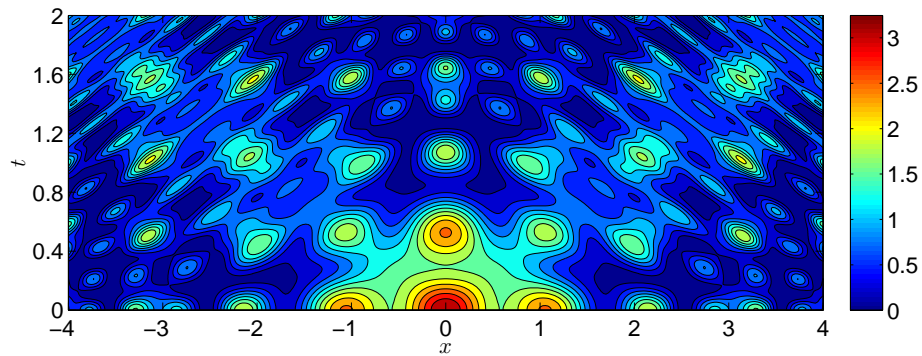


FIG. 2: Temporal evolution of the wave variance.

random variables, we can apply the convolution theorem and obtain the probability density function of the random wave ψ , as

$$p_{\Psi(x,t)}^{(a)} = \frac{1}{\sqrt{2\pi}|\sigma(x,t)|} e^{-a^2/[2\sigma(x,t)^2]}, \quad \sigma(x,t)^2 = G_{11}^2 + G_{12}^2 + G_{13}^2, \quad (65)$$

- 1 where G_{1i} are given in Eq. (56). This probability density is shown in figure 1 at different time instants for $U = 2$ and
- 2 an initial condition defined in terms of the functions

$$h_1(x) = \frac{3}{2}e^{-x^2/2}, \quad h_2(x) = \sin(x^2), \quad h_3(x) = \cos(3x). \quad (66)$$

- 3 According to Eq. (65) the mean wave is zero, while the variance $\sigma(x,t)^2$ evolves in time as shown in figure 2.

6.1 Analytical verification of some differential constraints

It is easy to prove that the probability density function (63) verifies the simple differential constraints (24) and (27), hereafter rewritten for convenience

$$\frac{\partial p_{\Psi\Psi_t\Psi_x}^{(a,b,c)}}{\partial x} = -c \frac{\partial p_{\Psi\Psi_t\Psi_x}^{(a,b,c)}}{\partial a}, \quad \frac{\partial p_{\Psi\Psi_t\Psi_x}^{(a,b,c)}}{\partial t} = -b \frac{\partial p_{\Psi\Psi_t\Psi_x}^{(a,b,c)}}{\partial a}.$$

To this end, let us first notice that the limit partial derivatives of the Jacobian determinant (61) with respect to x and t are identically zero, i.e. we have

$$\frac{\partial J}{\partial x} = 0, \quad \frac{\partial J}{\partial t} = 0. \quad (67)$$

In fact, by using the definitions (57) and (58) we obtain

$$\lim_{\substack{t', t'' \rightarrow t \\ x', x'' \rightarrow x}} \frac{\partial J}{\partial x} = G_{31} (G_{22}G_{33} - G_{32}G_{23}) + G_{32} (G_{31}G_{23} - G_{21}G_{33}) + G_{33} (G_{21}G_{32} - G_{31}G_{22}) = 0, \quad (68)$$

$$\lim_{\substack{t', t'' \rightarrow t \\ x', x'' \rightarrow x}} \frac{\partial J}{\partial t} = G_{21} (G_{22}G_{33} - G_{32}G_{23}) + G_{22} (G_{31}G_{23} - G_{21}G_{33}) + G_{23} (G_{21}G_{32} - G_{31}G_{22}) = 0. \quad (69)$$

At this point we can calculate the limit derivatives of the probability density (63) with respect to x and t

$$\frac{\partial p_{\Psi\Psi_t\Psi_x}^{(a,b,c)}}{\partial x} = -\frac{1}{J|J|} \frac{\partial J}{\partial x} p_{\xi_1 \xi_2 \xi_3}^{(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)} + \frac{1}{|J|} \frac{\partial p_{\xi_1 \xi_2 \xi_3}^{(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)}}{\partial \hat{\alpha}_i} \frac{\partial \hat{\alpha}_i}{\partial x} = \frac{1}{|J|} \frac{\partial p_{\xi_1 \xi_2 \xi_3}^{(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)}}{\partial \hat{\alpha}_i} \frac{\partial \hat{\alpha}_i}{\partial x}, \quad (70)$$

