Elsevier Editorial System(tm) for Journal of Computational and Applied Mathematics
Manuscript Draft

Manuscript Number:

Title: On mean-square-stability of two-step Maruyama methods for stochastic delay differential equations

Article Type: Research Paper

Section/Category: 65Cxx

Keywords: stochastic modeling; unconditional stability; two-step Maruyama methods

Corresponding Author: Dr. Wanrong Cao, Dr.

Corresponding Author's Institution:

First Author: Wanrong Cao, Dr.

Order of Authors: Wanrong Cao, Dr.; Zhongqiang Zhang

On mean-square-stability of two-step Maruyama methods for stochastic delay differential equations

Wanrong Cao[a,b,†] and Zhongqiang Zhang[b,‡]

a. Department of Mathematics, Southeast University, Nanjing 210096, P.R.China.
b. Division of Applied Mathematics, Brown University, Providence, RI, US 02912.

Abstract

In this paper we study the mean-square-stability of two-step Maruyama methods for stochastic differential equations with time delay. We propose a family of unconditionally mean-square stable schemes and prove stability for a model equation. Numerical results for linear and nonlinear equations show that this family of two-step Maruyama methods exhibit better stability than previous two-step Maruyama methods.

Key words: stochastic modeling, unconditionally stable, two-step Maruyama methods

1 Introduction

Stochastic delay differential equations (SDDEs) have been increasingly used to model the effects of noise and time delay on various types of complex systems, such as delayed visual feedback systems [3], control problems [11, 21], the dynamics of noisy bi-stable systems with delay [22], etc. SDDEs are also used in modeling deseases, for example, epidemic diseases[2], neurological diseases [4], etc, and also in finance SDDEs appear in models of stock markets [12].

1 † Corresponding author. *Email address: wrcao@seu.edu.cn.* This author's work was supported by the NSF of China (No.10901036).
2 ‡*Email address: Zhongqiang_zhang@brown.edu.* This author's work was supported by AIRFORCE MURI.

Some numerical methods and their convergence and stability properties have been established [1, 14, 10, 17, 23] recently, but most of them are on one-step methods. Instead of one-step methods we here focus on stochastic multi-step methods for SDDEs, which have been widely studied for solving stochastic ordinary differential equations (SODEs), i.e. with no time delay.

To extend the multi-step methods for SODEs to those for SDDEs is a non-trivial task and these extensions have not been investigated until recently. For a review of multi-step methods for SODEs, we refer to [13, 19]. Some more recent studies are as follows. In [7], certain stochastic linear multi-step methods are constructed; and mean-square convergence rates are obtained; and consistency conditions in the mean-square-sense are given for two-step Maruyama methods. Ewald and Temam [9] studied the convergence of a stochastic Adams-Bashforth scheme with application to geophysical applications. Adams-type methods for SODEs are also analyzed in [5], where first-order strong convergence conditions are given. For some special SODEs with additive noise, high order multi-step methods have been discussed in [8].

In this paper, we follow [6] and study two-step Maruyama schemes for the scalar equation

$$\begin{aligned}
&\mathrm{d}X(t) = f\Big(t, X(t), X(t-\tau)\Big)\mathrm{d}t + g\Big(t, X(t), X(t-\tau)\Big)\mathrm{d}W(t),\ t \in J,\\
&X(t) = \xi(t), t \in [-\tau, 0],
\end{aligned} \tag{1.1}$$

where τ is a positive fixed delay, $J = [0, T]$, $W(t)$ is a one-dimensional standard Wiener process and the functions $f : J\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $g : J\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$. We note that [6] is perhaps the only work on multi-step methods for SDDEs, wherein multi-step methods are proposed for m-dimensional systems of Itô SDDEs with d driving Wiener processes and multi-delay, and their properties are studied concerning consistency, numerical stability and convergence.

Instead of working with the general SDDE (1.1), we study the lineared test model (1.2), which can shed some lights on the general SDDE (1.1),

$$\begin{aligned}
&\mathrm{d}X(t) = [aX(t) + bX(t-\tau)]\mathrm{d}t + [cX(t) + dX(t-\tau)]\mathrm{d}W(t), t \geq 0,\\
&X(t) = \xi(t), t \in [-\tau, 0],
\end{aligned} \tag{1.2}$$

where $a, b, c, d \in \mathbb{R}$, τ is a positive fixed delay, $W(t)$ is a one-dimensional standard Wiener process and $\xi(t)$ is a $\mathrm{C}([-\tau, 0]; \mathbb{R})$-valued initial segment. In this work, we aim to drive mean-square-stable two-step Maruyama methods for the SDDE (1.2) .

The paper is organized in the following way. In Section 2 we provide some necessary notations and preliminaries on SDDEs, including some properties of analytical solutions to Eq.(1.2). Also, in this section the two-step Maruyama methods and their convergence properties are introduced. In Section 3 we derive a series of unconditionally mean-square-stable two-step Maruyama methods under certain conditions. Section 4 illustrates the mean-square-stability of these two-step Maruyama methods with numerical examples for the test model (1.2) and a nonlinear equation.

2 Notations and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$, which satisfies the usual conditions (increasing and right-continuous; each $\{\mathcal{F}_t\},\ t \geq 0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$).

Let $W(t), t \geq 0$ in Eq.(1.2) be $\mathcal{F}_t$-adapted and independent of $\mathcal{F}_0$. Assume $\xi(t), t \in [-\tau, 0]$ to be $\mathcal{F}_0$-measurable and right continuous, and $E\|\xi\|^2 < \infty$. Here $\|\xi\|$ is defined by $\|\xi\| = \sup_{-\tau\leq t\leq 0} |\xi(t)|$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}$. Throughout the paper, Eqs.(1.1) and (1.2) are interpreted in the Itô sense. Under these assumptions, Eq.(1.2) has a unique strong solution $X(t) : [-\tau, +\infty) \to \mathbb{R}$, which satisfies Eq.(1.2) and $X(t)$ is a measurable, sample-continuous and $\mathcal{F}_t$-adapted process; see [15, 20].

