The Effective Source Approach to the Self-force Problem.

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Outline

- Extreme Mass Ratio Inspirals
- Approaches to Self-force Calculations
- The Effective Source Approach
- Numerical Implementation
- Results
- Concluding Remarks

Extreme Mass Ratio Inspirals (EMRIs)

The inspiral of small compact objects into super massive black holes in the center of galaxies is expected to occur frequently in the universe.

Typical mass ratios around $\mu/M \approx 10^{-6}$ needs of the order 10^{6} cycles in the inspiral before the plunge and merger.

The large range of time and length scales makes brute force numerical relativity techniques impossible.

Instead perturbation theory has to be applied where the internal structure of the small object is neglected and it is treated as a point particle.

Given geodesic motion one can then calculate waveforms from the perturbations produced by the point particle.

This, however, ignores the back reaction of the perturbation on the orbit of the particle.

Corrections to the geodesic motion (the self-force) has to be included in order to accurately track the phase evolution of the waveform.

Since the field is singular at the location of the particle, extreme care must be employed to extract the regular part of the field that is responsible for the self-force.

Approaches to Self-force calculations

The basis for any self-force calculation is the fact that the retarded field can be decomposed as

$$\mathbf{\Phi}^{\mathrm{ret}} = \mathbf{\Phi}^{\mathrm{R}} + \mathbf{\Phi}^{\mathrm{S}}.$$

The individual modes in a spherical harmonic decomposition of the gradient of the retarded field $\mathbf{\Phi}^{\mathrm{ret}}$ defined as

$$\boldsymbol{\nabla}_{\alpha}\mathbf{\Phi}^{\mathrm{ret}} = \sum_{\ell}^{\infty}\left(\sum_{m=-\ell}^{\ell}(\boldsymbol{\nabla}_{\alpha}\mathbf{\Phi}^{\mathrm{ret}})_{\ell m}\right),$$

are finite at the location of the particle.

In the mode sum approach a careful analysis of the singular structure of the retarded field is performed in order to subtract the singular part of each mode.

The evolution of the individual modes can be done in either the time domain or the frquency domain.

Very high accuracy can be obtained for a prescribed orbit in the frequency domain but it is then unclear exactly how to adjust the orbit.

In a time domain approach it is expensive to calculate the self-force through the mode sum so adjusting the orbit at every timestep will be expensive.

The Effective Source Approach

In the effective source approach we take advantage of the decomposition of the retarded field as

$$\Phi^{\rm ret} = \Phi^{\rm R} + \Phi^{\rm S}.$$

For the scalar charge case the basic equation is

$$\Box\Phi^{\rm ret} = \Box(\Phi^{\rm R} + \Phi^{\rm S}) = -4\pi q \int \delta(x, x_0(\tau))\, d\tau.$$

We now introduce $\bar{\Phi}^{\rm S} = W\Phi^{\rm S}$ and define $\bar{\Phi}^{\rm R} := \Phi^{\rm ret} - W\Phi^{\rm S}$ such that

$$\Box\bar{\Phi}^{\rm R} = -4\pi q \int \delta(x, x_0(\tau))\, d\tau - \Box(W\Phi^{\rm S}) = \mathcal{S}_{\rm eff}(x, x_0, u_0).$$

Then $\bar{\Phi}^{\rm R} = \Phi^{\rm R}$ where $W = 1$ and $\bar{\Phi}^{\rm R} = \Phi^{\rm ret}$ where $W = 0$.

With a numerical solution for $\bar{\Phi}^{\rm R}$ the self-force can then be found as

$$F_\alpha = q\left(\nabla_\alpha \bar{\Phi}^{\rm R}\right)\Big|_{x=x_0}.$$

In practice we use an approximation to the singular field $\Phi^{\rm S} = \bar{\Phi}^{\rm S} + O(\epsilon^n)$.

The Effective Source Approach.