$$\frac{\partial p_{\Psi\Psi_t\Psi_x}^{(a,b,c)}}{\partial t} = -\frac{1}{J|J|} \frac{\partial J}{\partial t} p_{\xi_1 \xi_2 \xi_3}^{(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)} + \frac{1}{|J|} \frac{\partial p_{\xi_1 \xi_2 \xi_3}^{(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)}}{\partial \hat{\alpha}_i} \frac{\partial \hat{\alpha}_i}{\partial t} = \frac{1}{|J|} \frac{\partial p_{\xi_1 \xi_2 \xi_3}^{(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)}}{\partial \hat{\alpha}_i} \frac{\partial \hat{\alpha}_i}{\partial t}, \quad (71)$$

where we have used Eqs. (67). From Eqs. (64), (67) and the definitions (57) and (58) it follows that

$$\frac{\partial \hat{\alpha}_i}{\partial x} = -\frac{1}{J} H_{i1} c, \quad \frac{\partial \hat{\alpha}_i}{\partial t} = -\frac{1}{J} H_{i1} b. \quad (72)$$

Substituting these results back into (70) and (71) we obtain

$$\frac{\partial p_{\Psi\Psi_t\Psi_x}^{(a,b,c)}}{\partial x} = -\frac{c}{J|J|} \frac{\partial p_{\xi_1 \xi_2 \xi_3}^{(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)}}{\partial \hat{\alpha}_i} H_{i1}, \quad \frac{\partial p_{\Psi\Psi_t\Psi_x}^{(a,b,c)}}{\partial t} = -\frac{b}{J|J|} \frac{\partial p_{\xi_1 \xi_2 \xi_3}^{(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)}}{\partial \hat{\alpha}_i} H_{i1}. \quad (73)$$

At the same time we notice that

$$-c \frac{\partial p_{\psi_t \psi_x}^{(a,b,c)}}{\partial a} = -\frac{c}{J|J|} \frac{\partial p_{\xi_1 \xi_2 \xi_3}^{(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)}}{\partial \hat{\alpha}_i} \frac{\partial \hat{\alpha}_i}{\partial a} = -\frac{c}{J|J|} \frac{\partial p_{\xi_1 \xi_2 \xi_3}^{(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)}}{\partial \hat{\alpha}_i} H_{i1}, \quad (74)$$

$$-b \frac{\partial p_{\psi_t \psi_x}^{(a,b,c)}}{\partial a} = -\frac{b}{J|J|} \frac{\partial p_{\xi_1 \xi_2 \xi_3}^{(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)}}{\partial \hat{\alpha}_i} \frac{\partial \hat{\alpha}_i}{\partial a} = -\frac{b}{J|J|} \frac{\partial p_{\xi_1 \xi_2 \xi_3}^{(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)}}{\partial \hat{\alpha}_i} H_{i1}, \quad (75)$$

1 and therefore the differential constraints (24) and (27) are identically satisfied (simply compare Eqs. (74) and (75)
 2 with Eqs. (73)). Similarly, it is possible to show analytically that higher-order constraints such as (33), (12) or (31) are
 3 identically satisfied by the joint probability density function (63). We conclude this section by emphasizing that there
 4 exist many joint densities satisfying a single differential constraints, e.g. Eq. (12), for the same boundary and initial
 5 conditions. As an example, simply consider the family of probability densities in the form (63) where the functions $\hat{\alpha}_i$
 6 are translated by rather arbitrary functions $\chi_i(x', x'', t', t'')$ defined for $x', x'' \in \mathbb{R}$, $t', t'' \geq t_0$ and satisfying $\chi_i = 0$
 7 for $t' = t_0$ or $t'' = t_0$. This clearly shows that a boundary value problem for Eq. (12) alone is ill-posed, i.e. it admits
 8 an infinite number of solutions.

9 7. SUMMARY

10 We have determined and discussed new differential constraints satisfied by the joint probability density function of
 11 arbitrary random waves governed by Eq. (1) for random boundary conditions and random initial conditions. These
 12 differential constraints involve unusual limit partial differential operators, and they can be classified into two main
 13 groups: the first one depends on the stochastic wave equation while the second one includes a set of intrinsic relations
 14 determined by the structure of the joint probability density function of the wave and its derivatives. We have shown
 15 that in those cases where an evolution equation for the probability density function exists, e.g. for first-order nonlinear
 16 stochastic PDEs, the set of differential constraints is complete, i.e., it allows to determine the probability density
 17 function of the solution. We have applied the new theory to a random wave process in a one dimensional spatial
 18 domain and we have shown analytically that the differential constraints for the joint probability density function
 19 are identically satisfied. In addition, we have argued that a boundary value problem involving only one differential
 20 constraint is, in general, not well posed, i.e., it admits an infinite number of solutions. This brings us back to the
 21 fundamental question we have addressed at the beginning of this paper: *how many differential constraints do we need*
 22 *in order to have enough information to compute the probability density function of the system uniquely?* Most likely
 23 the full set of first-order differential constraints, which has been proved to be complete for first-order stochastic PDEs,
 24 already has the necessary information. However, a rigorous proof is still lacking. We conclude by emphasizing that

1 the sequence of steps that led us to formulate the set of differential constraints presented in this paper can be repeated
 2 with some modifications for other linear and nonlinear stochastic PDEs. In other words, we have provided a general
 3 methodology that allows to obtain an ensemble of conditions, in the form of partial differential equations, for the
 4 probability density function of the solution to an arbitrary stochastic PDE.

