# THE PROBLEM WITH BOUNDARIES IN GENERAL RELATIVITY

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## SOME THINGS I HAVE LEARNED FROM COMPUTATIONAL MATHEMATICIANS

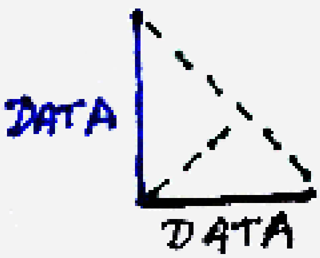
**BREAK COMPLICATED PROBLEMS UP INTO SIMPLER PROBLEMS**

**EMPLOY SIMPLE, DISCRIMINATING CODE TESTS**

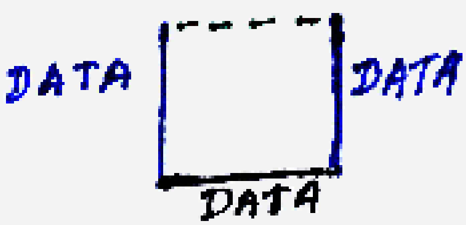
**FINITE PROPAGATION SPEED OF HYPERBOLIC SYSTEMS IMPLIES PROBLEMS CAN BE TREATED PIECEWISE**

1. The Cauchy problem


2. The half-space problem



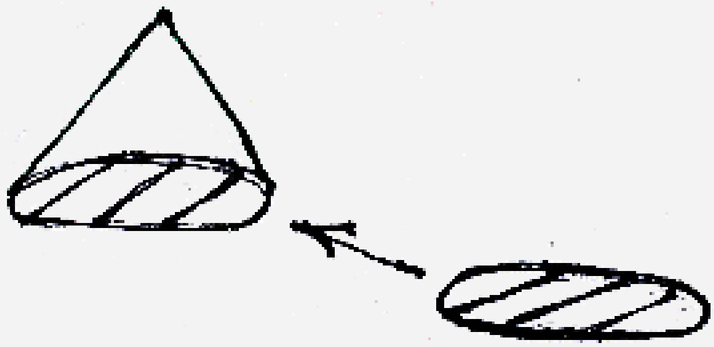

3. The strip problem




**WAVE EQUATIONS SHOULD BE TREATED IN SECOND ORDER FORM**

"Initial-boundary value problems for second order systems of partial differential equations", Kreiss, Ortiz, Petersson

# THE CAUCHY PROBLEM



DISEMBODIED CAUCHY DATA $h_{ab}$ $k_{ab}$
SUBJECT TO CONSTRAINTS

DETERMINES WELL POSED
GEOMETRICALLY UNIQUE SPACETIME

WELL POSEDNESS REQUIRES
STRONGLY HYPERBOLIC REDUCTION
OF EINSTEIN'S EQUATIONS

2ND ORDER WAVE EQUATIONS
(harmonic formulation, Choquet-Bruhat)

OR

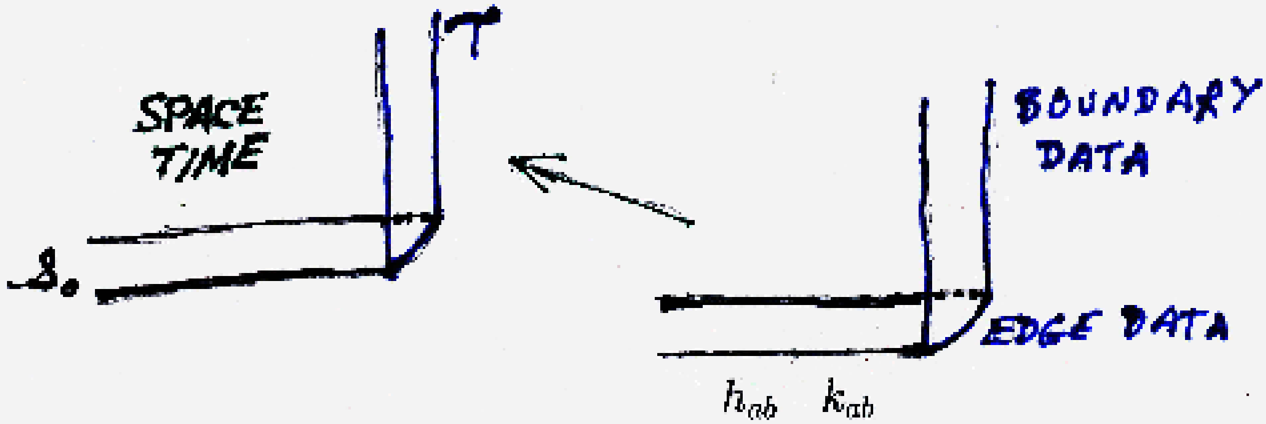
STRONGLY HYPERBOLIC FIRST ORDER
FORMULATION (BSSN)

BUT NOT

XXX WEAKLY HYPERBOLIC (ADM)

APPLES WITH APPLES
*Class. Quant. Grav.* 25, 125012 (2008)

# THE INITIAL-BOUNDARY VALUE PROBLEM




GEOMETRIC UNIQUENESS ???