Lemma 1 *If*

$$a < -|b| - (|c| + |d|)^2, \tag{2.1}$$

then the solution of Eq.(1.2) is mean-square-stable, that is

$$\lim_{t\to\infty} E|X(t)|^2 = 0. \tag{2.2}$$

By Corollary 3.2 in [16], the proof of this lemma is not difficult.

Applying the two-step Maruyama methods to Eq.(1.1) leads to the following

$$\begin{aligned}\sum_{j=-1}^{1} \alpha_j X_{i-j} = h \sum_{j=-1}^{1} \beta_j f(t_{i-j}, X_{i-j}, X_{i-m-j}) \\ + \sum_{j=0}^{1} \gamma_j g(t_{i-j}, X_{i-j}, X_{i-m-j})\Delta W_{i-j},\ i = 2, 3, \cdots, N,\end{aligned} \tag{2.3}$$

where $\alpha_j, \beta_j, \gamma_j, (j \in \{-1, 0, 1\})$ are parameters; $h > 0$ is the stepsize in time which satisfies $\tau = mh$ for a positive integer m, and $t_n = nh$, $N = T/h$. The increments $\Delta W_i := W(t_{i+1}) - W(t_i)$, are independent $\mathcal{N}(0, h)$-distributed

Gaussian random variables. Suppose that X_i is $\mathcal{F}_{t_i}$-measurable at the mesh-point t_i. Then X_i is an approximation to $X(t_i)$, where for $i \le 0$, X_i are given by the initial function.

Definition 2 *The function $u : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be uniform Lipschitz continuous if there exists a positive constant L_u, such that the function u satisfies*

$$|u(t, x_1, x_2) - u(t, y_1, y_2)| \le L_u(|x_1 - y_1| + |x_2 - y_2|) \tag{2.4}$$

for every x_1, x_2, y_1, $y_2 \in \mathbb{R}$ and $t \ge 0$,

When there exists a positive constant K, such that

$$|u(t, x, y)| \le K(1 + x^2 + y^2)^{\frac{1}{2}} \tag{2.5}$$

for x, $y \in \mathbb{R}$ and $t \ge 0$, we say that u satisfies a linear growth condition.

The characteristic polynomial of (2.3) is given by

$$\rho(\lambda) = \alpha_{-1}\lambda^2 + \alpha_0\lambda + \alpha_1.$$

Definition 3 *The method (2.3) is said to fulfill Dahlquist's root condition if i) the roots of $\rho(\lambda)$ lie on or within the unit circle; ii) the roots on the unit circle are simple.*

Here are the consistency and convergence properties of numerical methods (2.3).

Lemma 4 ([6]) *Assume that*

- *the coefficients f and g of the SDDE (1.1) are Lipschitz continuous in the sense of (2.4)and have first-order continuous partial derivatives with respect to the first variable and second-order continuous partial derivatives with respect to the second and third variables;*
- *these partial derivatives satisfy the linear growth condition (2.5);*
- *the coefficients of the stochastic linear two-step Maruyama scheme (2.3) satisfy Dahlquist's root condition,*
- *and the consistency conditions*

$$\sum_{j=-1}^{1} \alpha_j = 0,\ 2\alpha_{-1} + \alpha_0 = \sum_{j=-1}^{1} \beta_j,\ \alpha_{-1} = \gamma_0,\ \alpha_{-1} + \alpha_0 = \gamma_1. \tag{2.6}$$

Then the global error of the scheme (2.3) applied to (1.1) satisfies

$$\max_{i=2,\cdots,N} |X(t_i) - X_i| = O(h^{1/2}).$$

3 Mean-square-stability of the two-step Maruyama methods

In this section, we drive the following schemes in the class of two-step Maruyama methods for Eq. (1.2)

$$\begin{aligned}X_{i+1}+\alpha_0X_i+(-1-\alpha_0)X_{i-1}&=h(2+\alpha_0)(aX_{i+1}+bX_{i-m+1})\\&+(cX_i+dX_{i-m})\Delta W_i+(1+\alpha_0)(cX_{i-1}+dX_{i-m-1})\Delta W_{i-1},\end{aligned} \tag{3.1}$$

where a parameter $-1\le\alpha_0<0$. We will also determine a series of two-step Maruyama schemes, which are unconditionally stable, see Theorem 5.

Applying the two-step Maruyama methods (2.3) to Eq.(1.2) gives

$$\begin{aligned}\sum_{j=-1}^{1}\alpha_jX_{i-j}&=h\sum_{j=-1}^{1}\beta_j[aX_{i-j}+bX_{i-m-j}]\\&+\sum_{j=0}^{1}\gamma_j[cX_{i-j}+dX_{i-m-j}]\Delta W_{i-j},\ i=2,3,\cdots;\end{aligned} \tag{3.2}$$

for $i\le 0$, we have $X_i=\xi(t_i)$. For better mean square stability, we compute X_1 using implicit Milstein method, which is mean square stable for every $h=\tau/m$ and has convergence rate $O(h)$ in mean-square-sense. We also suppose that X_1 is $\mathcal{F}_{t_1}$-measurable at the mesh-point t_1.

By choosing the parameters of the two-step Maruyama method to satisfy the consistency condition (2.6) and

$$\alpha_{-1}=1,\ -1\le\alpha_0<0,\ \beta_0=\beta_1=0, \tag{3.3}$$

then we get

$$\alpha_1=-1-\alpha_0,\ \beta_{-1}=2+\alpha_0,\ \gamma_0=1,\ \gamma_1=1+\alpha_0. \tag{3.4}$$

Thus, we obtain a family of two-step Maruyama schemes (3.1) from (3.2).