5 ACKNOWLEDGMENTS

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8 APPENDIX A. FUNCTIONAL REPRESENTATION OF THE PROBABILITY DENSITY FUNCTION OF THE 9 SOLUTION TO STOCHASTIC PDES

10 Let us consider a physical system described in terms of partial differential equations subject to uncertain initial con-
 11 ditions, boundary conditions, physical parameters or external forcing terms. The solution to these types of boundary
 12 value problems is a random field whose regularity properties in space and time are strongly related to the type of
 13 nonlinearities appearing in the equations as well as on the statistical properties of the random input processes. In this
 14 paper we assume that the probability density function of the solution field exists. In order to fix ideas, let us consider
 15 the advection-diffusion equation

$$\frac{\partial \psi}{\partial t} + \xi(\omega) \frac{\partial \psi}{\partial x} = \nu \frac{\partial^2 \psi}{\partial x^2}, \quad (\text{A.1})$$

16 with deterministic boundary and initial conditions. The parameter $\xi(\omega)$ is assumed to be a random variable with
 17 known probability density function. The solution to Eq. (A.1) is a random field depending on the random variable
 18 $\xi(\omega)$ in a possibly nonlinear way. We shall denote such a functional dependence as $\psi(x, t; [\xi])$. The joint probability
 19 density of $\psi(x, t; [\xi])$ and ξ , i.e. the solution field at the space-time location (x, t) and the random variable ξ , admits
 20 the following integral representation (see Eq. (16) in [30])

$$p_{\psi(x,t)\xi}^{(a,b)} = \langle \delta(a - \psi(x, t; [\xi])) \delta(b - \xi) \rangle, \quad a, b \in \mathbb{R}. \quad (\text{A.2})$$

21 The average operator $\langle \cdot \rangle$, in this particular case, is defined as a simple integral with respect to the probability density
 22 of $\xi(\omega)$, i.e.

$$p_{\psi(x,t)\xi}^{(a,b)} = \int_{-\infty}^{\infty} \delta(a - \psi(x, t; [z])) \delta(b - z) p_{\xi}^{(z)} dz, \quad (\text{A.3})$$

1 where $p_{\xi}^{(z)}$ denotes the probability density of ξ , which could be compactly supported (e.g. a uniform distribution). The
 2 support of the probability density function $p_{\psi(x,t)\xi}^{(a,b)}$ is determined by the nonlinear transformation $\xi \rightarrow \psi(x, t; [\xi])$
 3 appearing within the delta function $\delta(a - \psi(x, t; [\xi]))$ (see, e.g., Ch. 3 in [31]). The simple representation (A.3) can
 4 be easily generalized to infinite dimensional random input processes. To this end, let us examine the case where the
 5 scalar field ψ is advected by a random velocity field U according to the equation

$$\frac{\partial \psi}{\partial t} + U(x, t; \omega) \frac{\partial \psi}{\partial x} = \nu \frac{\partial^2 \psi}{\partial x^2}, \quad (\text{A.4})$$

6 for some deterministic initial condition and boundary conditions. Disregarding the particular form of the random field
 7 $U(x, t; \omega)$, let us consider its collocation representation for a given discretization of the space-time domain. This
 8 gives us a certain number of random variables $\{U(x_i, t_j; \omega)\}$ ($i = 1, \dots, N, j = 1, \dots, M$). The random field ψ
 9 solving Eq. (A.4) at each one of these locations is, in general, a nonlinear function of *all* the variables $\{U(x_i, t_j; \omega)\}$.
 10 In order to see this, it is sufficient to write an explicit finite difference numerical scheme of Eq. (A.4). The joint
 11 probability density of the solution field ψ at (x_i, t_j) and the driving field U at (x_n, t_m) , admits the following integral
 12 representation

$$p_{\psi(x_i, t_j)U(x_n, t_m)}^{(a,b)} = \langle \delta(a - \psi(x_i, t_j; [U(x_1, t_1), \dots, U(x_N, t_M)])) \delta(b - U(x_n, t_m)) \rangle, \quad (\text{A.5})$$

where the average, in this case, is with respect to the joint probability of all the random variables $\{U(x_n, t_m; \omega)\}$
 ($n = 1, \dots, N, m = 1, \dots, M$). The notation $\psi(x_i, t_j; [U(x_1, t_1), \dots, U(x_N, t_M)])$ emphasizes that the solution field
 $\psi(x_i, t_j)$ is, in general, a nonlinear function of all the random variables $\{U(x_n, t_m; \omega)\}$. If we send the number of
 these variables to infinity, i.e., we refine the space-time mesh to the continuum level, we obtain a functional integral
 representation of the joint probability density

$$\begin{aligned} p_{\psi(x,t)U(x',t')}^{(a,b)} &= \langle \delta(a - \psi(x, t; [U])) \delta(b - U(x', t')) \rangle \\ &= \int \mathcal{D}[U] W[U] \delta(a - \psi(x, t; [U])) \delta(b - U(x', t')), \end{aligned} \quad (\text{A.6})$$