WELL POSEDNESS ???
REQUIRES SYMMETRIC HYPERBOLIC REDUCTION

USE Energy method - integration by parts

OR Fourier-Laplace method - pseudo-differential theory

H-O. Kreiss and J. Lorenz, "Initial-boundary value problems and the Navier-Stokes equations"

EARLY WORK - PARTIAL RESULTS

J. Stewart

G. Calabrese, J. Pullin, O. Reula, O. Sarbach, M. Tiglio

WELL POSED FORMULATIONS

Friedrich, Nagy: Energy treatment of frame-connection-curvature formulation

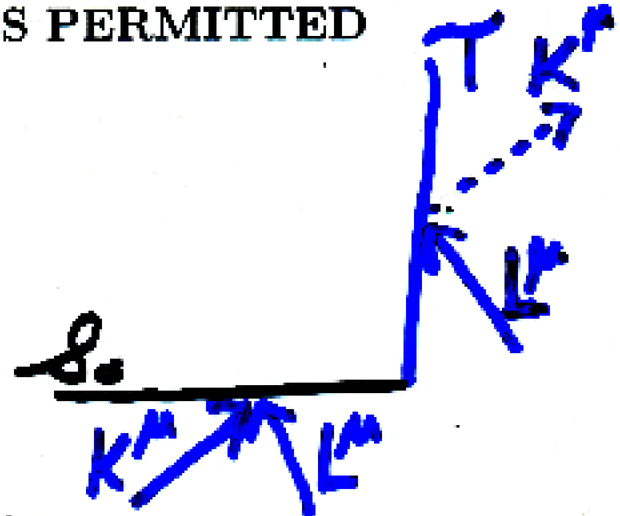
Kreiss, Winicour: Pseudo-differential treatment of harmonic formulation

Kreiss, Reula, Sarbach, Winicour: Energy treatment of harmonic formulation

A COMPREHENSIVE UNDERSTANDING REMAINS AN OUTSTANDING PROBLEM

## COMPLICATION OF BOUNDARY TREATMENT

## ONLY HALF THE DATA IS PERMITTED




For scalar wave $\Phi$ can prescribe

Dirichlet data: $\partial_T \Phi = q$

or

Neumann data: $\partial_N \Phi = q$

or

Sommerfeld data: $K^\mu \partial_\mu \Phi = (\partial_T + \partial_N)\Phi = q$
where $K^\mu$ is outgoing null direction

Same boundary data $q$ leads to different solutions depending upon the boundary condition

## CANNOT PRESCRIBE BOTH 3-METRIC AND EXTRINSIC CURVATURE

Instead, for example, give
Sommerfeld data $= K^\mu \partial_\mu g_{\rho\sigma}$

## THIS COMPLICATES CONSTRAINT ENFORCEMENT

Hamiltonian and momentum constraints cannot be enforced directly.

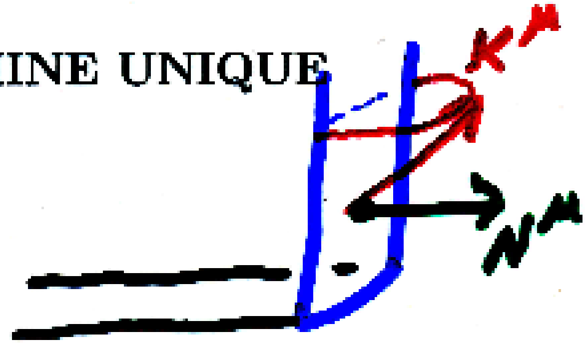
## DOMAIN OF DEPENDENCE OF BOUNDARY IS EMPTY