Next we determine the conditions on the parameters for mean square stability. It can be checked that the schemes (3.1) satisfy Dahlquist's root condition and all assumptions in Lemma 4. Thus, we have the following conclusion on mean-square-asymptotic stability.

Theorem 5 *Assume that the condition (2.1) holds. If the parameters of the two-step Maruyama method (3.2) satisfy the restricted conditions (3.3)and (3.4), then the method is unconditionally mean-square-stable. That is*

$$\lim_{n\to\infty} E|X_n|^2 = 0,$$

for every stepsize $h = \tau/m$.

The proof of this theorem needs the following lemma.

Lemma 6 *Under condition (2.1), the numerical solutions* $\{X_i,\, i \geq 2\}$ *produced by the two-step Maruyama method (3.1) satisfy*

$$E(X_i X_j \Delta W_k) = 0,\ \ i, j \leq k, \tag{3.5a}$$

$$E(X_i X_j \Delta W_k \Delta W_{k-1}) = 0,\ \ i, j \leq k, \tag{3.5b}$$

$$E(X_i X_j \Delta W_k^2) = hE(X_i X_j),\ \ i, j \leq k, \tag{3.5c}$$

$$E(X_i X_{i-1} \Delta W_{i-1}) < |c|hE(X_{i-1}^2) + |d|hE(|X_{i-1}X_{i-m-1}|), \tag{3.5d}$$

$$E(X_i X_{i-m-1} \Delta W_{i-1}) < |c|hE(|X_{i-1}X_{i-m-1}|) + |d|hE(X_{i-m-1}^2). \tag{3.5e}$$

Proof. Note that $E(\Delta W_k) = 0$, $E\Big[(\Delta W_k)^2\Big] = h$ and X_j and ΔW_j are $\mathcal{F}_{t_i}$-measurable if $j \leq i$, hence from properties of conditional expectation we get

$$\begin{aligned}
&E\Big(X_i X_j \Delta W_k\Big) = E\left[X_i X_j E\Big(\Delta W_k \Big| \mathcal{F}_{t_k}\Big)\right] = 0,\ \ i, j \leq k,\\
&E\Big(X_i X_j \Delta W_k \Delta W_{k-1}\Big) = E\left[X_i X_j \Delta W_{k-1} E\Big(\Delta W_k \Big| \mathcal{F}_{t_k}\Big)\right] = 0,\ \ i, j \leq k,\\
&E\Big[X_i X_j \Delta W_k^2\Big] = E\left[X_i X_j E\Big(\Delta W_k^2 \Big| \mathcal{F}_{t_k}\Big)\right] = hE\Big(X_i X_j\Big),\ \ i, j \leq k.
\end{aligned}$$

Now we prove the inequality (3.5d). From the scheme (3.1) we have

$$\begin{aligned}
X_{i+1} = &\frac{1}{(1-(2+\alpha_0)ah)}\Big[(2+\alpha_0)bhX_{i-m+1} + (1+\alpha_0)X_{i-1} - \alpha_0 X_i\\
&+(cX_i + dX_{i-m})\Delta W_i + (1+\alpha_0)(cX_{i-1} + dX_{i-m-1})\Delta W_{i-1}\Big].
\end{aligned}$$

and then

$$\begin{aligned}
X_i = &\frac{1}{(1-(2+\alpha_0)ah)}\Big[(2+\alpha_0)bhX_{i-m} + (1+\alpha_0)X_{i-2} - \alpha_0 X_{i-1}\\
&+(cX_{i-1} + dX_{i-m-1})\Delta W_{i-1} + (1+\alpha_0)(cX_{i-2} + dX_{i-m-2})\Delta W_{i-2}\Big].
\end{aligned}$$

Due to the condition (2.1) and $-1 \leq \alpha_0 < 0$, we get $1-(2+\alpha_0)ah > 1$. Using (3.5a)-(3.5c), it holds that

$$
\begin{aligned}
& E(X_iX_{i-1}\Delta W_{i-1}) \\
= & \frac{1}{(1-(2+\alpha_0)ah)}\Big[(2+\alpha_0)bhE(X_{i-m}X_{i-1}\Delta W_{i-1}) \\
& +(1+\alpha_0)E(X_{i-2}X_{i-1}\Delta W_{i-1}) - \alpha_0 E(X_{i-1}^2\Delta W_{i-1}) + cE(X_{i-1}^2\Delta W_{i-1}^2) \\
& +dE(X_{i-m-1}X_{i-1}\Delta W_{i-1}^2) + (1+\alpha_0)cE(X_{i-2}\Delta W_{i-2}X_{i-1}\Delta W_{i-1}) \\
& +(1+\alpha_0)dE(X_{i-m-2}\Delta W_{i-2}X_{i-1}\Delta W_{i-1})\Big] \\
= & \frac{1}{(1-(2+\alpha_0)ah)}\Big(chE(X_{i-1}^2) + dhE(X_{i-m-1}X_{i-1})\Big) \\
& < |c|hE(X_{i-1}^2) + |d|hE(|X_{i-m-1}X_{i-1}|).
\end{aligned}
$$

Inequality (3.5e) can be proved in the same way. This proves the lemma. □

Proof of Theorem 5. The explicit form of the scheme (3.1) is

$$
\begin{aligned}
\Big(1-(2+\alpha_0)ah\Big)X_{i+1} = & (2+\alpha_0)bhX_{i-m+1} + \Big((1+\alpha_0)X_{i-1} - \alpha_0 X_i\Big) \\
& +(cX_i + dX_{i-m})\Delta W_i + (1+\alpha_0)(cX_{i-1} + dX_{i-m-1})\Delta W_{i-1},
\end{aligned}
$$