13 where $W[U]$ is the probability density functional of random field $U(x, t; \omega)$ and $\mathcal{D}[U]$ is the usual functional integral
 14 measure [13–15]. Depending on the particular equation of motion, we will need to consider different joint probability
 15 density functions, e.g., the joint probability of a field and its derivatives with respect to space variables. The functional
 16 representation described above allows us to deal with these different situations in a very practical way. For instance,

1 the joint probability density of a field $\psi(x, y, t)$ and its first-order spatial derivatives at different space-time locations
 2 can be represented as

$$p_{\psi(x,y,t)\psi_x(x',y',t')\psi_y(x'',y'',t'')}^{(a,b,c)} = \langle \delta(a - \psi(x, y, t))\delta(b - \psi_x(x', y', t'))\delta(c - \psi_y(x'', y'', t'')) \rangle, \quad (\text{A.7})$$

3 where, for notational convenience, we have denoted by $\psi_x \stackrel{\text{def}}{=} \partial\psi/\partial x$, $\psi_y \stackrel{\text{def}}{=} \partial\psi/\partial y$ and we have omitted the
 4 functional dependence on the random input variables within each field. Similarly, the joint probability of $\psi(x, y, t)$ at
 5 two different spatial locations is

$$p_{\psi(x,y,t)\psi(x',y',t)}^{(a,b)} = \langle \delta(a - \psi(x, y, t))\delta(b - \psi(x', y', t)) \rangle. \quad (\text{A.8})$$

6 In order to lighten the notation further, sometimes we will drop the subscripts indicating the space-time variables and
 7 write, for instance

$$p_{\psi\psi'_x}^{(a,b)} = \langle \delta(a - \psi(x, y, t))\delta(b - \psi'_x(x', y', t')) \rangle, \quad (\text{A.9})$$

8 or even more compactly

$$p_{\psi\psi'_x}^{(a,b)} = \langle \delta(a - \psi)\delta(b - \psi'_x) \rangle. \quad (\text{A.10})$$

9 APPENDIX A.1 Representation of derivatives

The differentiation of the probability density function with respect to space and time variables involves generalized derivatives of the Dirac delta function and it can be carried out in a systematic manner. To this end, let us consider Eq. (A.2) and define the following linear functional

$$\int_{-\infty}^{\infty} p_{\psi(x,t)}^{(a)} \rho(a) da = \langle \int_{-\infty}^{\infty} \delta(a - \psi(x, t)) \rho(a) da \rangle = \langle \rho(\psi) \rangle, \quad (\text{A.11})$$

where $\rho(a)$ is a continuously differentiable and compactly supported function. A differentiation of Eq. (A.11) with respect to t gives

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\partial p_{\Psi(x,t)}^{(a)}}{\partial t} \rho(a) da &= \langle \Psi_t \frac{\partial \rho}{\partial \Psi} \rangle \\ &= \langle \Psi_t \int_{-\infty}^{\infty} \frac{\partial \rho}{\partial a} \delta(a - \Psi(x,t)) da \rangle \\ &= \int_{-\infty}^{\infty} -\frac{\partial}{\partial a} \langle \Psi_t \delta(a - \Psi(x,t)) \rangle \rho(a) da. \end{aligned} \quad (\text{A.12})$$

This equation holds for an arbitrary $\rho(a)$ and therefore we have the identity

$$\frac{\partial p_{\Psi(x,t)}^{(a)}}{\partial t} = -\frac{\partial}{\partial a} \langle \delta(a - \Psi(x,t)) \Psi_t(x,t) \rangle. \quad (\text{A.13})$$

Similarly,

$$\frac{\partial p_{\Psi(x,t)}^{(a)}}{\partial x} = -\frac{\partial}{\partial a} \langle \delta(a - \Psi(x,t)) \Psi_x(x,t) \rangle. \quad (\text{A.14})$$

Straightforward extensions of these results allow us to compute derivatives of joint probability density functions involving more fields, e.g., $\Psi(x, t)$ and its first-order spatial derivative $\Psi_x(x, t)$. For instance, we have

$$\frac{\partial p_{\Psi\Psi_x}^{(a,b)}}{\partial t} = -\frac{\partial}{\partial a} \langle \delta(a - \Psi(x,t)) \delta(b - \Psi_x(x,t)) \Psi_t(x,t) \rangle - \frac{\partial}{\partial b} \langle \delta(a - \Psi(x,t)) \delta(b - \Psi_x(x,t)) \Psi_{tx}(x,t) \rangle, \quad (\text{A.15})$$

¹ where $\Psi_{tx} \stackrel{\text{def}}{=} \partial^2 \Psi / \partial t \partial x$.

