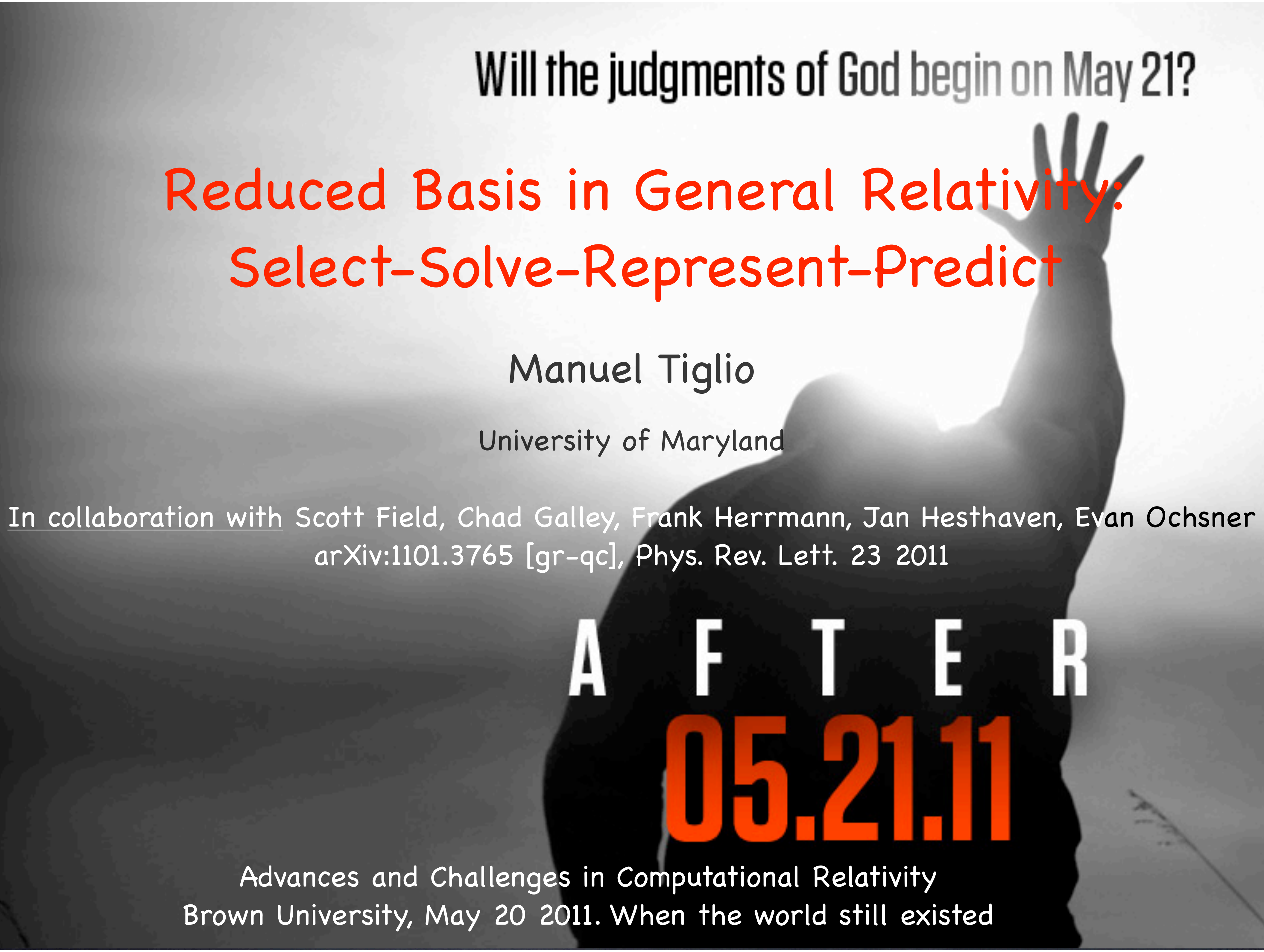

Reduced Basis in General Relativity: Select-Solve-Represent-Predict

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arXiv:1101.3765 [gr-qc], Phys. Rev. Lett. 23 2011

Advances and Challenges in Computational Relativity

Brown University, May 20 2011. When the world still existed

Select-Solve-Represent-Predict (SSRP)

- Full numerical relativity simulations of the Einstein equations are very expensive, even if only used to calibrate analytical models (Alessandra Buonanno's talk).

- There is a need of a way of efficiently selecting the most relevant configurations to solve for, even in computationally less demanding models.

- Even if dealing with analytical approximations of gravitational waves, such as those used in searches of inspiral sources, the number of templates is in general very large.

- Example I: for binary neutron stars, around 70,000 inspiral templates are needed just to account for the mass of each black hole at one percent accuracy.

- Example II: One-mode searches can lose up to 15% of intermediate mass black holes events. However, two-mode catalogs lead to more than 1,000,000

Select-Solve-Represent-Predict (SSRP)

- A compact yet high accuracy representation of gravitational waves is advantageous for:

  - Faster, cheaper data analysis for online searches and generate alerts for EM counterparts.

  - Being able to introduce spin effects.

- For any given any given number of optimal solutions one wants to predict other ones (interpolation) with high accuracy.

Our requirements

Being able to sequentially select "on the fly" the most relevant points in parameter space to solve for, in a nearly optimal way.

Exploit the smooth dependence of GR with respect to parameter variation to achieve exponential convergence in the number of solutions.

Yield nested catalogs that are hierarchically constructed, and which for increasing accuracy can be extended by adding members

$$C_N \subset C_{N+1} \subset C_{N+2} \cdots$$

Constant complexity -- the cost of selecting a parameter value to solve for has to be independent of N

Total cost to build C_N is O(N)

Being able to keep a tight control on the error of the catalog as an approximation of all possible solutions.

Independent of the solution method (time integration schemes, finite differences, spectral dG, etc in the case of PDEs)

Reduced Basis (RB)

- RB is an approach for the Select-(Solve)-Represent-Predict paradigm.
- The selection process usually follows a greedy algorithm.
- It constructs a compact, global basis to represent waveforms instead of using local methods ("application specific spectral expansion")
- Approximate H (space of waveforms) by "best" linear combination of waveforms -- reduced basis.

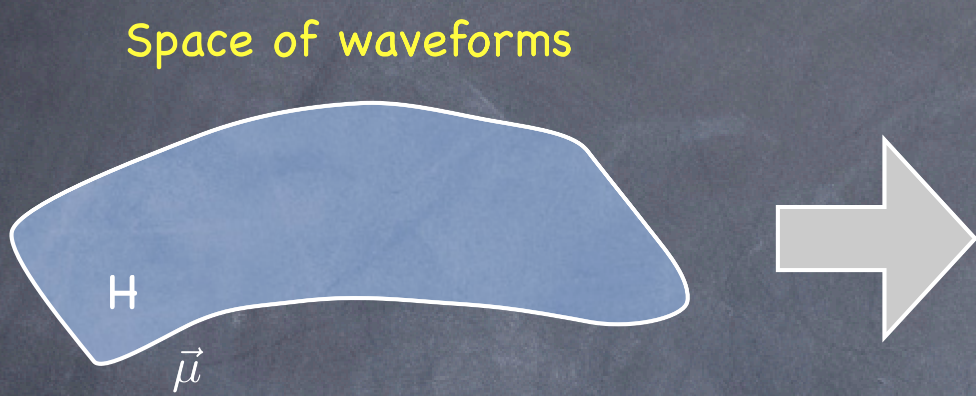
  Such waveforms can be optimally chosen so that the error in representing H by a reduced basis is minimized over the choice of N catalog members -- Kolmogorov N-width:

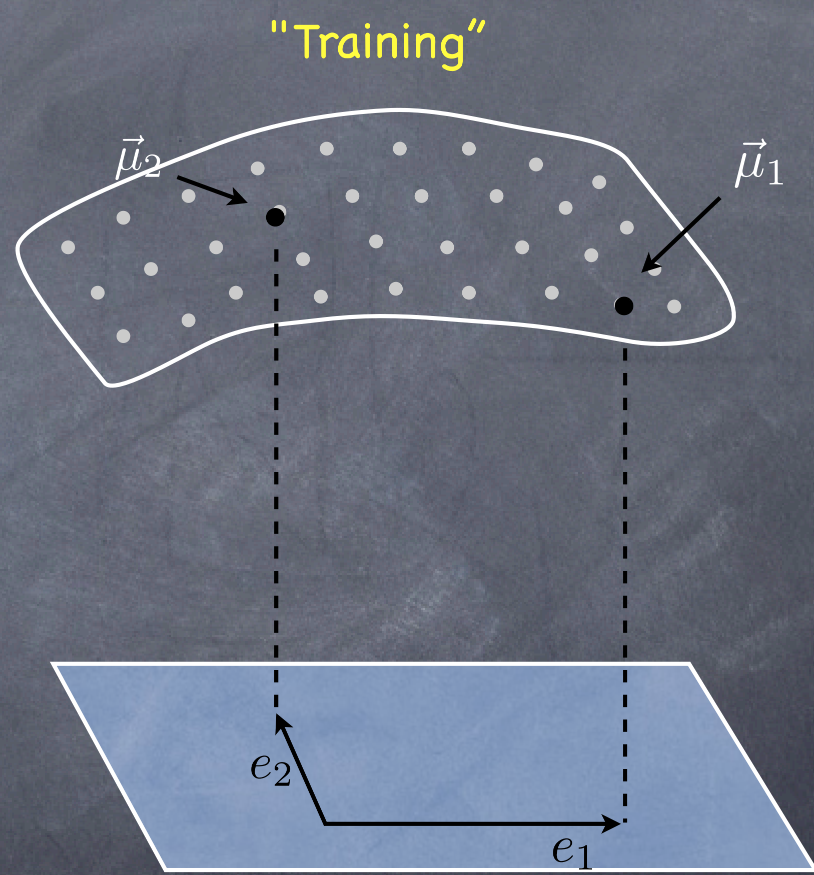
$$d_N(H) = \min_{C_N} \max_{\vec{\mu}} \min_{u \in W_N} ||u - h_{\vec{\mu}}|| \qquad \langle F, G \rangle \equiv \int_{f_L}^{f_H} \frac{F^*(f)G(f)}{S_n(f)}\, df$$

- For gravitational waves one expects the N-width to decay exponentially with N.
- However, finding a set of waveforms achieving the N-width is essentially a computationally intractable optimization problem.

Select and Represent: the Greedy algorithm (GA)

- The GA instead selects a set of solutions that <u>nearly</u> satisfies the N-width





1) Choose any parameter value from the training set and solve the eqs. for it

$$e_1 = h(\vec{\mu}_1) \qquad C_1 = \{h_{\vec{\mu}_1}\}$$

2) <u>Greedy sweep</u> - Find the parameter value that maximizes:

$$||h_{\vec{\mu}} - P_1(h_{\vec{\mu}})|| \ , \quad P_1(h_{\vec{\mu}}) = e_1 \langle e_1, h_{\vec{\mu}} \rangle$$

and solve the eqs. for it. Gram-Schmidt to get basis vector e_2

$$C_2 = \{h_{\vec{\mu}_1}, h_{\vec{\mu}_2}\} \ , \quad C_1 \subset C_2$$

3) Repeat: at each j-th step,

$$\vec{\mu}_j = \mathrm{argmax}_{\vec{\mu}} ||h(\vec{\mu}) - P_{j-1}(h(\vec{\mu}))||$$

Near optimality

- To what extent does the error in a catalog from the greedy algorithm approximate the Kolmogorov N-width catalog? [Binev et al. 2010]

$$d_N(H) \leq A e^{-cN^\alpha}$$

Kolmogorov N-width

$$\varepsilon_N \equiv \max_{\vec{\mu}} ||h_{\vec{\mu}} - P_N(h_{\vec{\mu}})|| \leq B e^{-dN^\beta}$$

$$P_N(h_{\vec{\mu}}) = \sum_{i=1}^{N} \langle e_i, h_{\vec{\mu}} \rangle e_i$$

"Greedy error"

- If the greedy error is of the order of double precision numerical round-off then the projection of the waveform onto the reduced basis is essentially equal to the waveform itself

$$h_{\vec{\mu}} = P_N(h_{\vec{\mu}}) + \delta h_{\vec{\mu}}(f) \ , \quad ||\delta h_{\vec{\mu}}(f)|| \leq \varepsilon_N$$

Idealized matched filter search with RB

- Using reduced bases to do a matched filter gravitational wave search would involve projecting the signal s onto the vector space

$$\langle s, h_{\vec{\mu}_j} \rangle = \sum_{i=1}^{N} \langle s, e_i \rangle \langle e_i, h_{\vec{\mu}_j} \rangle$$

- The inner products of e_i with the j^{th} waveform template are an output of our algorithm -- stored and known

- There are only N inner product integrals to do for a given signal, s

- N is much smaller than the number of points in the original catalog, significantly reducing the number of overlaps needed to be computed for a search

- Using RB does not increase the false alarm rate

Predict

- Need a global, high accuracy interpolation at the points picked up by the greedy algorithm

- A hierarchical process, where as more points are added, the previous ones and the associated computations to build the interpolation are reused -- constant complexity

- Inexpensive

- This is not standard polynomial interpolation. Instead, the bases are the members of the reduced basis.