Since the field $\bar{\Phi}^{\mathrm{R}}$ is regular everywhere, we can now use standard 3D numerical techniques in the time domain.

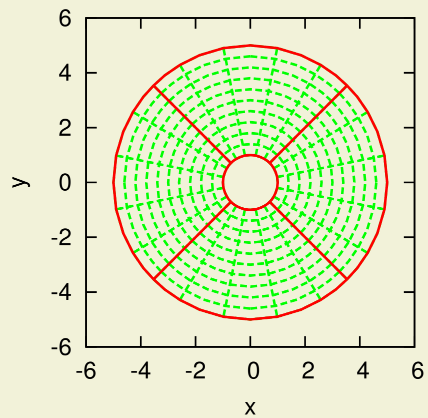
With the approximation used for the singular field our effective source is C^0 at the location of the particle and C^∞ everywhere else.

This will allow us to take the back reaction into account and have a self consistent evolution of both the field and the orbit.

The first successful tests of the method was done with an effective source specialized for circular orbits around a Schwarzschild black hole.

We have now developed an effective source valid for arbitrary orbits around a Schwarzschild black hole.

The Numerical Implementation



We use a 3D touching multi-block scalar wave evolution code in Kerr-Schild coordinates in order to use spherical coordinates without axis problems.

The computational domain is spherical with an inner region excised inside the black hole and a spherical outer boundary.

We use high order finite differencing operators that satisfy Summation By Parts

$$\langle u, Dv \rangle + \langle Du, v \rangle = [uv]_a^b$$

where

$$\langle u, v \rangle = h \sum_{i,j=0}^{N} u_i v_j \sigma_{ij}.$$

Combined with the use of penalty methods to communicate characteristic information between neighboring blocks we get a stable and convergent scheme for linear hyperbolic problems.

The Numerical Implementation

A crucial ingredient for the very long evolutions we expect to perform is to control the outer boundary. Even the slightest boundary contamination can mess up the very delicate self-force calculation.

In the current code we have solved this problem by smoothly transitioning from standard spatial slices in the interior to hyperboloidal slices towards the outer boundary.

