A Discontinuous Galerkin Method for BSSN-Type Systems

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Introduction

- Baumgarte-Shapiro-Shibata-Nakamura (BSSN) system is a popular formulation of the Einstein equations used for numerical evolutions
- Typical applications include binary black hole simulations
- We work with the Generalized BSSN (GBSSN) system\(^1\)

What are the differences from traditional BSSN?

\(^1\)David Brown, arXiv: 0501092
We may write the full spacetime metric as

\[ ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -(\alpha^2 - \gamma_{ij} \beta^i \beta^j) dt^2 + 2\gamma_{ij} \beta^j dt dx^i + \gamma_{ij} dx^i dx^j, \]

Lapse \( \alpha \), shift \( \beta^i \), and spatial metric \( \gamma_{ij} \)

- **Conformal spatial metric** (\( \chi \)’s weight to be specified)

\[ \gamma_{ij} \equiv \chi^{-1}\tilde{\gamma}_{ij} \]
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Traditional BSSN requires \( \tilde{\gamma} = 1 \), and so \( \chi = \gamma^{-1/3} \) is of weight \(-2/3\)

- Thus the conformal metric is of weight \(-2/3\)
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- Thus the conformal metric is of weight \(-2/3\)

Generalized BSSN introduces the scalar \( \chi = (\tilde{\gamma}/\gamma)^{1/3} \)

- Thus the conformal metric is a usual tensor
- Not necessarily unit determinant
- Must specify how the conformal metric’s determinant evolves

The GBSSN choice leads to... (a very small sampling)
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The GBSSN choice leads to... (a very small sampling)

\[
\mathcal{L}_n \bar{A}_{ij} = \frac{1}{3} \bar{A}_{ij} \mathcal{L}_n \ln \tilde{\gamma} + K \bar{A}_{ij} - 2 \bar{A}_{ik} \bar{A}^k_j + \chi \left( R_{ij} - \frac{1}{\alpha} D_i D_j \alpha \right)^{\text{TF}}
\]
To date BSSN-type codes are based on finite difference methods.

We present a high-order accurate discontinuous Galerkin scheme for GBSSN.

We directly discretize the second order spatial operators. Fewer variables and no extra constraints to worry about.

We will specialize to spherically symmetric solutions with comments towards a 3D solver.
The discontinuous Galerkin method: A hybrid of methods

- **Spectral methods**: approximate solutions by expanding them in a basis
- **Finite element methods**: integrate the residual against a set of test functions
- **Finite volume methods**: elements coupled via FV *numerical* fluxes, when the basis functions are constants dG formally is a FV method

Will develop the dG method in 4 steps, with 1 step per slide
Approximate physical domain $\Omega$ by subdomains $D^k$ such that $\Omega \sim \Omega_h = \bigcup_{k=1}^{K} D^k$

In general the grid is unstructured. We choose lines, triangles, and tetrahedrons for 1D, 2D, and 3D respectively.
DG method: solution (step 2 of 4)

- Local solution expanded in set of basis functions

\[ x \in D^k : \psi_h^k(x, t) = \sum_{i=0}^{N} \psi_h^k(x_i, t) l_i^k(x) \]

- Polynomials span the space of polynomials of degree \( N \) on \( D^k \).

- Global solution is a direct sum of local solutions

\[ \psi_h(x, t) = \bigoplus_{k=1}^{K} \psi_h^k(x, t) \]

- Solutions double valued along point, line, surface.
Consider a model PDE

$$L\Psi = \partial_t \Psi + \partial_x f = 0,$$

where $\Psi$ and $f = f(\Psi)$ are scalars.