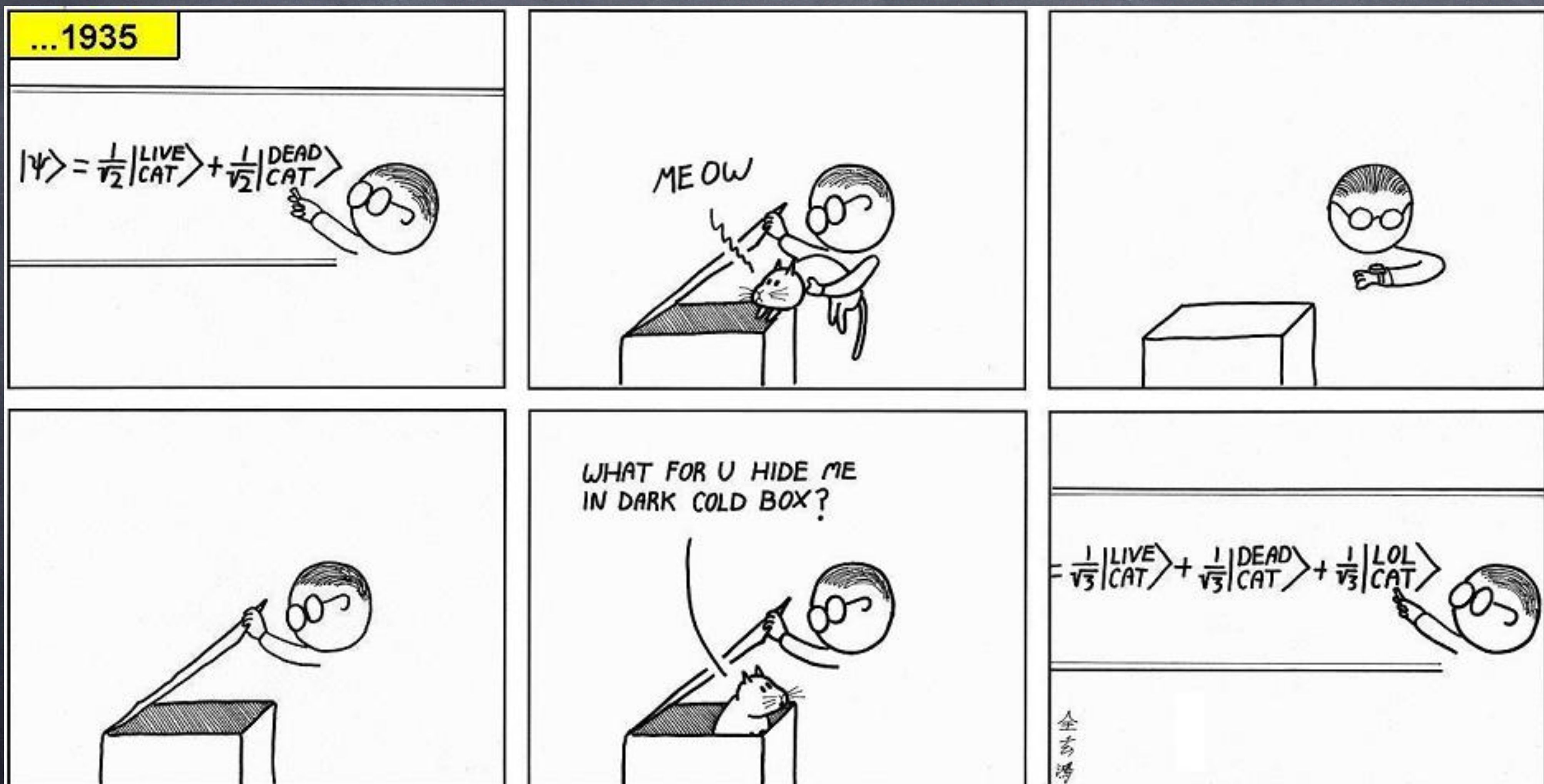
  The Empirical Interpolation Method (EIM, Barrault et al. 2004)

- If the error when building a reduced basis decays exponentially with the number of solutions, so does the interpolation one when using EIM (Maday et al. 2007).

- In the context of gravitational waves: Harbir Antil et al. (in progress)

What's the big deal with interpolation?

- What's the big deal? Even my cat knows how to compute a spline interpolant

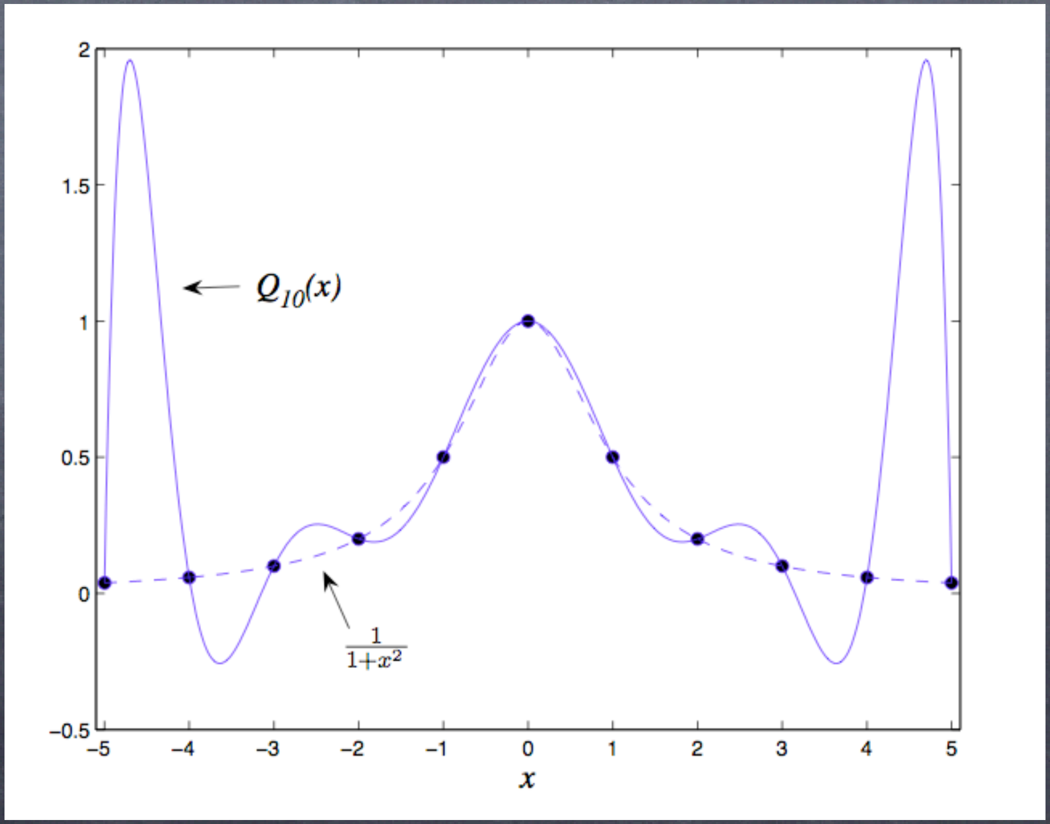


- First, in order to preserve the accuracy of the reduced basis, which is global, you need to do a global interpolation

- Second, you need to interpolate at an unstructured mesh (the greedy points), because that's where you solved for

- Ok, so what's the big deal with doing a global interpolation? My dog could do that