We use a transformed time coordinate

$$\tau = t - h(r)$$

and a compactified radial coordinate

$$r = \frac{\rho}{\Omega}, \qquad \text{with} \qquad \Omega = \Omega(\rho)$$

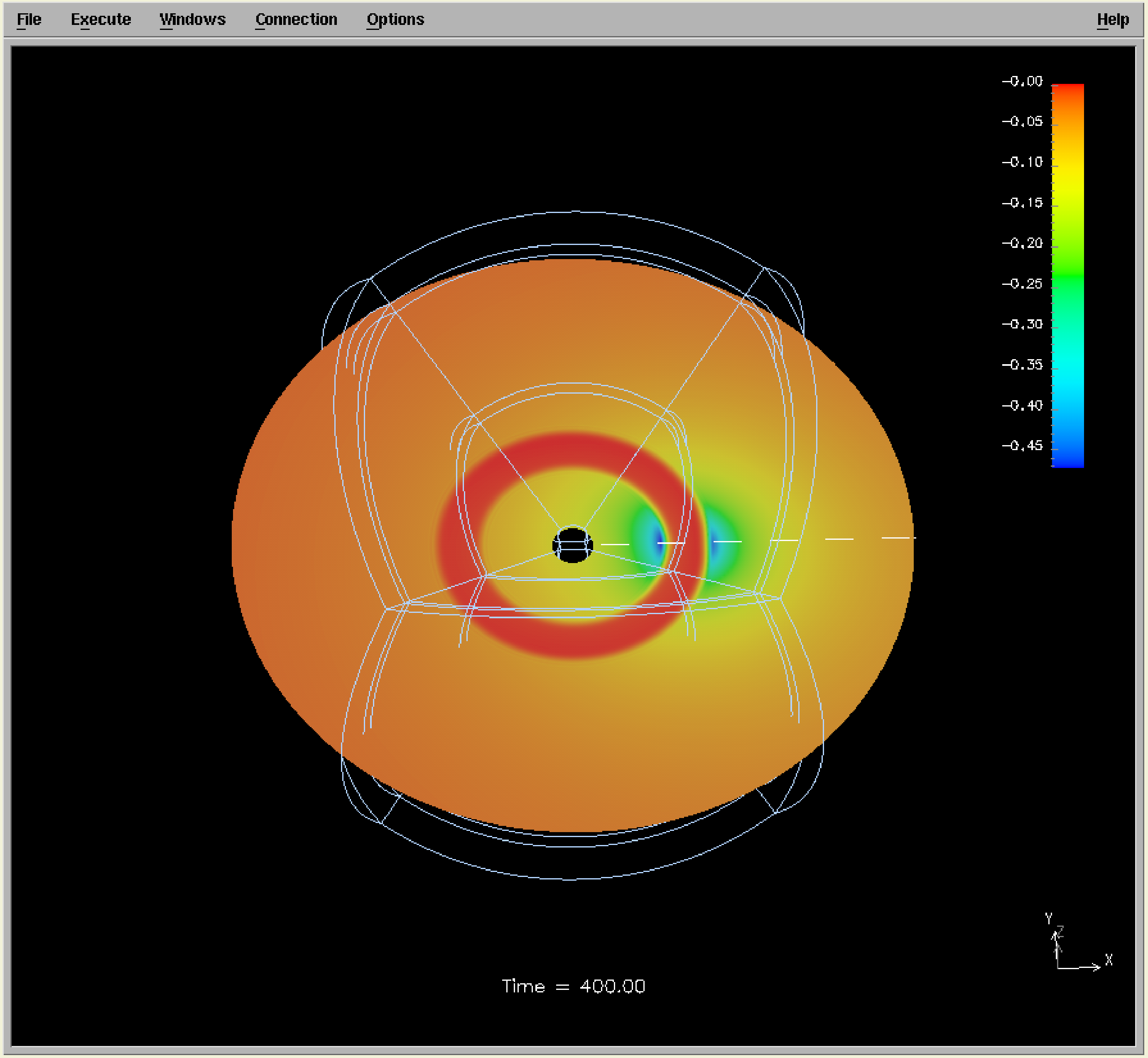
where $r \to \infty$ corresponds to $\Omega = 0$.

Choosing $h(r)$ and $\Omega(\rho)$ carefully we can guarantee that the outer boundary corresponds to $\mathscr{I}^+$ and that the equations are regular there.

We have only outgoing modes at the inner (excision) boundary and at the outer boundary ($\mathscr{I}^+$).