² APPENDIX A.2 Representation of averages

In this section we determine important formulae to compute the average of a product between Dirac delta functions and various fields. To this end, let us first consider the average $\langle \delta(a - \Psi) \Psi \rangle$. By applying well-known properties of Dirac delta functions it can be shown that

$$\langle \delta(a - \Psi) \Psi \rangle = a p_{\Psi(x,t)}^{(a)}. \quad (\text{A.16})$$

This result is a multidimensional extension of the following trivial identity that holds for one random variable ξ (with probability density $p_\xi^{(z)}$) and a nonlinear function $g(\xi)$ (see, e.g., Ch. 3 of [31] or [30])

$$\int_{-\infty}^{\infty} \delta(a - g(z))g(z)p_\xi^{(z)} dz = \sum_n \frac{1}{|g'(\hat{z}_n)|} \int_{-\infty}^{\infty} \delta(z - \hat{z}_n)g(z)p_\xi^{(z)} dz = \sum_n \frac{g(\hat{z}_n)p_\xi^{(\hat{z}_n)}}{|g'(\hat{z}_n)|}, \quad (\text{A.17})$$

where $\hat{z}_n = g^{-1}(a)$ are roots of $g(z) = a$. Since $g(\hat{z}_n) = g(g^{-1}(a)) = a$, from Eq. (A.17) it follows that

$$\int_{-\infty}^{\infty} \delta(a - g(z))g(z)p_\xi^{(z)} dz = a \sum_n \frac{p_\xi^{(\hat{z}_n)}}{|g'(\hat{z}_n)|} = a \int_{-\infty}^{\infty} \delta(a - g(z))p_\xi^{(z)} dz, \quad (\text{A.18})$$

which is the one-dimensional version of Eq. (A.16). Similarly, one can show that

$$\langle \delta(a - \psi)\delta(b - \psi_x)\psi \rangle = ap_{\psi\psi_x}^{(a,b)}, \quad (\text{A.19})$$

$$\langle \delta(a - \psi)\delta(b - \psi_x)\psi_x \rangle = bp_{\psi\psi_x}^{(a,b)}, \quad (\text{A.20})$$

and, more generally, that

$$\langle \delta(a - \psi)\delta(b - \psi_x)h(x, t, \psi, \psi_x) \rangle = h(x, t, a, b)p_{\psi\psi_x}^{(a,b)}, \quad (\text{A.21})$$

1 where, for the purposes of the present paper, $h(\psi, \psi_x, x, t)$ is any continuous function of ψ, ψ_x, x and t . The result
2 (A.21) can be generalized even further to averages involving a products of functions in the form

$$\langle \delta(a - \psi)\delta(b - \psi_x)h(x, t, \psi, \psi_x)g(x, t, \psi_{xx}, \psi_{xt}, \psi_{tt}, \dots) \rangle = h(x, t, a, b)\langle \delta(a - \psi)\delta(b - \psi_x)g(x, t, \psi_{xx}, \psi_{xt}, \psi_{tt}, \dots) \rangle. \quad (\text{A.22})$$

3 In short, the general rule is: *we are allowed to take out of the average all those functions involving fields for which*
4 *we have available a Dirac delta.* As an example, if ψ is a time-dependent random field in a two-dimensional spatial
5 domain we have

$$\langle \delta(a - \psi)\delta(b - \psi_x)\delta(c - \psi_y)e^{-(x^2+y^2)} \sin(\psi)\psi_x\psi_y^2\psi_{xx} \rangle = e^{-(x^2+y^2)} \sin(a)bc^2 \langle \delta(a - \psi)\delta(b - \psi_x)\delta(c - \psi_y)\psi_{xx} \rangle. \quad (\text{A.23})$$

1 APPENDIX B. HOPF CHARACTERISTIC FUNCTIONAL APPROACH

2 In this appendix we employ a Hopf characteristic functional approach [19, 32–34] to derive the differential constraint
 3 (12) for the probability density function of one-dimensional random waves satisfying Eq. (2). This allows us to show
 4 how the differential constraints for the probability density function can be obtained from general principles. The joint
 5 Hopf characteristic functional of the wave ψ and its derivatives with respect to space and time is defined as

$$F[\alpha, \beta, \gamma] \stackrel{\text{def}}{=} \langle e^{\mathcal{Z}[\alpha, \beta, \gamma]} \rangle, \quad (\text{B.1})$$

where

$$\begin{aligned} \mathcal{Z}[\alpha, \beta, \gamma] = & i \int_T \int_X \psi(X, \tau; \omega) \alpha(X, \tau) dX d\tau + i \int_T \int_X \psi_t(X, \tau; \omega) \beta(X, \tau) dX d\tau \\ & + i \int_T \int_X \psi_x(X, \tau; \omega) \gamma(X, \tau) dX d\tau \end{aligned} \quad (\text{B.2})$$