This couples the Cauchy problem with boundary problem

## COMPLICATIONS WITH SOMMERFELD CONDITION

$$\text{Sommerfeld data} = K^{\mu}\partial_{\mu}g_{\rho\sigma}$$

### BOUNDARY DOES NOT DETERMINE UNIQUE NULL DIRECTION




Resolution: Foliate boundary

### WHAT IS GEOMETRIC NATURE OF $\partial_{\mu}$ ?

Resolution: Use Cauchy data to introduce background metric

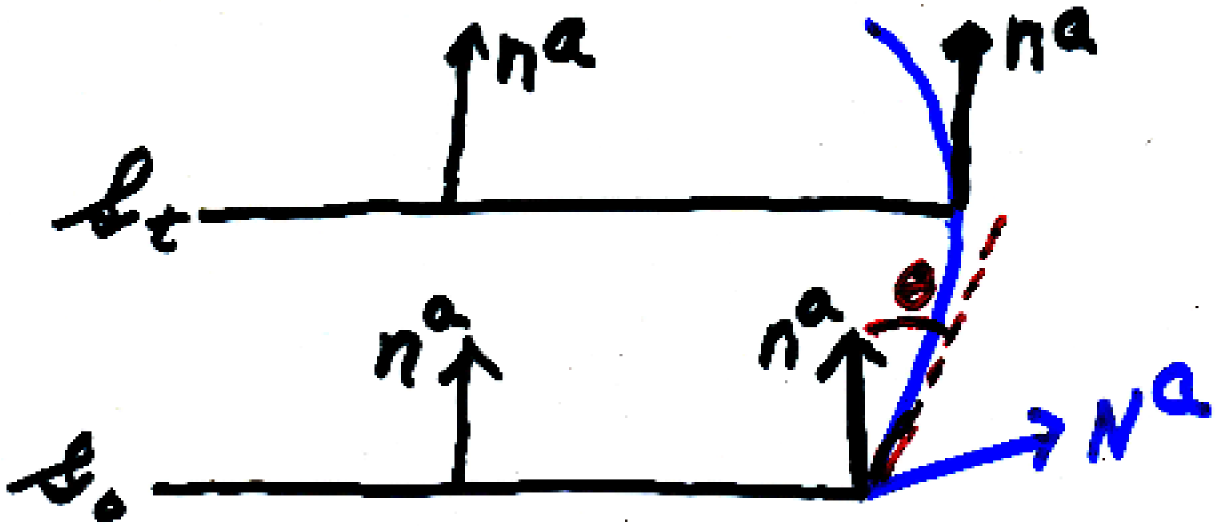
### WHAT IS GEOMETRIC CONTENT OF SOMMERFELD DATA?

Acceleration and shear of $K^{\mu}$ relative to background

### HOW DO YOU PRESCRIBE SOMMERFELD DATA?

For an isolated system, homogeneous data is a good approximation for a large spherical outer boundary

# COMPLICATIONS FROM THE MOTION OF THE BOUNDARY




## THE BOUNDARY MOVES RELATIVE TO THE CAUCHY HYPERSURFACES

$$N^a n_a = \sinh \Theta$$

Unit outward normal to boundary $N^a$

Unit future normal to Cauchy hypersurfaces $n_a = -\alpha \nabla_a t$ (shift $\alpha$)

## GEOMETRIC SPECIFICATION OF THE BOUNDARY REQUIRES NON-SOMMERFELD DATA

e.g. (Friedrich-Nagy) MEAN EXTRINSIC CURVATURE

Variables satisfying advective equation

$$n^a \partial_a \Phi = \ldots$$

pose difficulty at boundary if $\Theta \neq 0$

This forces a Dirichlet condition on the normal component of the shift in some 3+1 formulations

# SIMPLE EXAMPLE OF STRONG WELL-POSEDNESS

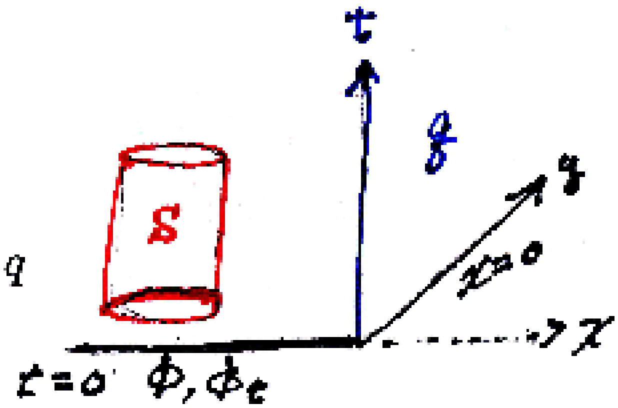
$$\phi_{tt} = \phi_{xx} + \phi_{yy} + S\,, \quad -\infty < x \leq 0$$

periodic in $y$

Cauchy data: $\phi|_{t=0}$, $\phi_t|_{t=0}$

Boundary condition: $(\phi_t + \alpha\phi_x + \beta\phi_y)|_{x=0} = q$

$\alpha > 0$




Energy norm: $E = \frac{1}{2}(||\phi_t^2|| + ||\phi_x^2|| + ||\phi_y^2||)$, $\quad ||F^2|| = \int F^2 dxdy$

Integration by parts:

$$\partial_t E = \int \{\phi_{tt}\phi_t + \phi_{xt}\phi_x + \phi_{yt}\phi_y\} dxdy$$

$$= \int \{(\phi_{xx} + \cancel{\phi_{yy}} + S)\phi_t + \phi_{xt}\phi_x + \cancel{\phi_{yt}\phi_y}\} dxdy$$

$$\leq \int_{x=0} \phi_t\phi_x dy + \frac{1}{2}(||\phi_t^2|| + ||S^2||)$$

$$\leq \int_{x=0} (-\alpha\phi_x - \beta\phi_y + q)\phi_x dy + E + \frac{1}{2}||S^2||$$

$$\leq \int_{x=0} (-\frac{\alpha}{2}\phi_x^2 + \frac{1}{2\alpha}q^2) dy + E + \frac{1}{2}||S^2|| - \beta \int_{x=0} \phi_y\phi_x dy$$