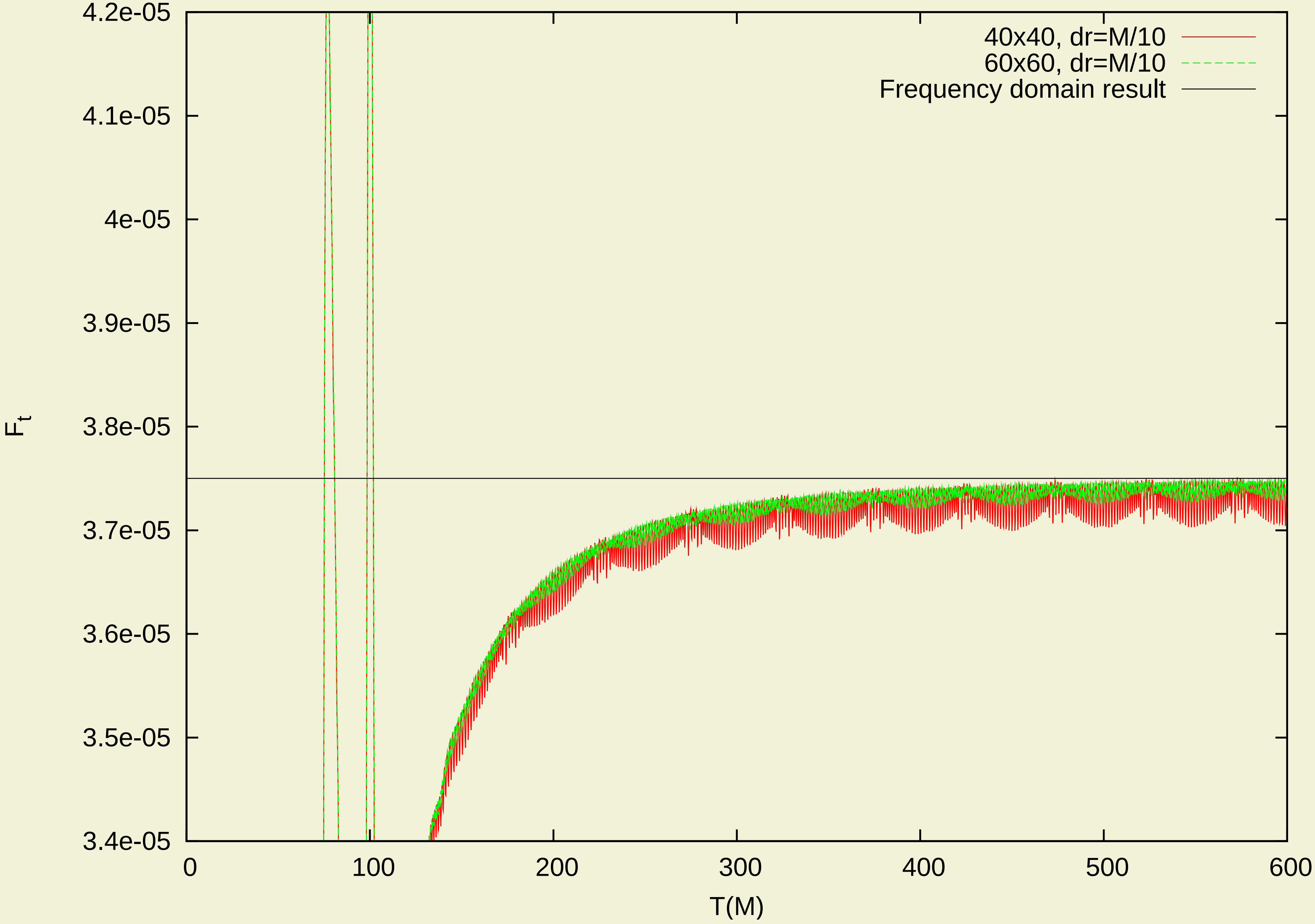
Results.

Circular orbit ($r = 10M$, $T = 400M$).