We square both sides of the last difference equation to obtain

$$
\begin{aligned}
& (1-(2+\alpha_0)ah)^2X_{i+1}^2 \\
= & (2+\alpha_0)^2b^2h^2X_{i-m+1}^2 + \Big((1+\alpha_0)X_{i-1} - \alpha_0 X_i\Big)^2 \\
& +(cX_i + dX_{i-m})^2\Delta W_i^2 + (1+\alpha_0)^2(cX_{i-1} + dX_{i-m-1})^2\Delta W_{i-1}^2 \\
& +2(2+\alpha_0)bhX_{i-m+1}\Big((1+\alpha_0)X_{i-1} - \alpha_0 X_i\Big) \\
& +2(2+\alpha_0)bhX_{i-m+1}(cX_i + dX_{i-m})\Delta W_i \\
& +2(2+\alpha_0)bhX_{i-m+1}(1+\alpha_0)(cX_{i-1} + dX_{i-m-1})\Delta W_{i-1} \\
& +2\Big((1+\alpha_0)X_{i-1} - \alpha_0 X_i\Big)(cX_i + dX_{i-m})\Delta W_i \\
& +2\Big((1+\alpha_0)X_{i-1} - \alpha_0 X_i\Big)(1+\alpha_0)(cX_{i-1} + dX_{i-m-1})\Delta W_{i-1} \\
& +2(1+\alpha_0)(cX_i + dX_{i-m})(cX_{i-1} + dX_{i-m-1})\Delta W_i\Delta W_{i-1}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&(1-(2+\alpha_0)ah)^2X_{i+1}^2\\
=&\,\alpha_0^2X_i^2+c^2\Delta W_i^2X_i^2+(1+\alpha_0)^2X_{i-1}^2+(1+\alpha_0)^2c^2\Delta W_{i-1}^2X_{i-1}^2\\
&+(2+\alpha_0)^2h^2b^2X_{i-m+1}^2+d^2\Delta W_i^2X_{i-m}^2+(1+\alpha_0)^2d^2\Delta W_{i-1}^2X_{i-m-1}^2\\
&-2\alpha_0(1+\alpha_0)X_iX_{i-1}-2\alpha_0(1+\alpha_0)c\Delta W_{i-1}X_iX_{i-1}\\
&-2\alpha_0(2+\alpha_0)bhX_iX_{i-m+1}-2\alpha_0(1+\alpha_0)d\Delta W_{i-1}X_iX_{i-m-1}\\
&+2(1+\alpha_0)(2+\alpha_0)bhX_{i-1}X_{i-m+1}+2cd\Delta W_i^2X_iX_{i-m}\\
&+2(1+\alpha_0)^2cd\Delta W_{i-1}^2X_{i-1}X_{i-m-1}+O_i(\Delta W_i,\Delta W_{i-1}),
\end{aligned}
\tag{3.6}
$$

where we define

$$
\begin{aligned}
&O_i(\Delta W_i,\Delta W_{i-1})\\
=&\,2(1+\alpha_0)^2(cX_{i-1}+dX_{i-m-1})X_{i-1}\Delta W_{i-1}\\
&+2(2+\alpha_0)bhX_{i-m+1}(cX_i+dX_{i-m})\Delta W_i\\
&+2(2+\alpha_0)bhX_{i-m+1}(1+\alpha_0)(cX_{i-1}+dX_{i-m-1})\Delta W_{i-1}\\
&+2\Big((1+\alpha_0)X_{i-1}-\alpha_0X_i\Big)(cX_i+dX_{i-m})\Delta W_i\\
&+2(1+\alpha_0)(cX_i+dX_{i-m})(cX_{i-1}+dX_{i-m-1})\Delta W_i\Delta W_{i-1}.
\end{aligned}
$$

By the linearity of expectation and (3.5a)-(**??**) in Lemma 6, we get

$$E\left(O_i(\Delta W_i,\Delta W_{i-1})\right)=0.$$

Let $Y_i=E(X_i^2),\quad i=0,1,2,\cdots$. Taking the expectation over both sides of (3.6), and using the inequality $2ab\le a^2+b^2$ and (3.5c)-(3.5e) in Lemma 6, it follows that

$$
\begin{gathered}
P_0Y_{i+1}\le P_1Y_i+P_2Y_{i-1}+P_3Y_{i-m+1}+P_4Y_{i-m}+P_5Y_{i-m-1},\\
i=2,3,\cdots,
\end{gathered}
\tag{3.7}
$$

where

$$
\begin{aligned}
P_0=&\,(1-(2+\alpha_0)ah)^2,\\
P_1=&\,\alpha_0^2+c^2h-\alpha_0(1+\alpha_0)-\alpha_0(2+\alpha_0)|b|h+|cd|h,\\
P_2=&\,(1+\alpha_0)^2c^2h+(1+\alpha_0)^2-\alpha_0(1+\alpha_0)-2\alpha_0(1+\alpha_0)c^2h\\
&-2\alpha_0(1+\alpha_0)|cd|h+(1+\alpha_0)(2+\alpha_0)|b|h+(1+\alpha_0)^2|cd|h,\\
P_3=&\,(2+\alpha_0)^2b^2h^2-\alpha_0(2+\alpha_0)|b|h+(1+\alpha_0)(2+\alpha_0)|b|h,\\
P_4=&\,d^2h+|cd|h,\\
P_5=&\,(1+\alpha_0)^2(d^2+|cd|)h-2\alpha_0(1+\alpha_0)d^2h-2\alpha_0(1+\alpha_0)|cd|h.
\end{aligned}
$$

Let $P=P(a,b,c,d,\alpha_0,h)=\Big(P_1+P_2+P_3+P_4+P_5\Big)/P_0$. It is obvious that

$$Y_{i+1} \leq P \max\Big\{Y_i, Y_{i-1}, Y_{i-m+1}, Y_{i-m}, Y_{i-m-1}\Big\}.\ i = 2, 3, \cdots. \tag{3.8}$$