So, if $\beta = 0$

$$\partial_t E + \tfrac{\alpha}{2}||\phi_x^2||_B \leq \tfrac{1}{2\alpha}||q^2||_B + E + \tfrac{1}{2}||S^2||$$

Similar estimates for $||\phi_t^2||_B$ and $||\phi_y^2||_B$ imply *strong well-posedness*:

$$E(T) + \int_0^T \left(||\phi_t^2||_B + ||\phi_x^2||_B + ||\phi_y^2||_B\right) dt$$
$$\leq const\{E(0) + \int_0^T (||q^2||_B + ||S^2||) dt\}$$

**WHAT HAPPENS WHEN $\beta \neq 0$?**

# MAXWELL'S EQUATIONS

**Symmetric hyperbolic system for $\bar{E}, \bar{B}$ with trivial constraint propagation**

$\mathcal{C}_B := \bar{\nabla} \cdot \bar{B}$

$\partial_t \mathcal{C}_B = -\bar{\nabla} \cdot \bar{\nabla} \times \bar{E} = 0$

**Similar to Friedrich, Nagy system, constraints propagate up the boundary**

## VECTOR POTENTIAL FORMULATION
## LORENTZ GAUGE

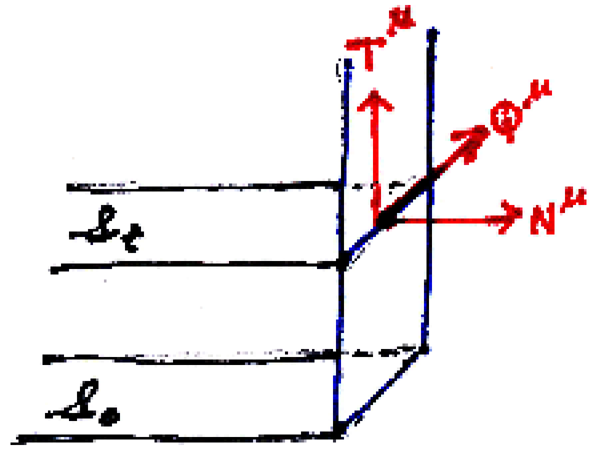
$\eta^{\alpha\beta}\partial_\alpha\partial_\beta A_\mu = 0$

$\mathcal{C} := \partial_\alpha A^\alpha$

$\eta^{\alpha\beta}\partial_\alpha\partial_\beta \mathcal{C} = 0$

**Cauchy problem well posed, constraint preserving**

## CONSTRAINT PRESERVING SOMMERFELD CONDITIONS




**Null tetrad adapted to boundary**

$\eta_{\mu\nu} = -K_{(\mu}L_{\nu)} + Q_{(\mu}\bar{Q}_{\nu)}$

$K^\mu = T^\mu + N^\mu, \quad L^\mu = T^\mu - N^\mu$

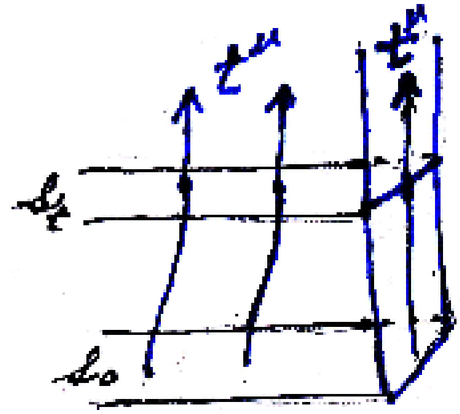
$$K^\mu\partial_\mu(K^\nu A_\nu) = q_K$$

$$K^\mu\partial_\mu(Q^\nu A_\nu) - Q^\mu\partial_\mu(K^\nu A_\nu) = q_Q$$

$$-2\mathcal{C} = K^\mu\partial_\mu(L^\nu A_\nu) + \left(L^\mu K^\nu - Q^\mu\bar{Q}^\nu - \bar{Q}^\mu Q^\nu\right)\partial_\mu A_\nu = 0$$

**CONSTRAINT INVOLVES SIDEWAYS DERIVATIVES ON BOUNDARY BUT THESE SOMMERFELD CONDITIONS HAVE HIERARCHICAL, UPPER TRIANGULAR FORM WHICH GIVES RISE TO STRONG WELL POSEDNESS**

# THE HARMONIC EINSTEIN SYSTEM



Fix an evolution field $t^\mu$

All choices of $t^\mu$ are related by diffeomorphism and this fixes gauge

Use initial Cauchy data to determine a background metric by Lie transport

$$\mathring{g}_{\mu\nu}|_{t=0} = g_{\mu\nu}|_{t=0} \quad \mathcal{L}_t \mathring{g}_{\mu\nu} = \mathcal{L}_t g_{\mu\nu}|_{t=0}$$