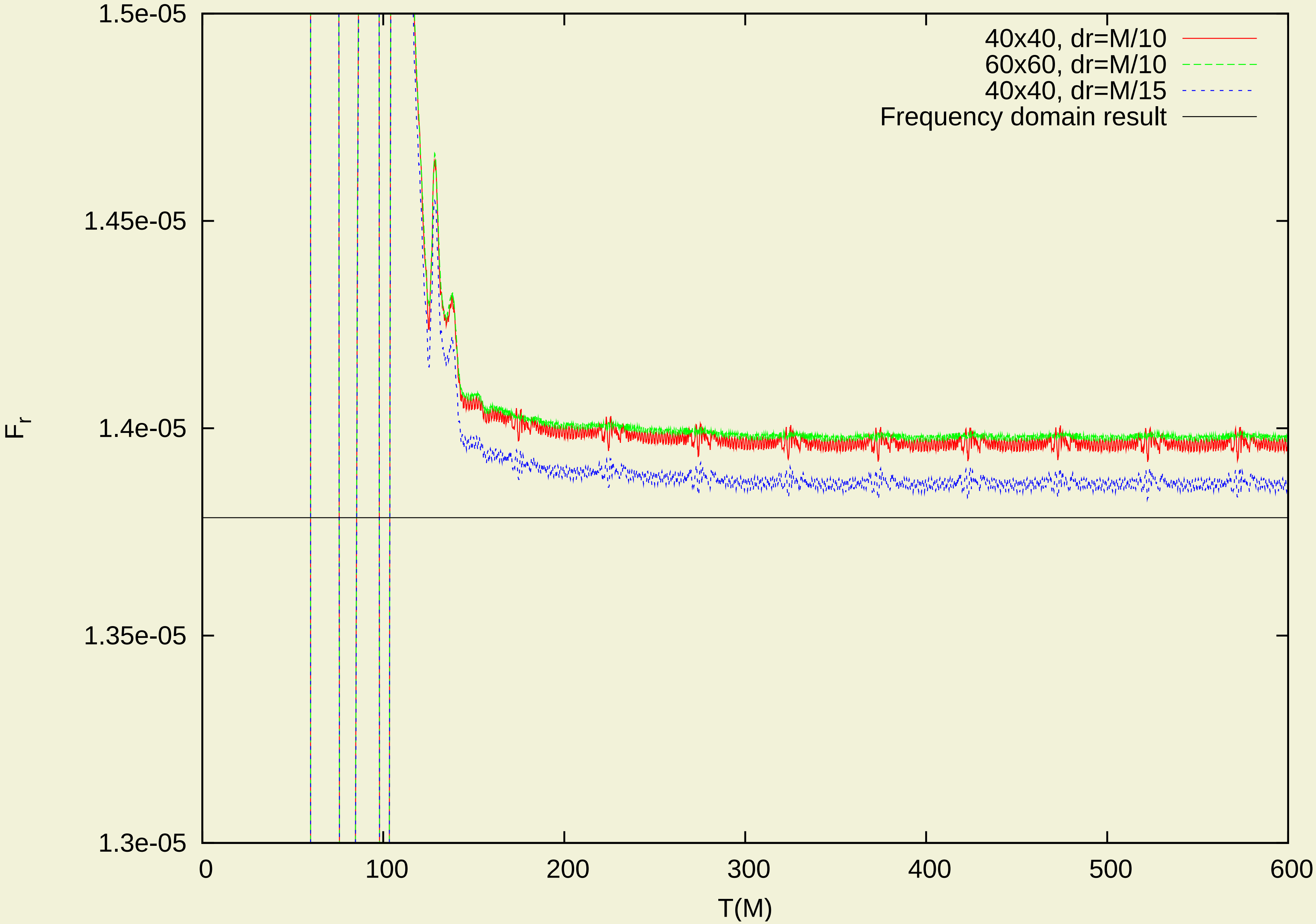
Results.

Circular orbit ($r = 10M$, time component).