We now claim that, for any stepsize $h = \tau/m$,

$$Y_{i+1} \leq \max\{P^{i+1}, P^i, \cdots, P^{[\frac{i-m-1}{m+2}]+1}\} E\|\xi\|^2. \tag{3.9}$$

Thus, if $P < 1$, then we can get $\lim_{i\to\infty} Y_i = 0$, as $E\|\xi\|^2 < \infty$. In fact, we have

$$\begin{aligned}
Y_{i+1} &\leq P \max\{Y_i, Y_{i-1}, Y_{i-m+1}, Y_{i-m}, Y_{i-m-1}\} \\
&\leq P^2 \max\{Y_{i-1}, Y_{i-2}, \cdots, Y_{i-2m-3}\} \\
&\qquad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
&\leq P^{[\frac{i-m-1}{m+2}]+1} \max\{Y_{i-[\frac{i-m-1}{m+2}]}, \cdots, Y_1, E\|\xi\|^2\} \\
&\qquad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
&\leq P^{i+1} \max\{E\|\xi\|^2, \frac{1}{P} E\|\xi\|^2, \cdots, \frac{1}{P^{i-[\frac{i-m-1}{m+2}]-1}} E\|\xi\|^2\} \\
&\leq \max\{P^{i+1}, P^i, \cdots, P^{[\frac{i-m-1}{m+2}]+1}\} E\|\xi\|^2.
\end{aligned}$$

Here we notice that in the process of iteration, Y_{i-m-1} $(i \geq m+1)$ will be the first term down to $Y_0 = E\|\xi\|^2$ and it calls for its previous $[\frac{i-m-1}{m+2}] + 1$ steps, where $[z]$ means the largest integer no more than a real number z. This proves the claim (3.9).

It only remains to verify that $P < 1$ i.e. $P_1 + P_2 + P_3 + P_4 + P_5 < P_0$. Recall that $-1 \leq \alpha_0 < 0$ and thus $2 - \alpha_0^2 \leq 2(2 + \alpha_0)$, then from (3.7), we have

$$\begin{aligned}
&P_1 + P_2 + P_3 + P_4 + P_5 - P_0 \\
=& 1 + \Big(2(2+\alpha_0)|b| + (2-\alpha_0^2)(|c|+|d|)^2\Big)h \\
&+ (2+\alpha_0)^2 b^2 h^2 - (1 - (2+\alpha_0)ah)^2 \\
=& (2+\alpha_0)^2 (b^2 - a^2) h^2 \\
&+ 2\Big((2+\alpha_0)a + (2+\alpha_0)b + \frac{1}{2}(2-\alpha_0^2)(|c|+|d|)^2\Big)h \\
\leq& (2+\alpha_0)^2 (b^2 - a^2) h^2 + 2(2+\alpha_0)\Big(a + |b| + (|c|+|d|)^2\Big)h.
\end{aligned}$$

Based on condition (2.1), we obtain that $P_1 + P_2 + P_3 + P_4 + P_5 - P_0 < 0$ holds for each stepsize $h = \tau/m$. This proves the theorem. □

4 Numerical examples

In all our numerical examples,

$$E(X_n^2) = \frac{1}{2000}\sum_{i=1}^{2000} |X_n(\omega_i)|^2,$$

are the sampled average over 2000 trajectories in Matlab.

Table 1 lists a number of two-step Maruyama schemes with different parameters under test here. Note that only the schemes in bold (TS3 and TS5) satisfy the required conditions in Theorem 5 and hence are unconditionally mean-square-stable. However, the other Maruyama schemes TS1, TS2, TS4, TS6 do not satisfy the conditions in Theorem 5 and thus may be only conditionally stable and even not stable as we show later on.

Table 1
parameters for different two-step Maruyama schemes

scheme	α_0	β_{-1}	β_0
two-step method 1 (TS1)	-1/2	1/4	5/4
two-step method 2 (TS2)	-3/2	1/2	1
two-step method 3 (TS3)	**-1/2**	**3/2**	**0**
two-step method 4 (TS4)	-1/3	1/3	4/3
two-step method 5 (TS5)	**-2/3**	**4/3**	**0**
two-step method 6 (TS6)	-4/3	0	2/3

Example 1. We consider the linear test model

$$\begin{aligned} &\mathrm{d}X(t) = [aX(t) + bX(t-\tau)]\mathrm{d}t + [cX(t) + dX(t-\tau)]\mathrm{d}W(t), t \geq 0, \\ &X(t) = t + \tau, t \in [-\tau, 0] \end{aligned} \tag{4.1}$$

to illustrate the mean-square-stability of the two-step Maruyama schemes in Table 1.

We choose the parameters as $a = -4$, $b = 2$, $c = 0.5$, $d = 0.5$ and $\tau = 1$, which ensures that the exact solution of the equation (4.1) is mean-square-stable by Lemma 1. From Fig. 1, we observe that TS1 is not mean-square-stable for both large stepsize $h = 1/4$ and small stepsize $h = 1/64$. From Figs. 2 and 3, we see that TS3 and TS5 are mean-square-stable even for a large stepsize $h = 1/4$; Fig. 2 illustrates that TS4 is conditionally mean-