6 and the average operator is a functional integral with respect to the joint probability functional of the random initial
 7 condition and the random boundary conditions. The Volterra functional derivative [35, 36] of Eq. (B.1) with respect to
 8 $\psi(x, t)$, i.e. the Gâteaux differential [37, 38] of $F[\alpha, \beta, \gamma]$ with respect to α evaluated at $z(X, t) = \delta(t - \tau)\delta(x - X)$,
 9 is

$$\frac{\delta F[\alpha, \beta, \gamma]}{\delta \psi(x, t)} \stackrel{\text{def}}{=} i \langle \psi(x, t; \omega) e^{\mathcal{Z}[\alpha, \beta, \gamma]} \rangle. \quad (\text{B.3})$$

10 Similarly, the functional derivatives of (B.1) with respect to $\psi_t(x, t)$ and $\psi_x(x, t)$ are

$$\frac{\delta F[\alpha, \beta, \gamma]}{\delta \psi_t(x, t)} = i \langle \psi_t(x, t; \omega) e^{\mathcal{Z}[\alpha, \beta, \gamma]} \rangle, \quad (\text{B.4})$$

11

$$\frac{\delta F[\alpha, \beta, \gamma]}{\delta \psi_x(x, t)} = i \langle \psi_x(x, t; \omega) e^{\mathcal{Z}[\alpha, \beta, \gamma]} \rangle. \quad (\text{B.5})$$

12 Note that, by construction,

$$\frac{\partial}{\partial x} \left(\frac{\delta F[\alpha, \beta, \gamma]}{\delta \psi(x, t)} \right) = \frac{\delta F[\alpha, \beta, \gamma]}{\delta \psi_x(x, t)}, \quad \text{and} \quad \frac{\partial}{\partial t} \left(\frac{\delta F[\alpha, \beta, \gamma]}{\delta \psi(x, t)} \right) = \frac{\delta F[\alpha, \beta, \gamma]}{\delta \psi_t(x, t)}. \quad (\text{B.6})$$

1 If we differentiate (B.3) twice with respect to t and x

$$\frac{\partial^2}{\partial t^2} \left(\frac{\delta F[\alpha, \beta, \gamma]}{\delta \psi(x, t)} \right) = i \langle \psi_{tt}(x, t; \omega) e^{\mathcal{Z}[\alpha, \beta, \gamma]} \rangle, \quad (\text{B.7})$$

2

$$\frac{\partial^2}{\partial x^2} \left(\frac{\delta F[\alpha, \beta, \gamma]}{\delta \psi(x, t)} \right) = i \langle \psi_{xx}(x, t; \omega) e^{\mathcal{Z}[\alpha, \beta, \gamma]} \rangle. \quad (\text{B.8})$$

3 and we take into account Eq. (2), we easily determine the following functional differential equation governing for the
4 dynamics of the Hopf characteristic functional

$$\frac{\partial^2}{\partial t^2} \left(\frac{\delta F[\alpha, \beta, \gamma]}{\delta \psi(x, t)} \right) = U^2 \frac{\partial^2}{\partial x^2} \left(\frac{\delta F[\alpha, \beta, \gamma]}{\delta \psi(x, t)} \right). \quad (\text{B.9})$$

We notice that up to this point we have made no use of the fields β and γ appearing in (B.1). Indeed, we can safely set $\beta = 0$ and $\gamma = 0$ in Eqs. (B.2) and (B.1) and determine exactly the same evolution (B.9). The need for β and γ clearly appears when one tries to extract a pointwise equation for the probability density function of the random wave from Eq. (B.9). To this end, let us first remark that the functional differential equation (B.9) holds for *arbitrary* test functions α , β and γ . In particular, it holds for

$$\alpha^+(X, \tau) = a\delta(t - \tau)\delta(x - X), \quad (\text{B.10})$$

$$\beta^+(X, \tau) = b\delta(t' - \tau)\delta(x' - X), \quad (\text{B.11})$$

$$\gamma^+(X, \tau) = c\delta(t'' - \tau)\delta(x'' - X). \quad (\text{B.12})$$

5 An evaluation of Eq. (B.9) for $\alpha = \alpha^+$, $\beta = \beta^+$ and $\gamma = \gamma^+$ yields the condition

$$i \langle (\psi_{tt}(x, t; \omega) - U^2 \psi_{xx}(x, t; \omega)) e^{i\psi(x, t; \omega)a + i\psi_t(x', t'; \omega)b + i\psi_x(x'', t''; \omega)c} \rangle = 0. \quad (\text{B.13})$$