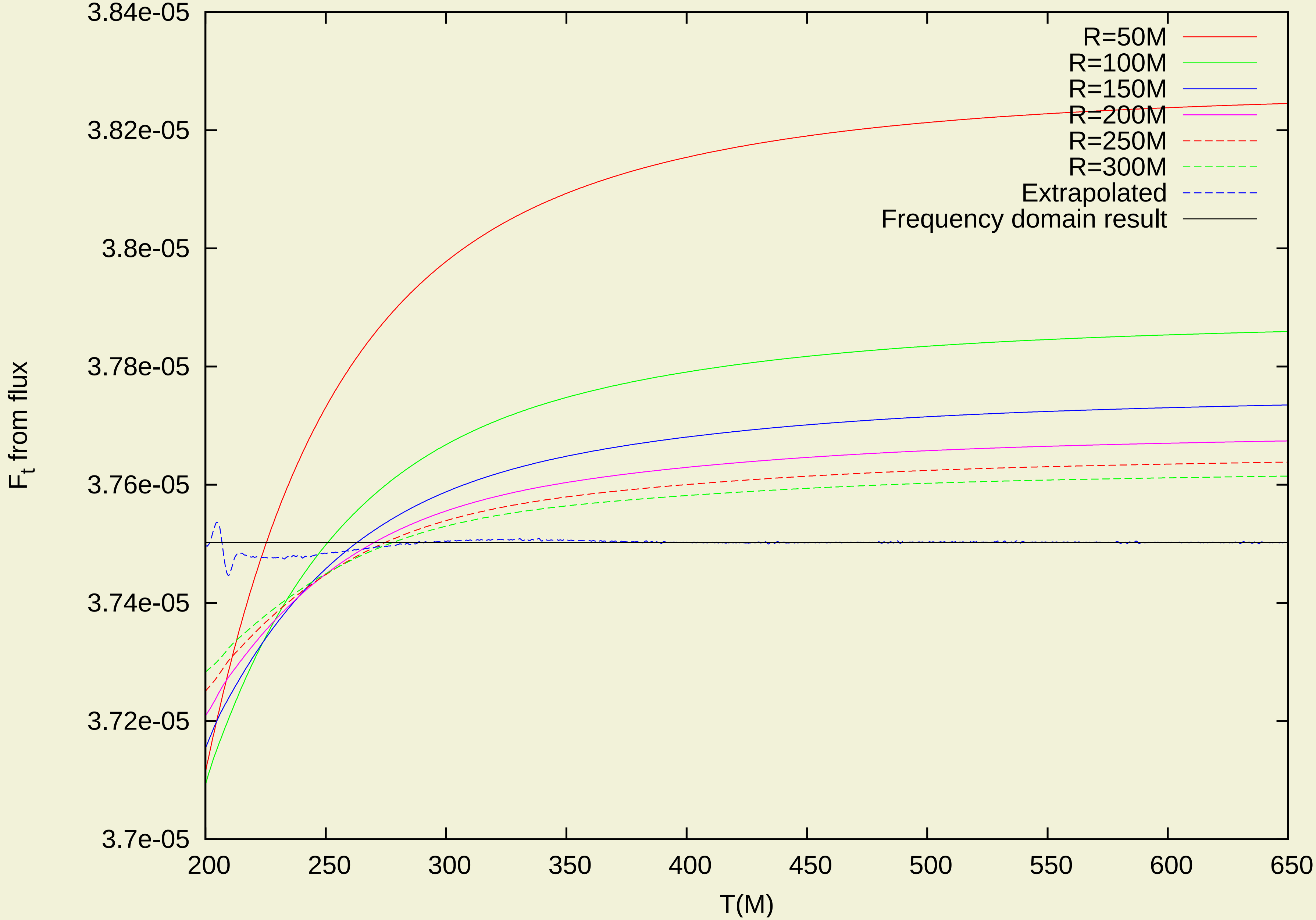
Results.

Circular orbit ($r = 10M$, radial component).