Integrate the residual $L\Psi_h$ against all basis functions on $D^k$

$$\int_{D^k} (L\Psi_h) l_i^k(x) dx = 0 \quad \forall i \in [0, N]$$

We still must couple the subdomains $D^k$ to one another...
DG method: numerical flux (step 4 of 4)

- To couple elements first perform IBPs

\[
\int_{D^k} \left( l^k_i \partial_t \psi_h - f(\psi_h) \partial_x l^k_i \right) \, dx = - \int_{\partial D^k} l^k_i \hat{n} \cdot f^*(\psi_h)
\]

where the *numerical* flux is \( f^*(\psi_h) = f^*(\psi^+, \psi^-) \)

- \( \psi^+ \) and \( \psi^- \) are the solutions exterior and interior to subdomain \( D^k \), restricted to the boundary

- **Example**: Central flux \( f^* = \frac{f(\psi^+)+f(\psi^-)}{2} \)

- Passes information between elements, implements boundary conditions, and ensures stability of scheme

- Choice of \( f^* \) is, in general, problem dependent
Remark: The term ‘nodal discontinuous Galerkin’ should now be clear. We seek a global discontinuous solution interpolated at nodal points and demand this solution satisfy a set of integral (Galerkin) conditions.
Final comments

- Timestep with a classical $4^{th}$ order Runge-Kutta
- Robust for hyperbolic equations as we *directly* control the scheme’s stability through a numerical flux choice
- For a smooth enough solution, numerical error decays exponentially with polynomial order $N$
Stable treatment of second order operators
Semi-discrete stability

Generic framework in place, specification of numerical flux remains
Semi-discrete stability

Generic framework in place, specification of numerical flux remains

- We should hope the result is semi-discrete stable
  - Stability after spatial discretization
- Extensive literature on fully first order hyperbolic systems (e.g. Lax-Friedrichs flux)
- Strange terms like $\chi''$, $(\chi')^2$, and $\alpha'\chi'$. What to do?
Second order operators: Key new feature

Consider a model problem ($a \geq 1$ for real speeds)

\[ \partial_t u = u' + av - u^3 \]
\[ \partial_t v = u'' + v' - (u + v)(u')^2 + v^2 u^2, \]

- Techniques used to treat this system used for GBSSN
Second order operators: Key new feature

Consider a model problem \((a \geq 1\) for real speeds)

\[
\begin{align*}
\partial_t u &= u' + av - u^3 \\
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\end{align*}
\]

- Techniques used to treat this system used for GBSSN
- First rewrite as

\[
\begin{align*}
\partial_t u &= Q + av - u^3 \\
\partial_t v &= Q' + v' - (u + v)Q^2 + v^2 u^2 \\
Q &= u' & \text{\textit{Q not evolved}}
\end{align*}
\]
Second order operators: Key new feature

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\end{align*}
\]

- Techniques used to treat this system used for GBSSN
- First rewrite as, and we presently specialize to

\[
\begin{align*}
\partial_t u &= Q + av - u^3 \\
\partial_t v &= Q' + v' - (u + v)Q^2 + v^2 u^2 \\
Q &= u' \quad Q \text{ not evolved}
\end{align*}
\]
Follow the previous discontinuous Galerkin construction

\[
\int_{D^k} l_i^k \partial_t u_h = \int_{D^k} l_i^k (Q_h + av_h)
\]

\[
\int_{D^k} l_i^k \partial_t v_h = -\int_{D^k} l_i^{k'} (Q_h + v_h) + \int_{\partial D^k} l_i^k (Q^* + v^*)
\]

\[
\int_{D^k} l_i^k Q_h = -\int_{D^k} l_i^{k'} u_h + \int_{\partial D^k} l_i^k u^*,
\]

- \(Q_h\) is constructed and substituted, thus we see \(Q\) is not evolved
- The key is specifying a form for \(Q^*\), \(u^*\), and \(v^*\), such that the resulting scheme is stable.
Stability of the continuum system

Notice that the continuum system (with $Q = u'$) with periodic boundary conditions satisfies

$$\frac{1}{2} \partial_t \int_{\Omega} \left[ av^2 + Q^2 \right] = 0.$$

- Mimic this estimate for our dG scheme. We seek...
Stability of the continuum system