6 This condition is equivalent to a differential constraint involving the *joint characteristic function* of the wave and its
7 derivatives

$$\Phi_{\psi \psi'_x \psi''_t}^{(a, b, c)} \stackrel{\text{def}}{=} \langle e^{i\psi(x, t; \omega)a + i\psi_t(x', t'; \omega)b + i\psi_x(x'', t''; \omega)c} \rangle, \quad (\text{B.14})$$

In order to see this, let us notice that

$$\begin{aligned} \frac{\partial^2 \Phi_{\Psi \Psi'_x \Psi''_t}^{(a,b,c)}}{\partial t^2} &= ia \langle \Psi_{tt}(x, t; \omega) e^{i\Psi(x,t;\omega)a + i\Psi_t(x',t';\omega)b + i\Psi_x(x'',t'';\omega)c} \rangle \\ &\quad - a^2 \langle \Psi_t(x, t; \omega)^2 e^{i\Psi(x,t;\omega)a + i\Psi_t(x',t';\omega)b + i\Psi_x(x'',t'';\omega)c} \rangle. \end{aligned} \tag{B.15}$$

If we take the limit of this expression for $(t', t'') \rightarrow t$ and $(x', x'') \rightarrow x$ we obtain

$$\frac{\partial^2 \Phi_{\Psi \Psi_x \Psi_t}^{(a,b,c)}}{\partial t^2} = ia \langle \Psi_{tt}(x, t; \omega) e^{i\Psi(x,t;\omega)a + i\Psi_t(x,t;\omega)b + i\Psi_x(x,t;\omega)c} \rangle + a^2 \frac{\partial^2 \Phi_{\Psi \Psi_x \Psi_t}^{(a,b,c)}}{\partial b^2}, \tag{B.16}$$

where ∂ denotes the limit partial derivative operator. Similarly,

$$\frac{\partial^2 \Phi_{\Psi \Psi_x \Psi_t}^{(a,b,c)}}{\partial x^2} = ia \langle \Psi_{xx}(x, t; \omega) e^{i\Psi(x,t;\omega)a + i\Psi_x(x,t;\omega)b + i\Psi_t(x,t;\omega)c} \rangle + a^2 \frac{\partial^2 \Phi_{\Psi \Psi_x \Psi_t}^{(a,b,c)}}{\partial c^2}. \tag{B.17}$$

1 At this point we can combine the results above to obtain the following differential constraint for the joint characteristic
 2 function

$$\frac{\partial^2 \Phi_{\Psi \Psi_x \Psi_t}^{(a,b,c)}}{\partial t^2} - a^2 \frac{\partial^2 \Phi_{\Psi \Psi_x \Psi_t}^{(a,b,c)}}{\partial b^2} = U^2 \frac{\partial^2 \Phi_{\Psi \Psi_x \Psi_t}^{(a,b,c)}}{\partial x^2} - U^2 a^2 \frac{\partial^2 \Phi_{\Psi \Psi_x \Psi_t}^{(a,b,c)}}{\partial c^2}. \tag{B.18}$$

3 The inverse Fourier transform of this relation with respect to a, b and c gives us exactly Eq. (12). In order to prove
 4 this, we simply recall the definition of $p_{\Psi \Psi_x \Psi_t}^{(a,b,c)}$ as the inverse Fourier transform of the characteristic function $\Phi_{\Psi \Psi_x \Psi_t}^{(a,b,c)}$

$$p_{\Psi \Psi_x \Psi_t}^{(a,b,c)} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ia\alpha - ib\beta - ic\eta} \Phi_{\Psi \Psi_x \Psi_t}^{(\alpha,\beta,\eta)} d\alpha d\beta d\eta, \tag{B.19}$$

and two simple relations arising from Fourier transformation theory of a one dimensional function $f(x)$

$$\int_{-\infty}^{\infty} e^{-iax} \frac{d^n f(x)}{dx^n} dx = (ia)^n \int_{-\infty}^{\infty} e^{-iax} f(x) dx, \tag{B.20}$$

$$\int_{-\infty}^{\infty} e^{-iax} x^n f(x) dx = i^n \frac{d^n}{da^n} \int_{-\infty}^{\infty} e^{-iax} f(x) dx. \tag{B.21}$$

5 APPENDIX C. PERTURBATION EXPANSIONS OF DIFFERENTIAL CONSTRAINTS

In this appendix we show that the differential constraint (12) can be expanded in a perturbation series leading to an equation governing the local behavior of the probability density function in the neighborhood of the hyperplane