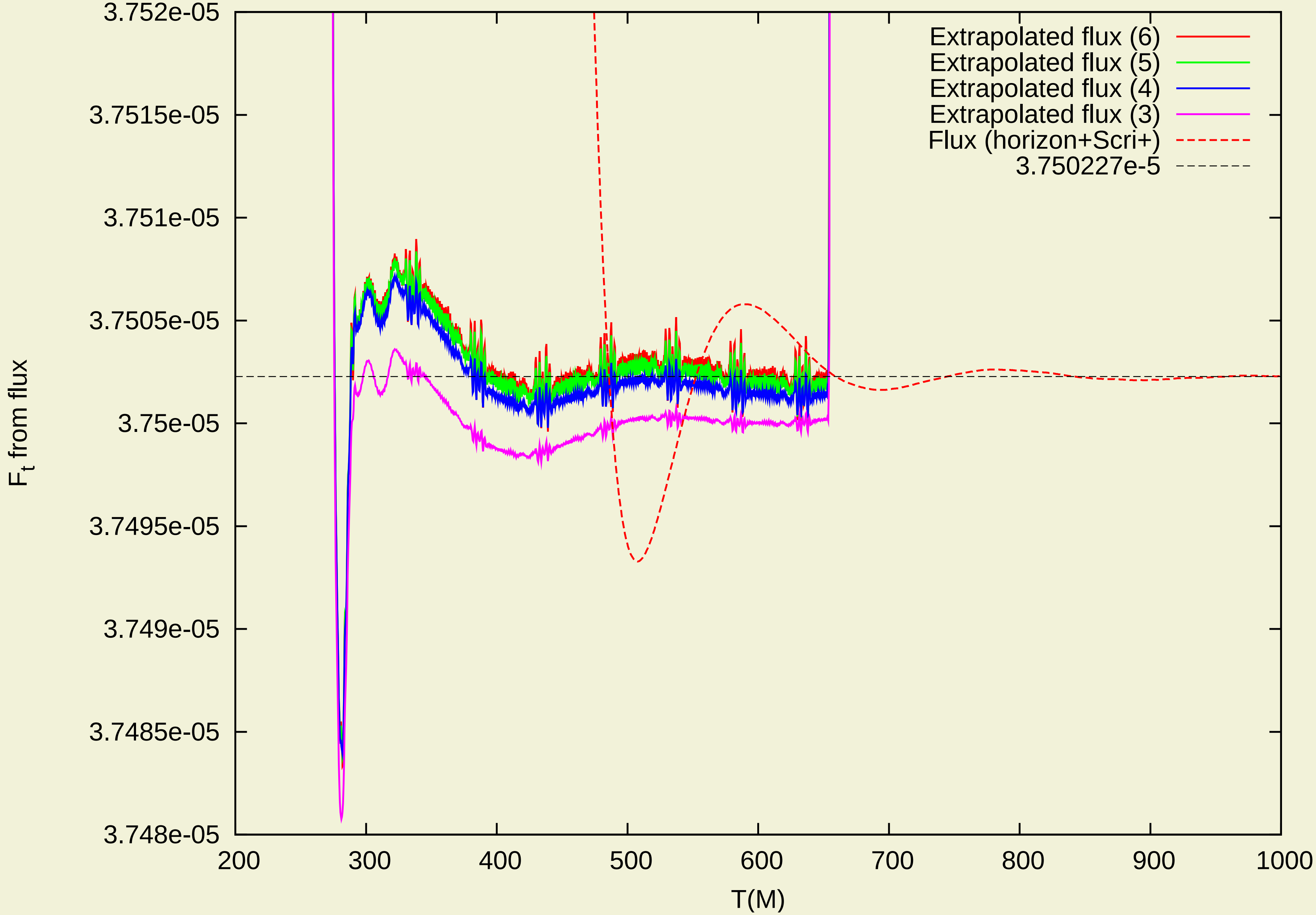
Results.

Circular orbit ($r = 10M$, flux).