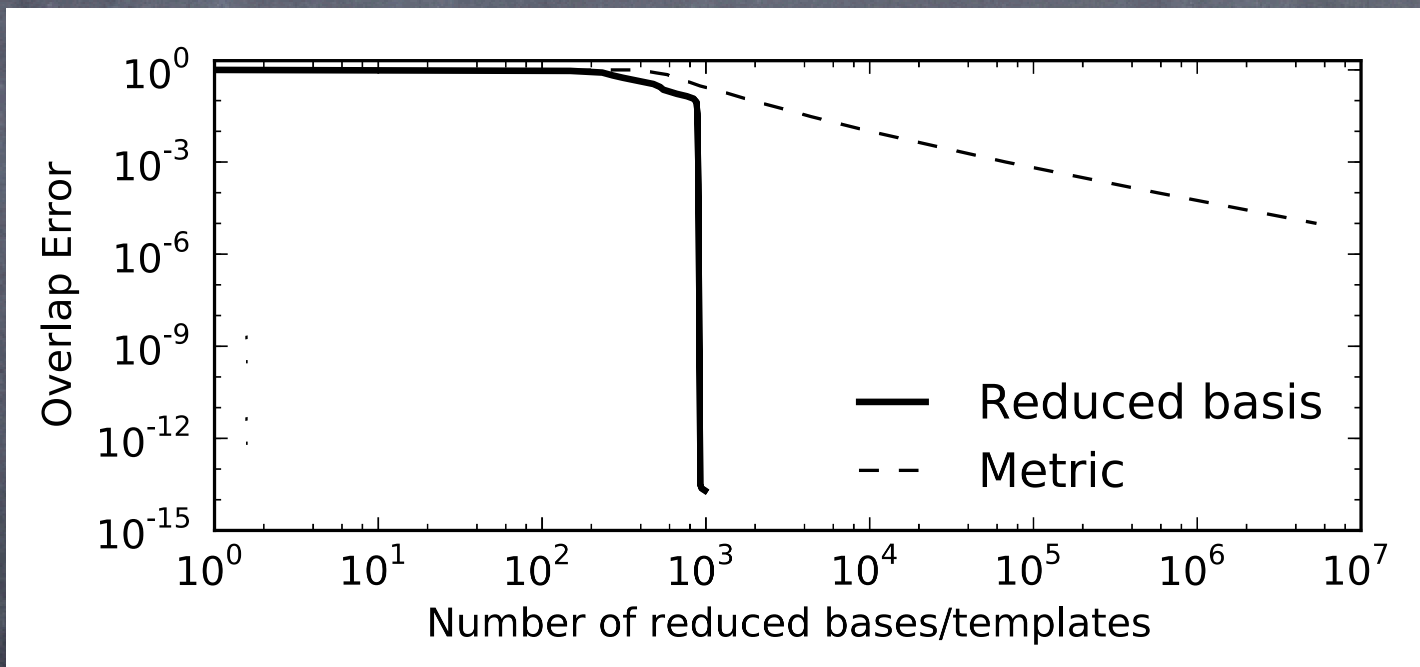
Runge's example



- Global interpolation on unstructured meshes: Narayan et al. (in preparation)

RB catalogs for inspiral, non-spinning binaries

- 2PN stationary-phase-approximation (SPA) "chirp" waveforms. These are known in closed form, no need to solve for them.



$$\varepsilon_N = Be^{-dN^{\beta}} \qquad B = 0.0311,\ d = 0.299,\ \beta = 1.25$$

$$\varepsilon_N^2 = \max_{\vec{\mu}} \left(1 - \mathrm{Re}\langle h_{\vec{\mu}}, P_N(h_{\vec{\mu}})\rangle\right) = 1 - MM$$

Reduced Basis catalogs

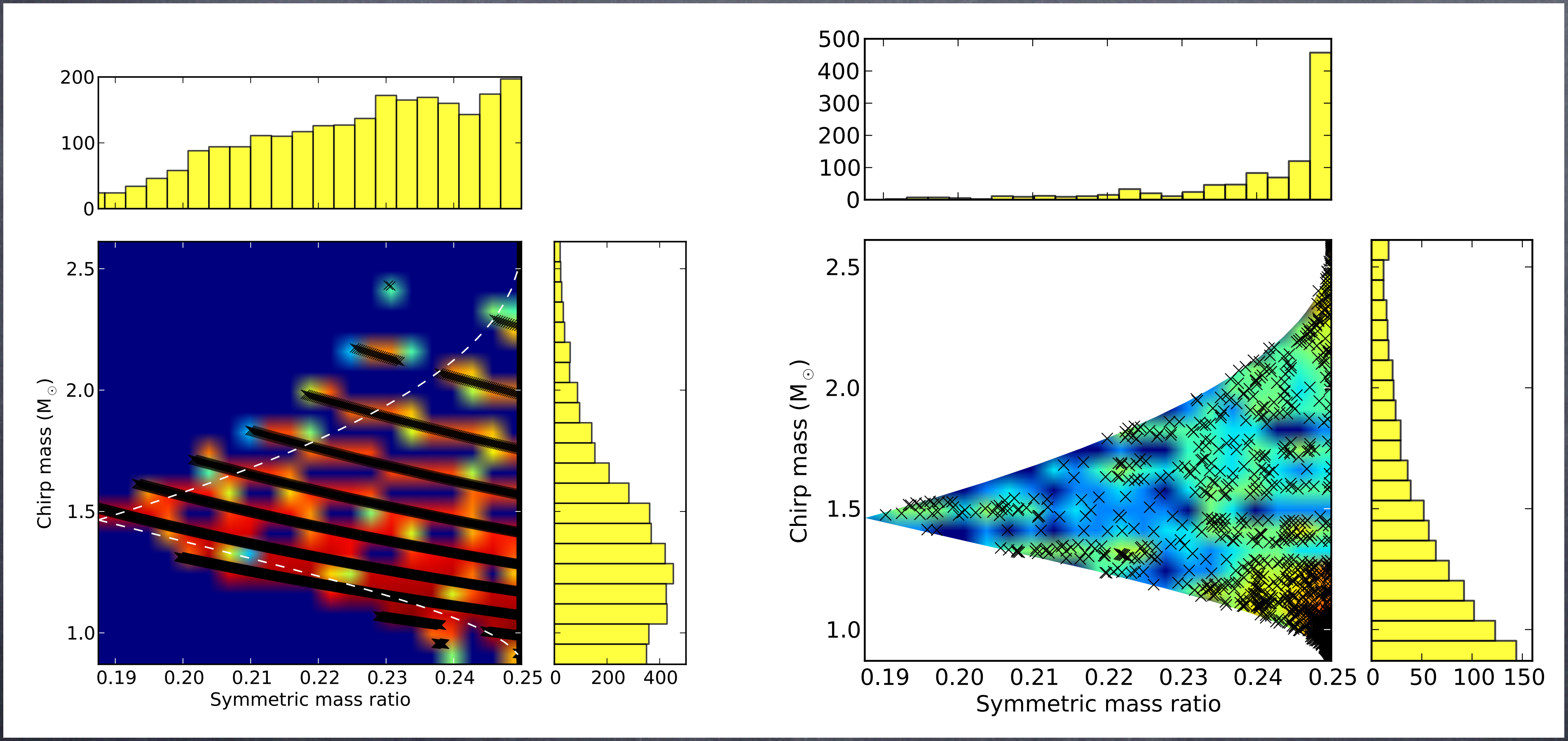
- BNS: Binary Neutron Stars, individual masses in the range [1,3] Solar Masses
- BBH: Binary Black Holes, individual masses in the range [3,30] Solar Masses

Detector	Overlap Error	BBH		BNS	
		RB	TM	RB	TM
InitLIGO	10^{-2}	165	$2,450$	898	$10,028$
	10^{-5}	170	1.2×10^6	904	4.3×10^6
	2.5×10^{-13}	182	5.9×10^{12}	917	1.4×10^{13}
AdvLIGO	10^{-2}	$1,058$	$19,336$	$5,395$	$72,790$
	10^{-5}	$1,687$	1.5×10^7	$8,958$	4.9×10^7
	2.5×10^{-13}	$1,700$	2.3×10^{14}	$8,976$	5.6×10^{14}
AdvVirgo	10^{-2}	$1,395$	$42,496$	$7,482$	$156,127$
	10^{-5}	$1,690$	3.1×10^7	$8,960$	8.3×10^7
	2.5×10^{-13}	$1,703$	4.8×10^{14}	$8,977$	6.0×10^{14}