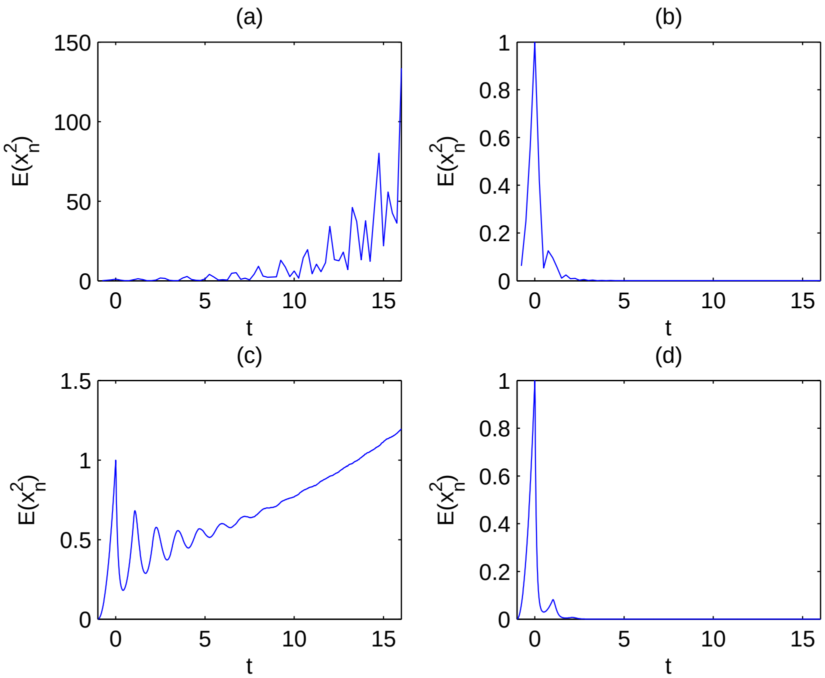

Fig. 1. Simulations with TS1 and TS3. **(a):** TS1, $h = 1/4$; **(b):** TS3, $h = 1/4$; **(c):** TS1, $h = 1/64$; **(d):** TS3, $h = 1/64$.

square-stable; TS4 is mean square stable if the stepsize h is small enough (like $h = 1/64$ here).

In Fig. 3, we fix the stepsize $h = 1/8$ and show that to great extent the mean square stability of the implicit scheme TS3 is better than the explicit two-step scheme TS6. If we test TS6 for a rather long time interval, then the numerical solution will oscillate and will finally diverge. From Figs. 1, 2 and 3, we observe that the numerical solution from TS6 blows up earlier than any other unstable implicit methods shown in Fig 1 and 2. On the other hand, numerical solutions of both TS3 and TS5 converse to zero very fast.

In Fig. 4 we choose different parameters for the equation (4.1): $a = -3, b = 1, c = 0.5$, $d = 0.5$ and $\tau = 1$ when (4.1) is mean square stable. Here we use a very large stepsize $h = 1/2$ for TS1, TS2 and TS3. The results show that the scheme TS3 maintains the mean-square-stability even with large stepsize h.

Example 2. We test the proposed two-step Maruyama methods for the following nonlinear SDDE (Example 5.2.1, [18]):

$$\begin{aligned} \mathrm{d}X(t) &= -\frac{a}{1+t}X(t) + \frac{b}{1+t}X(t)\sin(X(t-\tau))\mathrm{d}W(t), \quad t \geq 0, \\ X(t) &= t + \tau, \quad t \in [-\tau, 0]. \end{aligned} \tag{4.2}$$

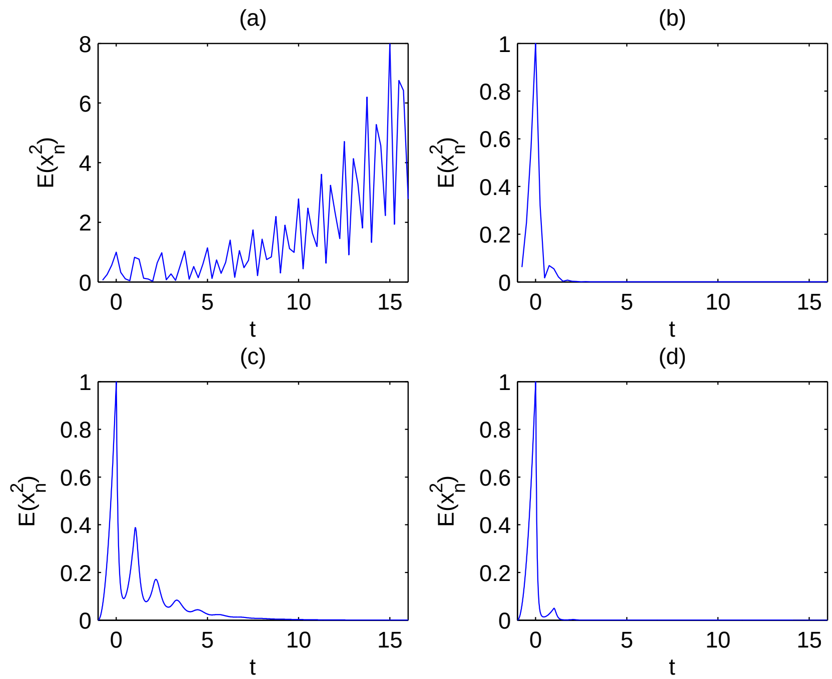

Fig. 2. Simulations with TS4 and TS5. **(a):** TS4, $h = 1/4$; **(b):** TS5, $h = 1/4$; **(c):** TS4, $h = 1/64$; **(d):** TS5, $h = 1/64$.