The difference $f_{\mu\nu} = g_{\mu\nu} - \mathring{g}_{\mu\nu}$ has homogeneous Cauchy data

$$f_{\mu\nu}|_{t=0} = 0\,, \quad \mathcal{L}_t f_{\mu\nu}|_{t=0} = 0$$

The difference in Christoffel symbols is tensor field

$$C^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \mathring{\Gamma}^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \mathring{\nabla}_\mu f_{\nu\sigma} + \mathring{\nabla}_\nu f_{\mu\sigma} - \mathring{\nabla}_\sigma f_{\mu\nu} \right)$$

Assume generalized harmonic formulation with constraints

$$C^\rho := g^{\mu\nu} \left( \Gamma^\rho_{\mu\nu} - \mathring{\Gamma}^\rho_{\mu\nu} \right) = -\frac{1}{\sqrt{-g}} \mathring{\nabla}_\nu (\sqrt{-g} g^{\rho\nu})$$

HARMONIC EINSTEIN EQUATIONS REDUCE TO QUASILINEAR WAVE SYSTEM

$$g^{\rho\sigma} \mathring{\nabla}_\rho \mathring{\nabla}_\sigma f_{\mu\nu} = \text{LOWER ORDER TERMS}$$

BIANCHI IDENTITIES GOVERN CONSTRAINT PROPAGATION

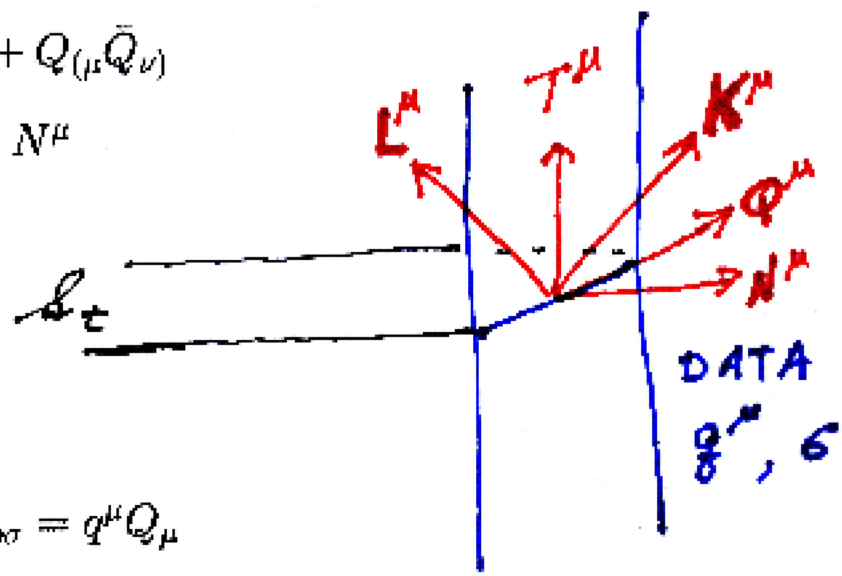
$$\nabla^\rho \nabla_\rho C^\mu + R^\mu_\rho C^\rho = 0$$

CAUCHY PROBLEM IS WELL POSED

# HIERARCHY OF SOMMERFELD BOUNDARY CONDITIONS

**Null tetrad** $g_{\mu\nu} = -K_{(\mu}L_{\nu)} + Q_{(\mu}\bar{Q}_{\nu)}$

$K^\mu = T^\mu + N^\mu, \quad L^\mu = T^\mu - N^\mu$




$$\tfrac{1}{2}K^\rho K^\sigma K^\mu \mathring{\nabla}_\mu f_{\rho\sigma} = q^\mu K_\mu$$

$$(Q^\rho K^\sigma K^\mu - \tfrac{1}{2}K^\rho K^\sigma Q^\mu)\mathring{\nabla}_\mu f_{\rho\sigma} = q^\mu Q_\mu$$

$$(L^\rho K^\sigma K^\mu - \tfrac{1}{2}K^\rho K^\sigma L^\mu)\mathring{\nabla}_\mu f_{\rho\sigma} = q^\mu L_\mu$$

$$(\tfrac{1}{2}Q^\rho Q^\sigma K^\mu - Q^\rho K^\sigma Q^\mu)\mathring{\nabla}_\mu f_{\rho\sigma} = 2\sigma$$

**REMAINING SOMMERFELD CONDITIONS ENFORCE HARMONIC CONSTRAINTS $C^\mu = 0$ ON BOUNDARY**

$$-2C^\mu K_\mu = \left(Q^\rho \bar{Q}^\sigma K^\mu + K^\rho K^\sigma L^\mu - K^\rho \bar{Q}^\sigma Q^\mu - K^\rho Q^\sigma \bar{Q}^\mu\right)\mathring{\nabla}_\mu f_{\rho\sigma} = 0$$

$$-2C^\mu Q_\mu = \left(L^\rho Q^\sigma K^\mu + K^\rho Q^\sigma L^\mu - K^\rho L^\sigma Q^\mu + Q^\rho Q^\sigma \bar{Q}^\mu\right)\mathring{\nabla}_\mu f_{\rho\sigma} = 0$$