Distribution of selected parameters

Metric placement
MM = 0.97
Owen (1996)

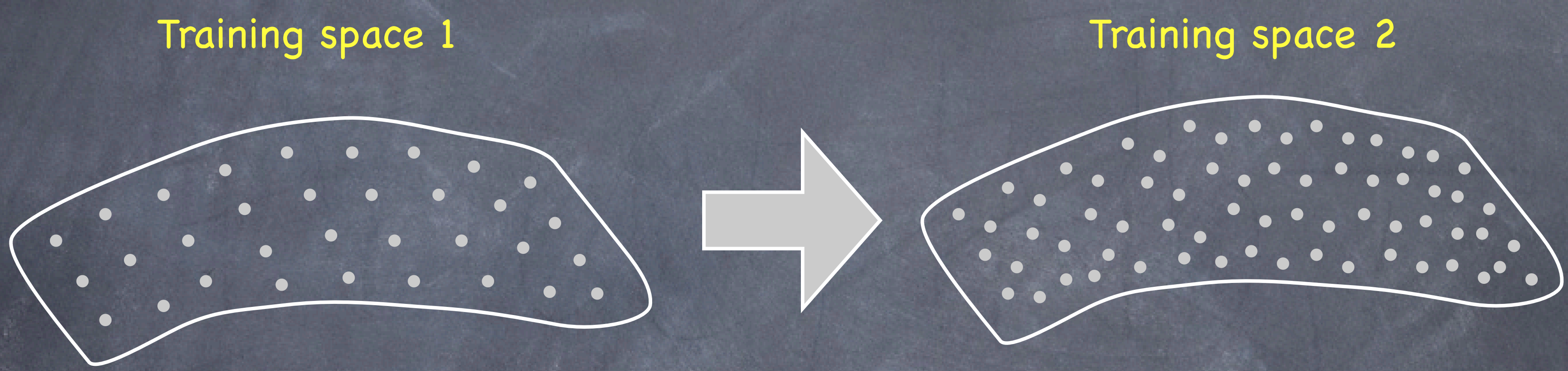
Reduced Basis
MM = 1 - 2.5 x 10^{-13}



(Inspiraling BNS, Initial LIGO)

A(nother) surprising result

- What can we learn through RB at this stage, if anything, about fundamental properties of gravitational waves?
- Imagine increasing the number of points in the training space

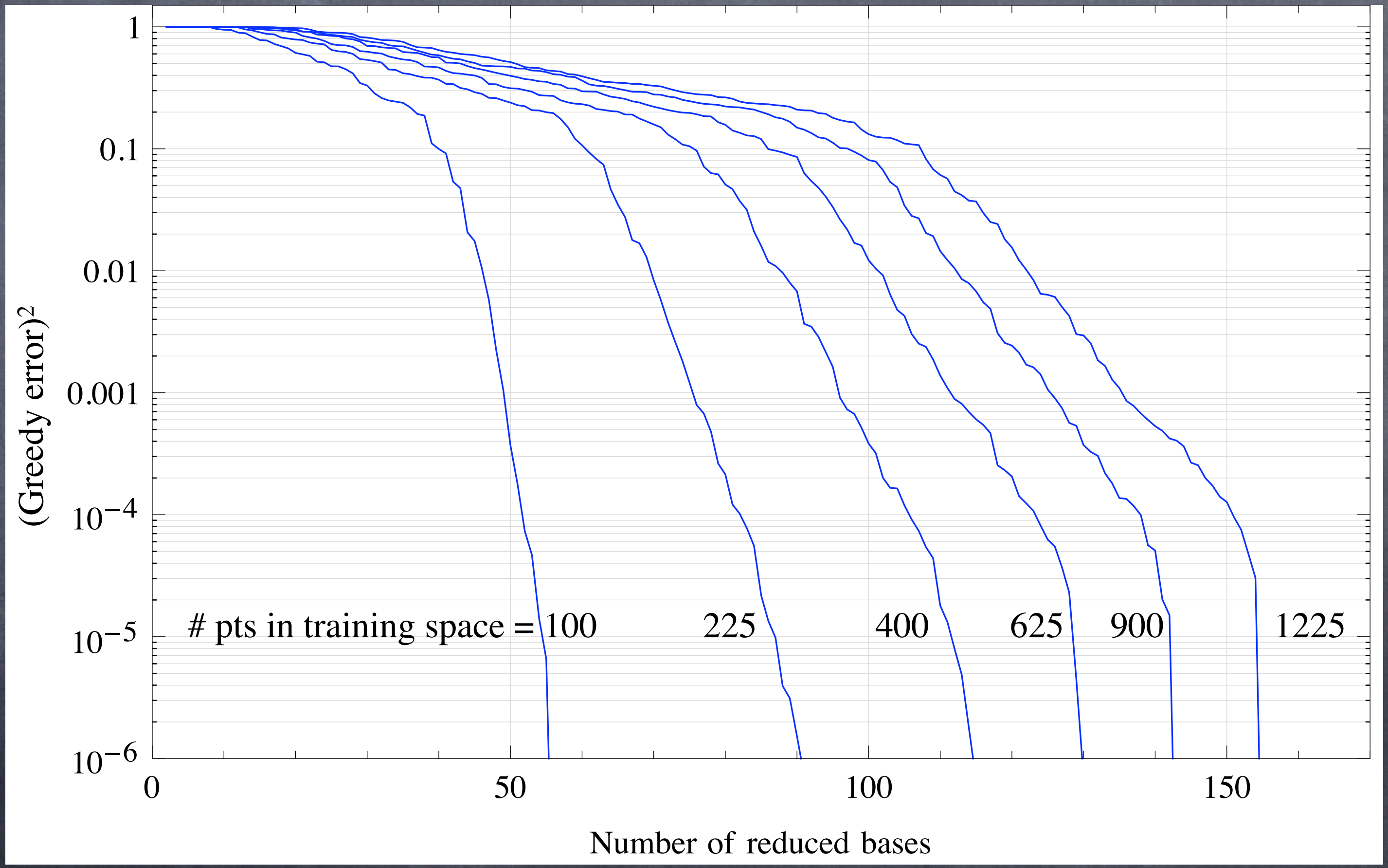


There is consequently a different and larger # of RBs

- However, in all cases we found that for any finite range of masses the number of RBs asymptotes to a finite value as the number of points in the training space tends to infinity.
- We conjecture that this is in general true.