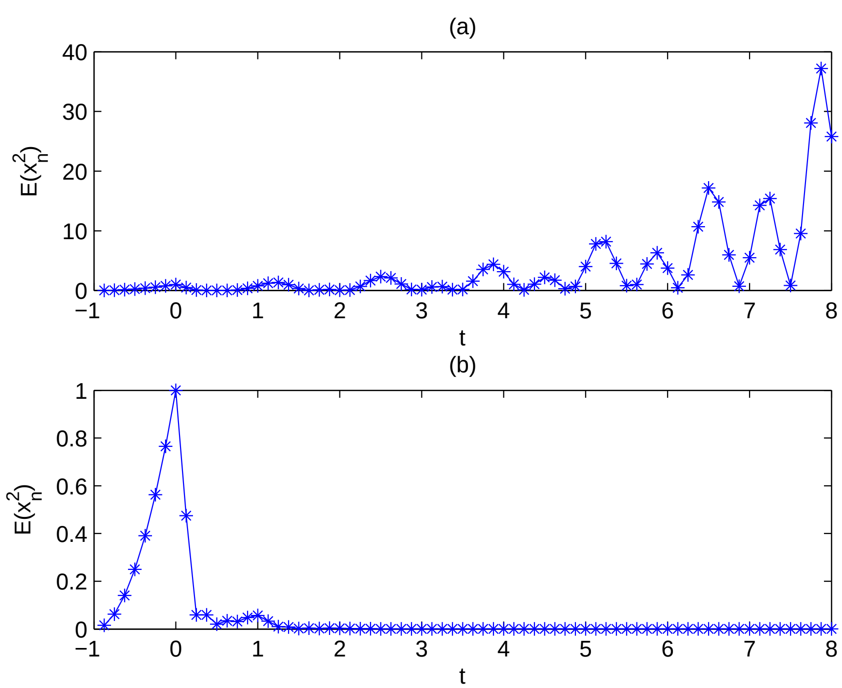

Fig. 3. Simulations with TS3 and the explicit method TS6. **(a):** TS6; **(b):** TS3. $h = 1/8$.

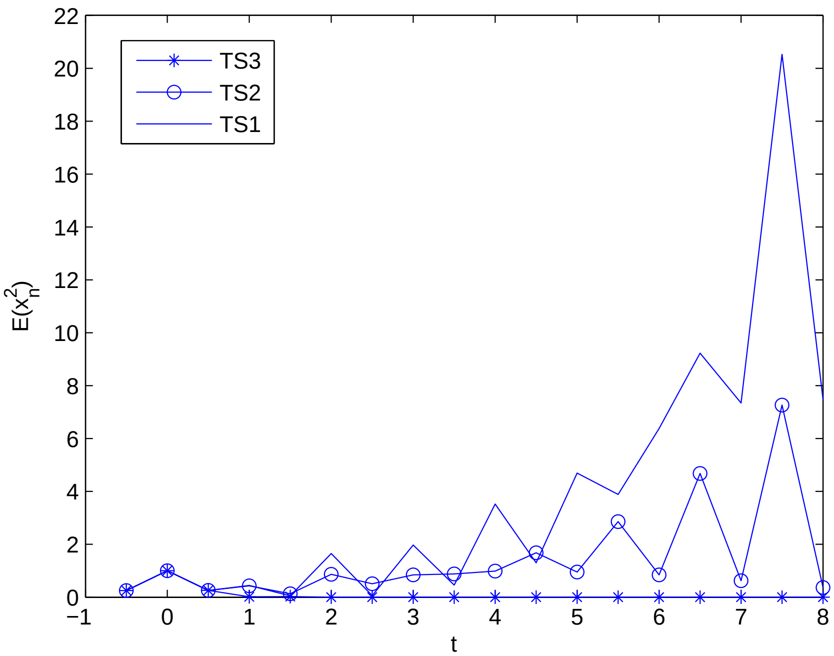

Fig. 4. Simulations with TS1, TS2 and TS3 for fixed stepsize $h = 1/2$.

The solution of Eq.(4.2) is mean square stable if $2a - 1 \geq b^2$ and $b \neq 0$ [18].

We take $a = 100,\ b = 10$ and $\tau = 1$.

It is shown from Figs. 5 and 6 that the schemes TS3 and TS5 maintain their mean-square-stability for nonlinear SDDE (4.2); TS1 and TS4 are conditionally mean-square-stable. It is interesting to mention that in some range of stepsize h, smaller h leads to a greater instability, comparing to (a),(c) in Fig. 5 and (a),(c) in Fig. 6.

5 Conclusion

We have studied the mean-square-stability of two-step Maruyama methods and proposed a family of unconditionally mean-square-stable two-step Maruyama schemes. Numerical tests that this family of numerical methods exhibits of mean-square-stability for both linear and some particular nonlinear SDDE. The numerical results suggest that our proposed scheme can be adopted for general nonlinear SDDEs, but further numerical studies are required.

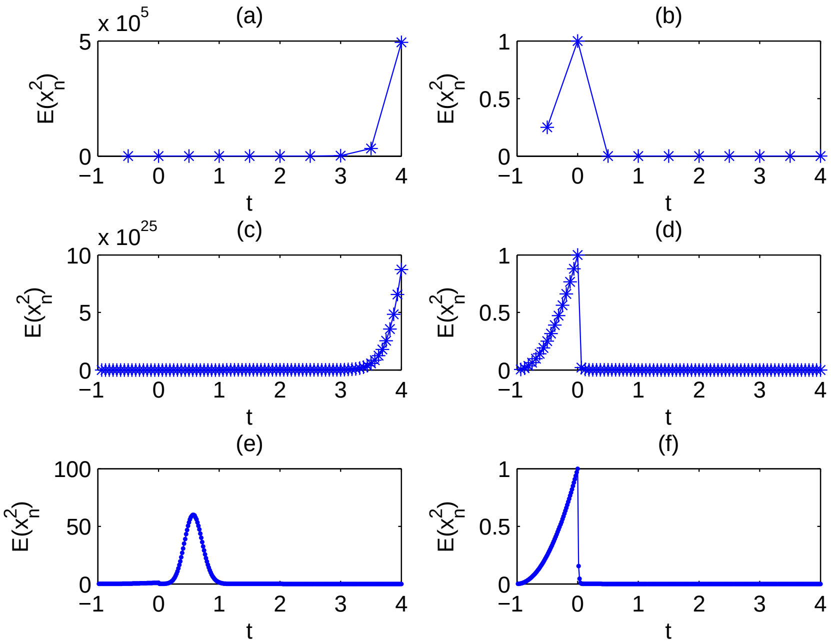

Fig. 5. Simulations with TS1 and TS3. **(a):** TS1, $h = 1/2$; **(b):** TS3, $h = 1/2$; **(c):** TS1, $h = 1/16$; **(d):** TS3, $h = 1/16$; **(e):** TS1, $h = 1/64$; **(f):** TS3, $h = 1/64$