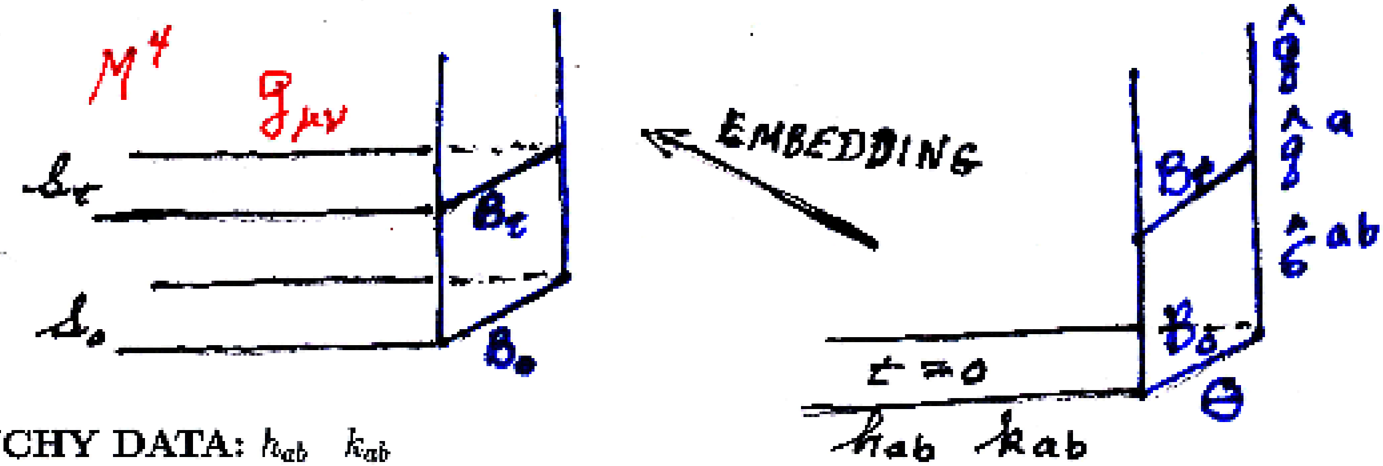
$$-2C^\mu L_\mu = \left(L^\rho L^\sigma K^\mu + Q^\rho \bar{Q}^\sigma L^\mu - \bar{Q}^\rho L^\sigma Q^\mu - Q^\rho L^\sigma \bar{Q}^\mu\right)\mathring{\nabla}_\mu f_{\rho\sigma} = 0$$

**SEQUENTIAL ORDER**

$(KK), (QK), (LK), (QQ), (Q\bar{Q}), (LQ), (LL)$

**OF COMPONENTS $K^\mu \mathring{\nabla}_\mu f_{\rho\sigma}$ ENSURES A STRONGLY WELL-POSED INITIAL-BOUNDARY VALUE PROBLEM**

# DISEMBODIED HARMONIC DATA FOR A GEOMETRICALLY UNIQUE SPACETIME




CAUCHY DATA: $h_{ab}$ $k_{ab}$

BOUNDARY DATA: Foliation $\mathcal{B}_t$ determined by evolution field $t^a$

$\hat{q}$, $\hat{q}^a$, $\hat{\sigma}^{ab}$ (rank 2: $\hat{\sigma}^{ab}\nabla_b t = 0$)

EDGE DATA $\sinh\Theta$: Initial velocity of boundary

**DISEMBODIED DATA DETERMINES UNIQUE SPACETIME UP TO DIFFEOMORPHISM**

## 4D GEOMETRIC INTERPRETATION OF SOMMERFELD DATA

Outgoing null direction $K^\mu$ Boundary normal $N^\mu$

2-metric of $\mathcal{B}_t$ $Q_{\mu\nu} = Q_{(\mu}\bar{Q}_{\nu)}$

Evolution field $t^\mu$ (gauge), $\mathcal{L}_t t = 1$

Background metric $\mathring{g}_{\mu\nu}$ determined by Cauchy data

**SOMMERFELD DATA IS ACCELERATION AND SHEAR OF $K^\mu$ RELATIVE TO BACKGROUND**

$$q^\mu := \hat{q}N^\mu + \hat{q}^\mu = K^\nu(\nabla_\nu - \mathring{\nabla}_\nu)K^\mu$$

$$\sigma = Q^\mu Q^\nu \hat{\sigma}_{\mu\nu} = Q^\mu Q^\nu(\nabla_\mu - \mathring{\nabla}_\mu)K_\nu$$

**PHYSICAL CONTENT OF BOUNDARY DATA CLARIFIED BY CONSIDERING LINEARIZED PLANE WAVE**

$q^\mu$ related to gauge freedom

$\sigma$ describes the incoming gravitational radiation entering the boundary

Phys. Rev. D 80, 1204043 (2009) Gen. Rel. Gravit. 41, 1909 (2009)