$x'' = x' = x$, $t'' = t' = t$. To this end, let us consider again Eq. (8)

$$\begin{aligned} \frac{\partial^2 p_{\Psi \Psi'_t \Psi''_x}^{(a,b,c)}}{\partial t^2} - U^2 \frac{\partial^2 p_{\Psi \Psi'_t \Psi''_x}^{(a,b,c)}}{\partial x^2} &= \frac{\partial^2}{\partial a^2} \langle \delta(a - \Psi) \Psi_t^2 \delta(b - \Psi'_t) \delta(c - \Psi''_x) \rangle \\ &+ U \frac{\partial^2}{\partial a^2} \langle \delta(a - \Psi) \Psi_x^2 \delta(b - \Psi'_t) \delta(c - \Psi''_x) \rangle \end{aligned} \quad (\text{C.1})$$

and look for an approximation to both averages appearing at the right hand side. The simplest one consists in expanding Ψ_t^2 and Ψ_x^2 in a first-order Taylor series around (x', t') and (x'', t'') , respectively, i.e.

$$\Psi_t^2 = \Psi_t'^2 + 2\Psi_t' \Psi'_{tx}(x - x') + 2\Psi_t' \Psi'_{tt}(t - t') + \dots, \quad (\text{C.2})$$

$$\Psi_x^2 = \Psi_x''^2 + 2\Psi_x'' \Psi''_{xx}(x - x'') + 2\Psi_x'' \Psi''_{xt}(t - t'') + \dots. \quad (\text{C.3})$$

A substitution of Eqs. (C.2) and (C.3) into the averages in Eq. (C.1) yields

$$\begin{aligned} \langle \delta(a - \Psi) \Psi_t^2 \delta(b - \Psi'_t) \delta(c - \Psi''_x) \rangle &= \langle \delta(a - \Psi) \Psi_t'^2 \delta(b - \Psi'_t) \delta(c - \Psi''_x) \rangle \\ &+ 2(x - x') \langle \delta(a - \Psi) \Psi_t' \Psi'_{tx} \delta(b - \Psi'_t) \delta(c - \Psi''_x) \rangle \\ &+ 2(t - t') \langle \delta(a - \Psi) \Psi_t' \Psi'_{tt} \delta(b - \Psi'_t) \delta(c - \Psi''_x) \rangle, \end{aligned} \quad (\text{C.4})$$

$$\begin{aligned} \langle \delta(a - \Psi) \Psi_x^2 \delta(b - \Psi'_t) \delta(c - \Psi''_x) \rangle &= \langle \delta(a - \Psi) \Psi_x''^2 \delta(b - \Psi'_t) \delta(c - \Psi''_x) \rangle \\ &+ 2(x - x'') \langle \delta(a - \Psi) \Psi_x'' \Psi''_{xx} \delta(b - \Psi'_t) \delta(c - \Psi''_x) \rangle \\ &+ 2(t - t'') \langle \delta(a - \Psi) \Psi_x'' \Psi''_{xt} \delta(b - \Psi'_t) \delta(c - \Psi''_x) \rangle. \end{aligned} \quad (\text{C.5})$$

At this point we can use the property (A.22) and write Eqs. (C.4) and (C.5) as

$$\langle \delta(a - \Psi) \Psi_t^2 \delta(b - \Psi'_t) \delta(c - \Psi''_x) \rangle = b^2 p_{\Psi \Psi'_t \Psi''_x}^{(a,b,c)} + 2b(x - x') \int_{-\infty}^b \frac{\partial p_{\Psi \Psi'_t \Psi''_x}^{(a,b',c)}}{\partial x'} db' + 2b(t - t') \int_{-\infty}^b \frac{\partial p_{\Psi \Psi'_t \Psi''_x}^{(a,b',c)}}{\partial t'} db', \quad (\text{C.6})$$

$$\langle \delta(a - \Psi) \Psi_x^2 \delta(b - \Psi'_t) \delta(c - \Psi''_x) \rangle = c^2 p_{\Psi \Psi'_t \Psi''_x}^{(a,b,c)} + 2c(x - x'') \int_{-\infty}^c \frac{\partial p_{\Psi \Psi'_t \Psi''_x}^{(a,b,c')}}{\partial x''} dc' + 2c(t - t'') \int_{-\infty}^c \frac{\partial p_{\Psi \Psi'_t \Psi''_x}^{(a,b,c')}}{\partial t''} dc'. \quad (\text{C.7})$$

1 A substitution of Eqs. (C.6) and (C.7) into Eq. (C.1) yields a fourth-order (nine-dimensional) partial differential
2 equation that characterizes locally, i.e. in the neighborhood of the hyperplane $x = x' = x''$, $t = t' = t''$ the evolution
3 of the probability density function of the random wave. Equivalently, we can say that we have determined a local
4 (non-pointwise) realization of the functional differential equation (B.9). An alternative method to obtain this result
5 consists in evaluating the functional differential equation (B.9) for test functions α , β and γ that are very concentrated
6 near specific space-time locations. These types of test functions can be any of those belonging to a delta sequence
7 (see, e.g., appendix A in [39]).

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