Notice that the continuum system (with \( Q = u' \)) with periodic boundary conditions satisfies

\[
\frac{1}{2} \partial_t \int_{\Omega} [a v^2 + Q^2] = 0.
\]

- Mimic this estimate for our dG scheme. We seek...

\[
\frac{1}{2} \partial_t \sum_{k=1}^{k_{\text{max}}} \int_{D^k} (Q_h^2 + a v_h^2) \leq 0
\]
At each subdomain interface

\[
\{\{v_h\}\} = \frac{1}{2} \left(v_{k+1/2}^L + v_{k+1/2}^R\right)
\]

\[
[[v_h]] = v_{k+1/2}^L - v_{k+1/2}^R.
\]

Consider numerical fluxes of the form (No need to diagonalize!)

\[
Q^* = \{\{Q_h\}\} - \frac{\tau Q}{2} [[Q_h]]
\]

\[
v^* = \{\{v_h\}\} - \frac{\tau v}{2} [[v_h]]
\]

\[
u^* = \{\{u_h\}\} - \frac{\tau u}{2} [[u_h]]
\]
Integrate to internal boundaries

\[ \frac{1}{2} \partial_t \sum_{k=1}^{k_{\text{max}}} \int_{D^k} (Q_h^2 + av_h^2) = \sum_{k=1}^{k_{\text{max}}-1} (\text{interface terms})|_{l^{k+1/2}} \]

With our choice of numerical flux each subdomain interface term is

\[ -\frac{a\tau_v}{2} [[v_h]]^2 - \frac{a(\tau_u + \tau_Q)}{2} [[Q_h]] [[v_h]] - \frac{\tau_u}{2} [[Q_h]]^2. \]
Role of penalties

Figure: The left ($\tau_v = 10^{-6}$) and right ($\tau_v = 1 + \sqrt{2}$) plots depict stable choices (determined empirically) of $\tau_u$ and $\tau_Q$ for the linear model system. The stable regions are colored black, but the jagged edges result from the discretization of the ($\tau_u$, $\tau_Q$)-plane.
We work with the spherically symmetric version to demonstrate the general method. Discontinuous Galerkin method directly applies

- Introduce *locally* constructed auxiliary variables
  - For example $Q_\chi = \chi'$
- Solution is a sum over interpolating polynomials
- Integrate against test functions
- We use the same penalty choice as discussed for model problem
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Choices

- Evolution for conformal metric’s determinant $\partial_t \tilde{\gamma} = 0$
  - Used to replace $\mathcal{L}_n \ln \tilde{\gamma}$ throughout system
- $1 + \log$ and Gamma-driver evolution for the lapse and shift (standard choice)
Conformal Kerr-Schild Initial data

Physical metric is

\[ ds^2 = -\alpha^2 dt^2 + (1 + 2M/R)(dR + \beta^R dt)^2 + R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2 \]

the lapse \( \alpha = (1 + 2M/R)^{-1/2} \) and shift \( \beta^R = 2M/(R + 2M) \)

- Conformal metric determinant \( \bar{\gamma} \) is not unity
- Spherically symmetric, analytic, coordinates pass through the horizon
- Inner boundary is outflow, singularity treated by excision
Stability and convergence (with polynomial order $N$)

$\Omega = [.3, 4]$ and $M = 1$, left boundary inside the event horizon

\[ \| \Delta A_{rr} \|_{\infty} \]

\[ \| \Delta K \|_{\infty} \]

\[ \max (\| \Delta B^r \|_{\infty}, \| \Delta \Gamma^r \|_{\infty}) \]
Stability and convergence

- Other fields show similar convergence
- Hamiltonian, momentum, and conformal connection constraints converge
- A variety of $M$ were tested, similarly a variety of domain sizes and locations
- Perturbing all fields leads to a stable scheme

**Main result:** We conclude that the scheme is stable in 1D
Time dependent solutions

We can perturb the initial data

\[ \alpha = \alpha_{KS} + \frac{1}{10} \exp \left( -\frac{1}{2}(R - 50)^2 \right) + \frac{1}{10} \exp \left( -\frac{1}{2}(R - 70)^2 \right) \]
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Future work: Puncture evolutions

If one does not use excision...