## NUMERICAL APPLICATION

**IMPLEMENTATION IN HARMONIC CODES IS STRAIGHTFORWARD AND ROBUST**

### APPLES WITH APPLES BOUNDARY TESTS

**PITT harmonic code - Babiuc, Kreiss, Szilágyi, Winicour**

**AEI harmonic code - Seiler, Szilágyi, Pollney, Rezzolla**

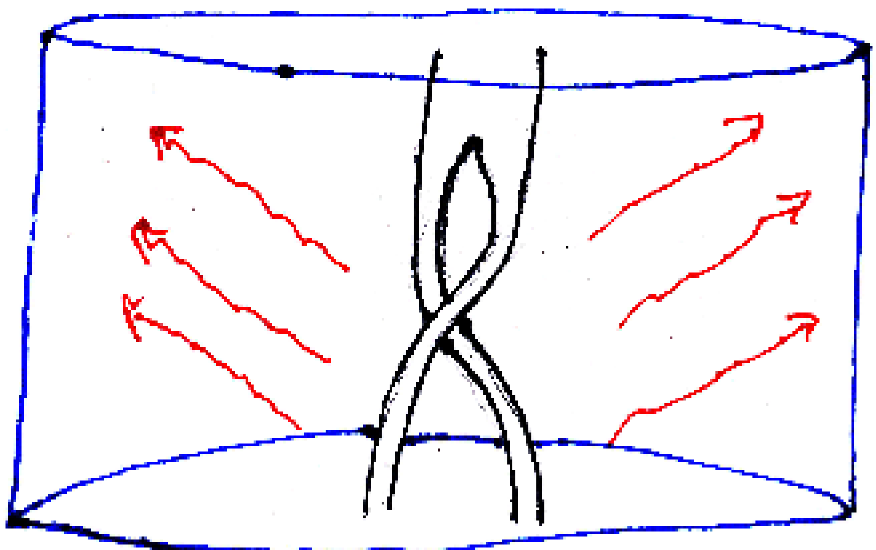
**Caltech spectral harmonic code - Rinne, Lindblom, Scheel**

$\psi_0$ boundary condition - Buchman, Sarbach
Ruiz, Rinne, Sarbach

2nd order Sommerfeld condition

$\psi_0 = K^\alpha \partial_\alpha \sigma + \ldots = Q^\mu Q^\nu (K^\alpha \partial_\alpha)^2 g_{\mu\nu} + \ldots$

### APPLICATION TO OUTER BOUNDARY OF ISOLATED SYSTEM



### SET SOMMERFELD DATA TO ZERO

**Setting $\psi_0 = 0$ gives $O(\frac{1}{R})$ less backreflection from outer boundary than setting $\sigma = 0$**

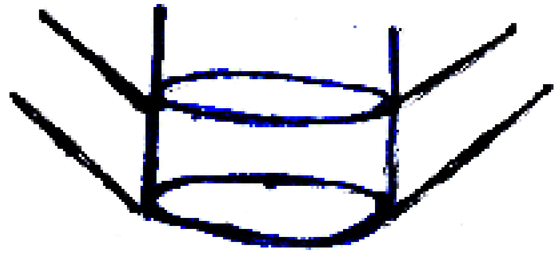
## APPLICATION TO OTHER FORMULATIONS

Geometric nature of Sommerfeld conditions allows formal application to any other metric formulation BUT ...

CHARACTERISTIC CODES - BONDI-SACHS SYSTEM

BASED UPON DIRICHLET BOUNDARY CONDITIONS

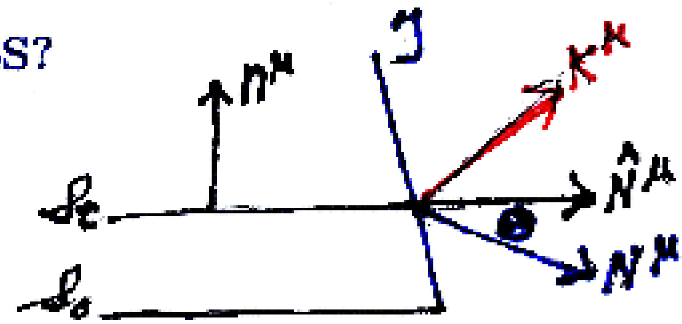
WELL POSEDNESS???