- Example:

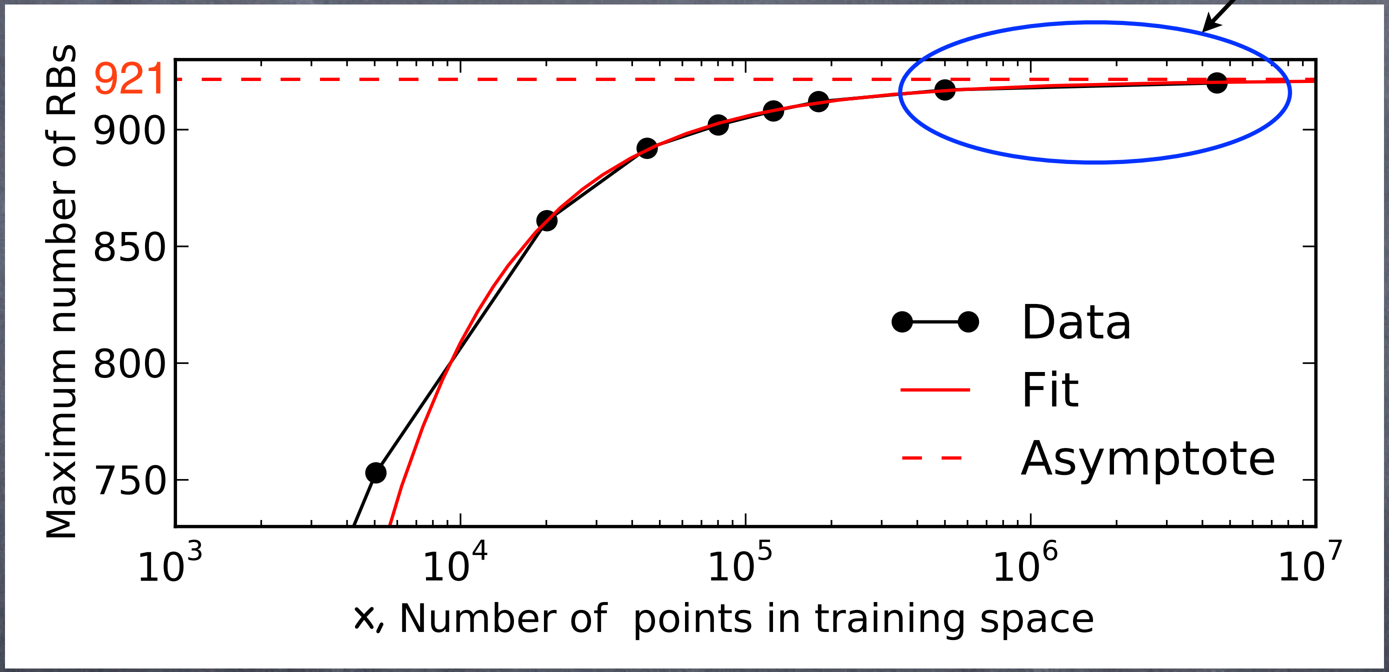
Here shown for inspiraling BBH with component masses of [3-30] solar masses, 2PN SPA waveforms, and Initial LIGO



- Example:

For BNS with components of [1-3] solar masses, 2PN SPA waveforms, and for Initial LIGO

Not included in fit



Fit: $N = a + bx^{-1/2} + cx^{-1}$

$$a = 921,\ b = -2090,\ c = -9.18 \times 10^5$$

- The implication is that the space of waveforms is essentially a finite dimensional space since it can be represented by a finite number of RBs that yield errors smaller than double precision numerical round-off
  Similar findings more recently reported by Cannon, Kanna, Keppel (arXiv: 1101.4939)

- Example:

  - Space of inspiral 2PN SPA waveforms of BNS for Initial LIGO can be codified in a 921-dimensional vector space

  - Consequently a matched-filter search will only involve the computation of 921 overlap integrals

$$\langle s, h_{\vec{\mu}} \rangle = \sum_{i=1}^{921} \langle s, e_i \rangle \langle e_i, h_{\vec{\mu}} \rangle$$

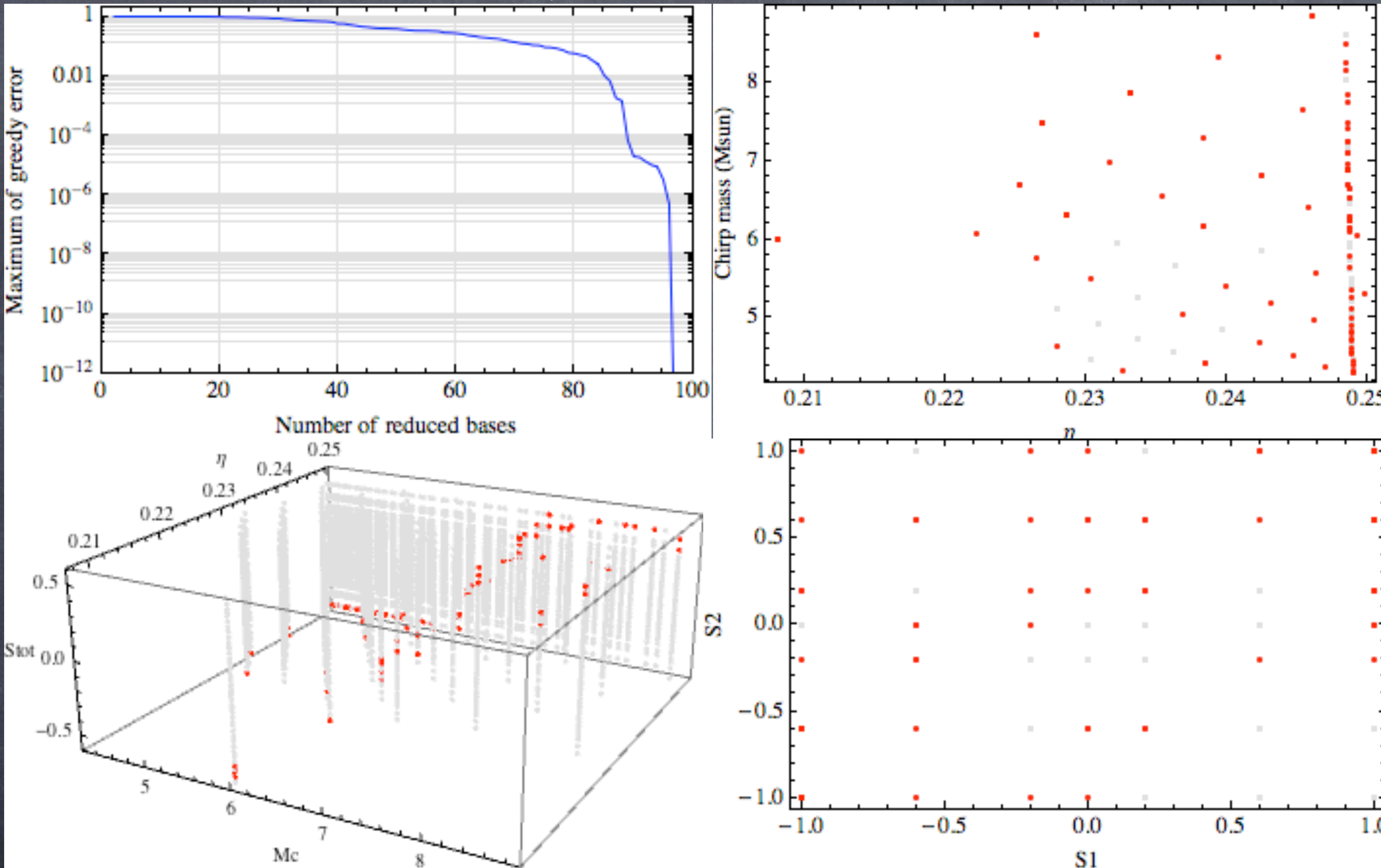
Output from
RB algorithm

Including spin in non-precessing inspirals

(in progress)

- Chirp waveforms, each spin aligned or anti-aligned with the orbital angular momentum
- Masses chosen by the metric template placement approach with MM=0.97, Init. LIGO, for the individual masses
- For each mass pair, we uniformly sample the dimensionless spin parameter at Ns values in the interval [-1,1]
- Shown next: individual masses in the range [5,10] Solar Masses each, Ns=7

VERY preliminary results



- 4,900 templates, 97 RBs
- Selected waveforms mostly have small chirp mass and anti-aligned spins, or large mass and aligned spins.