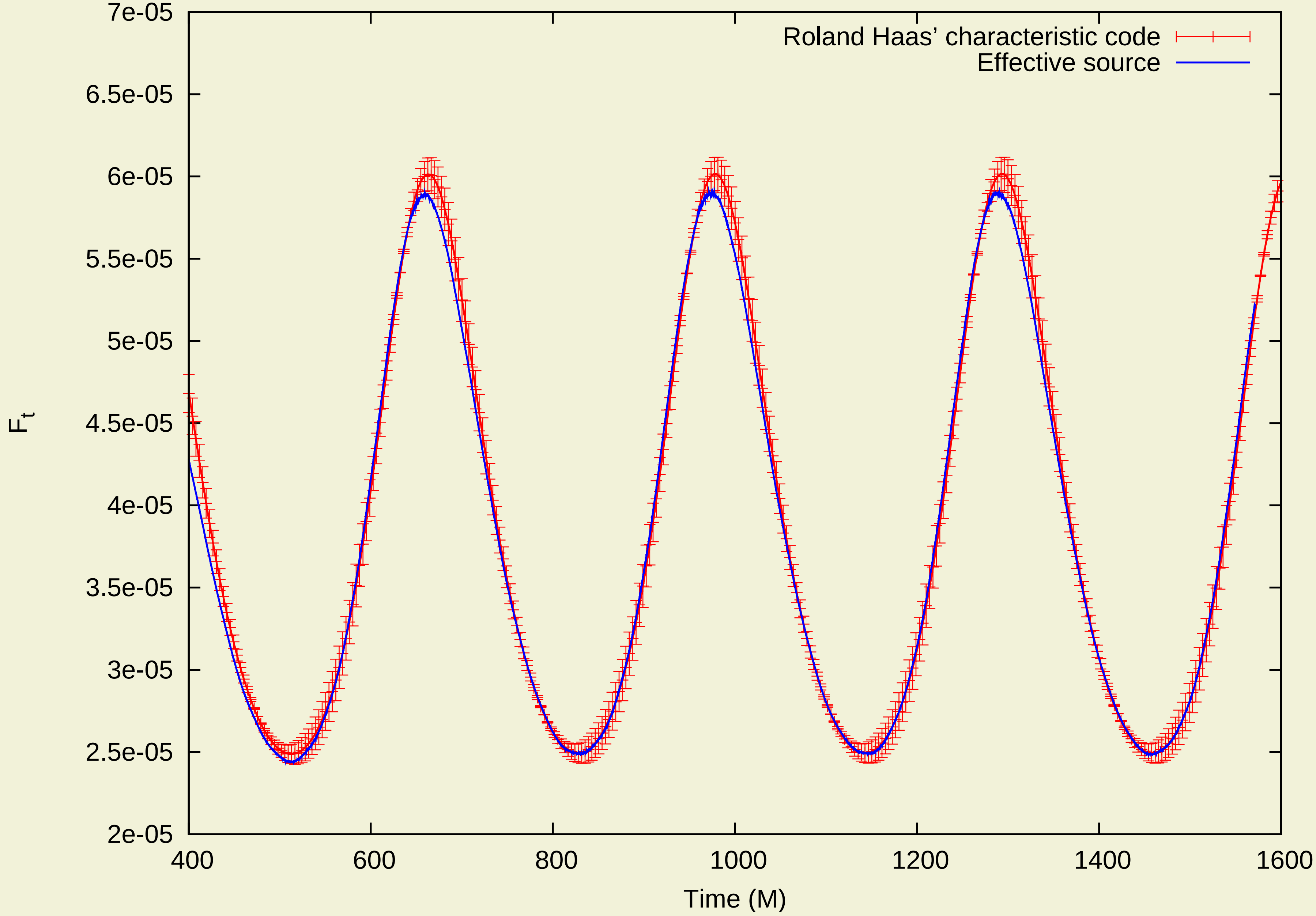
Results.

Circular orbit ($r = 10M$, flux).



Results.

Eccentric orbit ($e = 0.1$, $R_{\rm min} = 9$, $R_{\rm max} = 11$).



Concluding remarks

- With the current numerical techniques we can calculate the self-force locally from the regular field at the 1% accuracy level using a source specialized for circular orbits.
- The accuracy and convergence is limited by the non-smoothness of the effective source at the particle location.
- This can be improved on the analytical side by using a higher order approximation to the singular field or by improving the numerical techniques.
- Our first comparison to other results for eccentric orbits look promising but we are still in the process of validating the new source for generic geodesics around a Schwarzschild black hole.
- The next step will be to take the back reaction into account and evolve the field and the orbit consistently.
- Having sorted out various subtleties in the implementation of the generic source going to the Kerr metric should be straightforward.
- Moving on to the gravitational case will require a lot of work both on the analytical and numerical side, but should be feasible.