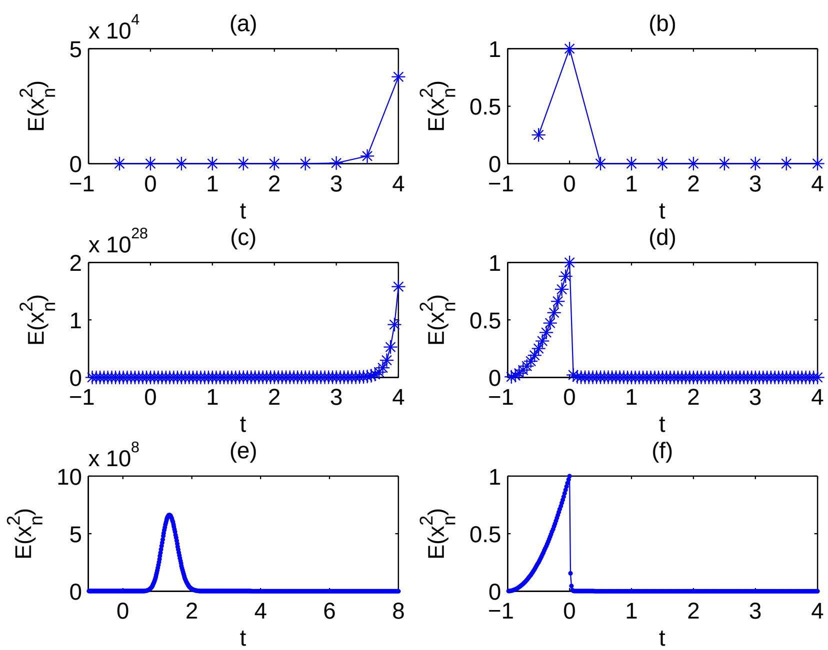

Fig. 6. Simulations with TS4 and TS5. **(a):** TS4, $h = 1/2$; **(b):** TS5, $h = 1/2$; **(c):** TS4, $h = 1/16$; **(d):** TS5, $h = 1/16$; **(e):** TS4, $h = 1/64$; **(f):** TS5, $h = 1/64$.

References

[1] C.T.H. Baker, E. Buckwar, Exponential stability in p-th mean of solutions, and of convergent Euler-type solutions, of stochastic delay differential equations, J. Comput. Appl. Math., 184(2005)404-427.

[2] E. Beretta, V. B. Kolmanovskii, L Shaikhet, Stability of epidemic model with time delays influenced by stochastic perturbations, Math. Comput. Simulation, 45(1998)269-277.

[3] A. Beuter, K. Vasilakos, Effects of noise on a delayed visual feedback system, J. Theor. Biology, 165(1993)389-407.

[4] A. Beuter, J. Belair, Feedback and delays in neurological diseases: a modeling study using dynamical systems, bull Math. biology, 55(1993)525-541.

[5] L. Brugnano, K. Burrage, P. M. Burrage, Adams-type methods for the numerical solution of stochastic ordinary differential equations, BIT. 40(2000)451-470.

[6] E. Buckwar, R. Winkler, Multi-step Maruyama methods for stochastic delay differential equations. Stochastic Analysis and Applications, 25(2007)933-959.

[7] E. Buckwar, R. Winkler, Multi-step methods for SDEs and their application to problems with small noise, SIAM J. Numer. Anal. 44(2006)779-803.

[8] G. Denk, S. Schäffler, Adams methods for the efficient solution of stochastic differential equations with additive noise, Computing, 59(1996)153-161.

[9] B. D. Ewald, R. Témam, Numerical analysis of stochastic schemes in geophysics, SIAM J. Numer. Anal. 42(2005)2257-2276.

[10] Z. Fan, Waveform relaxation method for stochastic differential equations with constant delay, Applied Numerical Mathematics, 61(2011)229-240.

[11] M. Grigoriu, Control of time delay linear systems with Gaussian white noise, Probabilistic Engineering Mechanics, 12(1997)89-96.

[12] D. G. Hobson, L. C. G. Rogers, Complete models with stochastic balatility, Math. Finance, 8(1998)27-48.

[13] P. E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, Springer, Berlin, 1992.

[14] R. Li, Convergence and stability of numerical solutions to SDDEs with Markovian switching, Applied Mathematics and Computation, 175(2006)1080-1091.

[15] X. Mao, Stochastic Differential Equations and Applications, Horwood, 1997.

[16] X. Mao, Razumikhin-Type theorems on exponential stability of stochastic functional differential equations, Stochastic Processes and their Applications, 65(1996)233-250.

[17] X. Mao, Numerical solutions of stochastic differential delay equations under the generalized Khasminskii-type conditions, Appl. Math. Comput.,

217(2011)5512-5524.
[18] X. Mao, Stability of Stochastic Differential Equations with Respect to Semimartingales, Longman Scientific&Technical, 1991.
[19] G. N. Milstein, Numerical Integration of Stochastic Differential Equations, Kluwer Academic Publishers Group, Dordrecht, 1995.
[20] S. E. A. Mohammed, Stochastic Functional Differential Equations, Research Notes in Mathematics, Pitman, London, 99(1984).
[21] M. D. Paola, A. Pirrotta, Time delay induced effects on control of linear systems under random excitation, Probabilistic Engineering Mechanics, 16(2001)43-51.
[22] L. S. Tsimring, A. Pikovsky, Noise-induced dynamics in bistable systems with delay, Phys. Rev. Lett. 87(2001)250602-1-25062-4.
[23] S. Zhou, F. Wu, Convergence of numerical solutions to neutral stochastic delay differential equations with Markovian switching, J. Comput. Appl. Math., 229(2009)85-96.