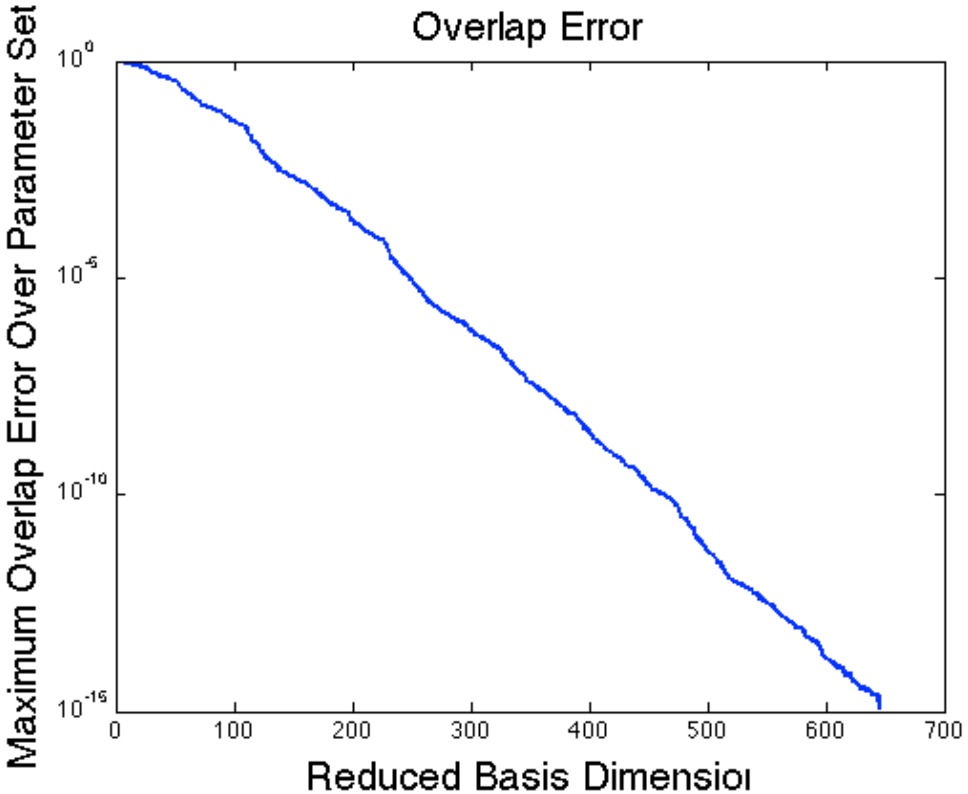
Two mode single black hole ringdown waveforms

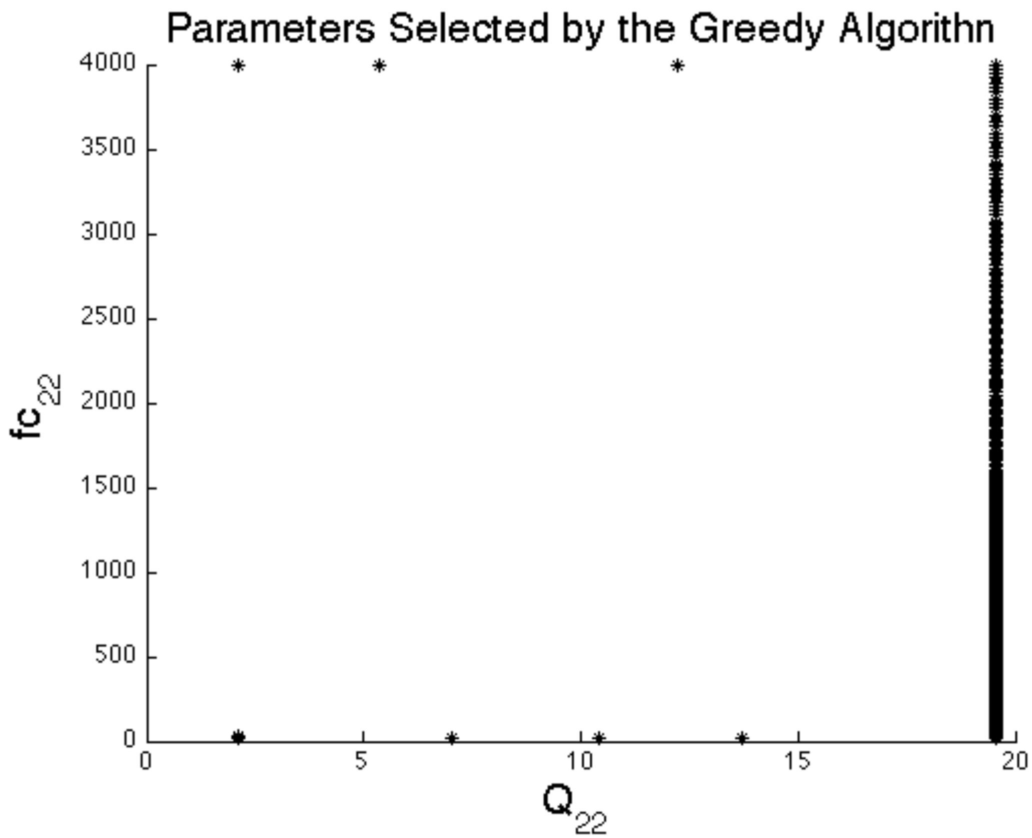
(Sarah Caudill, S. Field, C. Galley, F. Herrmann, MT, in progress)

- Current ringdown searches use one mode (l=2,m=2) catalogs, which are rather small compared to the inspiral case.

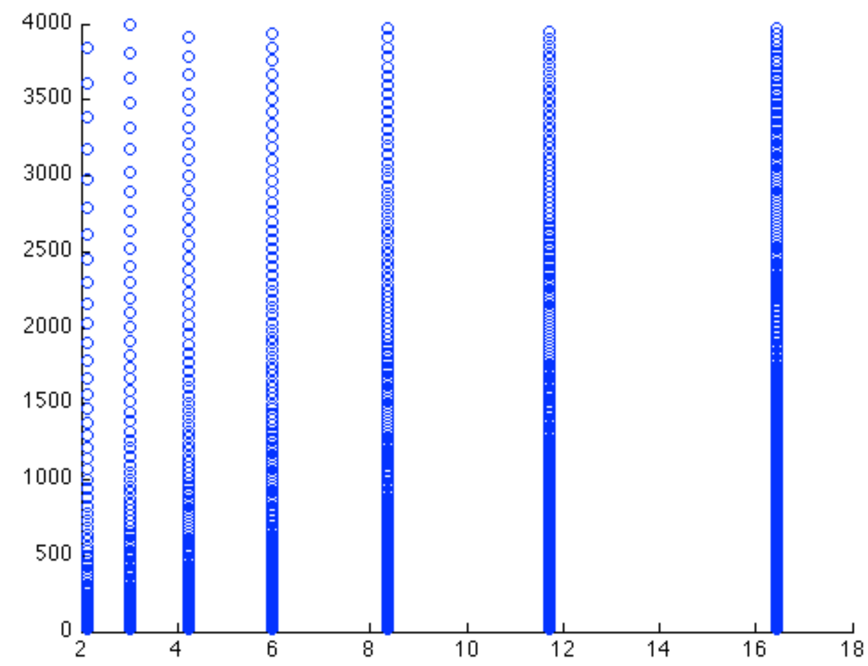
$$h(t) = Ae^{-\pi f t/Q} \sin\left(2\pi f t - \phi\right)$$

- For example, for MM=0.01 and Adv LIGO, there are 1,956 templates, 17,599 for MM=0.001, etc.
- For MM=e-4, there are 170,454 templates. And around 659 reduced bases are needed to represent them within double precision numerical roundoff.