WHAT ABOUT 3+1 CODES?

$$g_{\mu\nu} = -n_\mu n_\nu + h_{\mu\nu}$$
$$= -n_\mu n_\nu + \hat{N}_\mu \hat{N}_\nu + Q_{(\mu}\bar{Q}_{\nu)}$$

so that $K^\rho = e^{-\Theta}(n^\rho + \hat{N}^\rho)$




Sommerfeld data $(\hat{q}^a, \sigma)$ determines boundary values for components of $K^\rho \partial_\rho g_{\mu\nu}$ corresponding to all components of extrinsic curvature $k_{\mu\nu}$ except $Q^\mu \bar{Q}^\nu k_{\mu\nu}$

Remaining piece of Sommerfeld data $\hat{q} = q^\mu N_\mu$ determines the time derivative of the normal component of the shift

$$\beta_N = \beta^\mu N_\mu \qquad t^\mu = \alpha n^\mu + \beta^\mu$$

The harmonic constraint $C^\mu K_\mu$ determines the missing Sommerfeld data for $Q^\mu \bar{Q}^\nu k_{\mu\nu}$

HOW DOES THIS JIVE WITH 3+1 CONSTRAINT PRESERVATION, WELL POSEDNESS AND GAUGE CONDITIONS?

3+1 CODES ONLY EVOLVE 6 COMPONENTS OF EINSTEIN'S EQUATIONS

# CONSTRAINT PRESERVATION

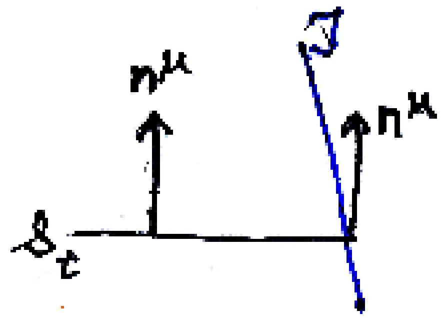
**CONSTRAINTS** $H = G_{\mu\nu} n^\mu n^\nu \qquad P^i = h^{i\nu} n^\gamma G_{\nu\gamma} \qquad x^\mu = (t, x^i)$

**ADM EVOLUTION SYSTEM** $\quad h^\rho_\mu h^\sigma_\nu R_{\rho\sigma} = 0$

**Bianchi identity $\nabla_\nu G^\nu_\mu = 0$ gives symmetric hyperbolic constraint propagation system**

$$n^\mu \partial_\mu H - \partial_j P^j = AH + A_i P^i$$

$$n^\mu \partial_\mu P^i - h^{ij} \partial_j H = B^i H + B^i_j P^j$$




**Only one boundary condition allowed if $\beta_N \leq 0$, i.e boundary moves *inward* relative to Cauchy hypersurfaces**

**All constraints preserved if $H + P^i N_i = 0$ at boundary**

**Via evolution system, this is equivalent to outgoing Raychaudhuri equation**

$$G_{\mu\nu} K^\mu K^\nu = K^\mu \partial_\mu \theta + \frac{1}{2}\theta^2 + \sigma\bar{\sigma} = 0 \qquad \theta = Q^\mu \bar{Q}^\nu \nabla_\mu K_\nu$$

**So constraint preservation enforced by Sommerfeld condition for $\theta$, which supplies boundary values for the missing $Q^\mu \bar{Q}^\nu k_{\mu\nu}$ component of extrinsic curvature**

**The constraint system is exactly what you would like**

**BUT ADM IS CATASTROPHICALLY UNSTABLE**

## BAUMGARTE-SHAPIRO-SHIBATA-NAKAMURA SYSTEM

**EVOLUTION SYSTEM** $h^\rho_\mu h^\sigma_\nu R_{\rho\sigma} - \frac{2}{3}h_{\mu\nu}H = 0$

**Cauchy problem well-posed (Beyer, Sarbach)**

**PROBLEMS WITH BOUNDARY TREATMENT**

**SIGN OF $\beta_N$ DETERMINES ALLOWED NUMBER OF BOUNDARY CONDITIONS**

**FORCES DIRICHLET CONDITION ON $\beta_N$, e.g. $\beta_N = 0$**

**BIANCHI IDENTITY GIVES CONSTRAINT SYSTEM**

$$n^\gamma\partial_\gamma H - \partial_j P^j = AH + A_i P^i$$
$$n^\gamma\partial_\gamma P^i + \frac{1}{3}h^{ij}\partial_j H = B^i H + B^i_j P^j$$

**NOT SYMMETRIC HYPERBOLIC!**

**REMEDY (Nunez, Sarbach): FURTHER MODIFY EVOLUTION SYSTEM BY MIXING IN AUXILIARY CONSTRAINTS $\mathcal{Z}$, WHICH LEADS TO A LARGER SYMMETRIC HYPERBOLIC CONSTRAINT SYSTEM**

**CONSTRAINT SYSTEM THEN IMPLIES INGOING RAYCHAUDHURI EQUATION**

$$G_{\mu\nu}L^\mu L^\nu = \mathcal{Z}$$

**BIZARRE! BUT NEVERTHELESS IT SUPPLIES BOUNDARY VALUE FOR MISSING $Q^\mu \bar{Q}^\nu k_{\mu\nu}$ COMPONENT OF EXTRINSIC CURVATURE**

**THERE MUST BE A BETTER WAY TO DO 3+1**