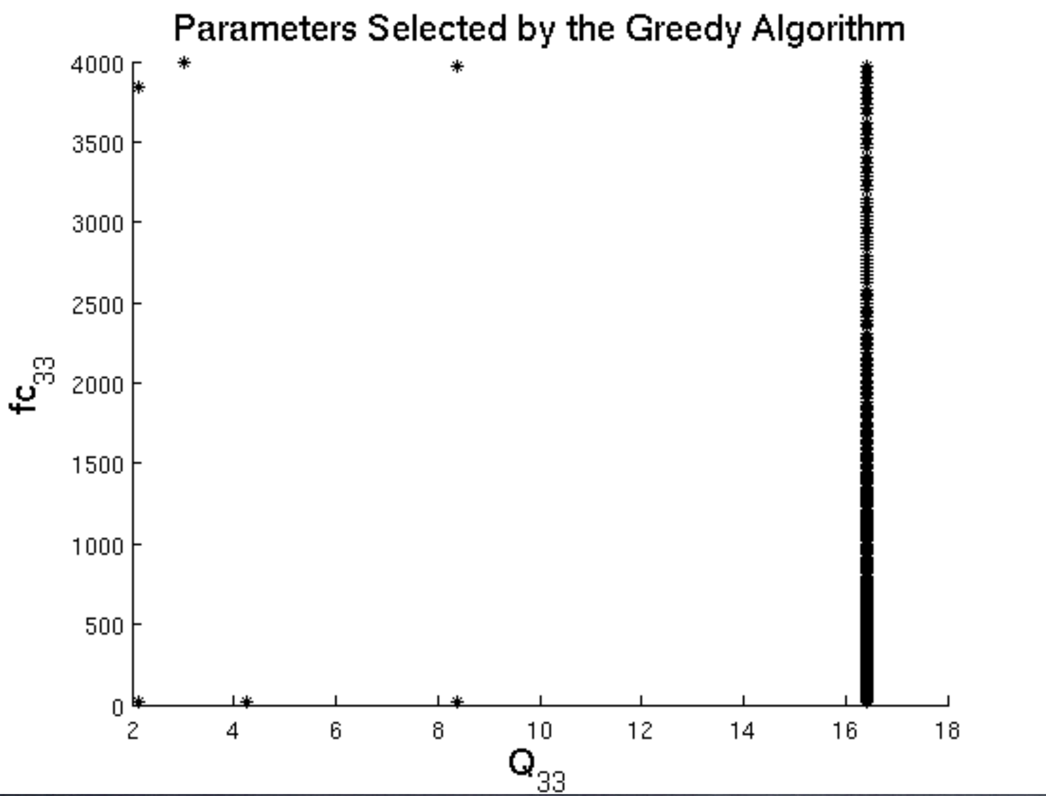


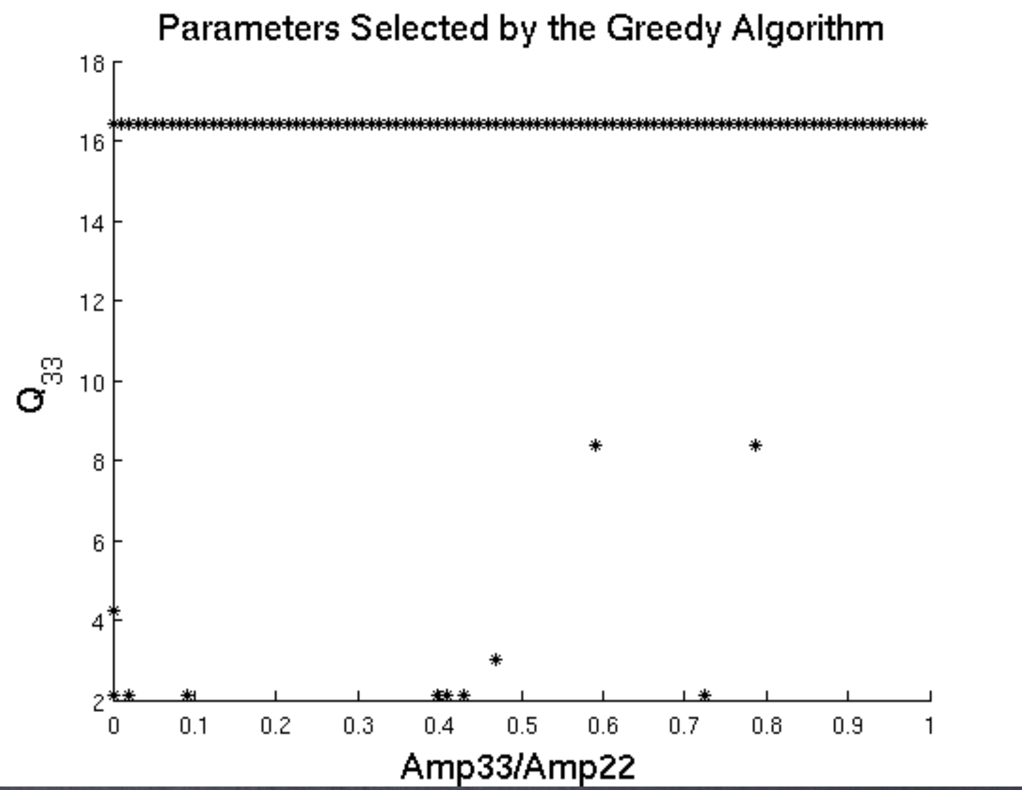
To illustrate: metric template training set from a coarser MM ->



Two-mode catalogs

- From the metric template placement for the (2,2) mode at MM=0.01 add a (3,3) one with Namp relative amplitudes in [0,1]





MM=0.01 x 99 amplitudes = 193,664 templates. 578 reduced bases to represent them within double precision numerical roundoff

Summary and roadmap

- RB provides an attractive method to efficiently generate hierarchical, compact, and nearly optimal waveform template banks when the waveforms are either known or must be computed "on the fly"
- Provides a framework for selecting the optimal parameters for a future NRAR-type collaboration
- Due to the exponential convergence, the number of RBs needed to achieve an error of the order of numerical round-off is very small.
- The space of waveforms is essentially finite dimensional.
- Roadmap for the "near" to medium term future:

  Interpolation

  Inspiral-Merger-Ringdown EOB and phenomenological waveforms

  Include precession at the post-Newtonian level

  Gradually start looking at full numerical relativity RB simulations