- Quantities diverge like powers of $1/r$ near a singularity
- Very successful in finite difference codes

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\textsuperscript{2} Work being carried out with Michael Wagman
Future work: Puncture evolutions

If one does not use excision...

- Quantities diverge like powers of $1/r$ near a singularity
- Very successful in finite difference codes

As we use subdomains, $r = 0$ should be included. Some ideas to try$^2$

- Gauss-Radau points remove the $r = 0$ node, likely a 1D trick
- When our basis functions are constants, dG is a first order finite volume method
- Turducken (smooth stuffing) around the singularity, perhaps repeatedly

These may require singularity tracking (vanishing lapse, distribution of solution’s modes, etc)

$^2$Work being carried out with Michael Wagman
Future work: 3D solver

- Both theory and applications are well-developed for 3D hyperbolic problems
  - Open source projects like HEDGE are available (Andreas’ Sunday talk)
- To-do list: punctures and generalization of our numerical flux choice
- Questions...
  - What elements to use? Cubes? Tetrahedrons? Spheres?
  - Polynomial or tensor product basis?
Potential benefits of a 3D solver

Potentially useful when...

- High-order accuracy needed
- Matter fields are present (including shocks)$^3$
- Different length scales are present, can use local timestepping techniques
  - $\Delta t$ might be different in each subdomain

$^3$David Radice and Luciano Rezzolla, arXiv: 1103.2426
What has been done

- Brief remarks on (G)BSSN system
- Introduced a discontinuous Galerkin method
- Developed a stable and exponentially convergent scheme
  - Key part is treatment of second order spatial operators
- Highlighted potential future work and challenges
QUESTIONS?
Metric in ADM form

We may write the full spacetime metric as

\[ ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -(\alpha^2 - \gamma_{ij} \beta^i \beta^j) dt^2 + 2\gamma_{ij} \beta^j dt dx^i + \gamma_{ij} dx^i dx^j, \]

\( \gamma_{ij} \) is the **spatial metric** for 3D spatial slice

\( \alpha \) is the **lapse**

\( \beta^i \) is the **shift**

**Extrinsic Curvature:**

\[ K_{ij} \equiv -\frac{1}{2} \mathcal{L}_n \gamma_{ij} = -\frac{1}{2} \frac{1}{\alpha} (\partial_t - \mathcal{L}_\beta) \gamma_{ij} \]
(G)BSSN variables

1. **Conformal metric** *(χ’s weight to be specified)*

\[ \gamma_{ij} \equiv \chi^{-1} \bar{\gamma}_{ij} \]
(G)BSSN variables

1. **Conformal metric** (*χ’s weight to be specified*)

\[ \gamma_{ij} \equiv \chi^{-1} \gamma_{ij} \]

2. Decompose \( K_{ij} \) into **trace** \( K \) and **traceless** \( \tilde{A}_{ij} \) parts

\[ K_{ij} = \chi^{-1} \left( \tilde{A}_{ij} + \frac{1}{3} \gamma_{ij} K \right) \]
(G)BSSN variables

1. **Conformal metric** (χ’s weight to be specified)

   \[ \gamma_{ij} \equiv \chi^{-1} \tilde{\gamma}_{ij} \]

2. Decompose \( K_{ij} \) into **trace** \( K \) and **traceless** \( \tilde{A}_{ij} \) parts

   \[ K_{ij} = \chi^{-1} \left( \tilde{A}_{ij} + \frac{1}{3} \tilde{\gamma}_{ij} K \right) \]

3. **Conformal connection functions**

   \[ \bar{\Gamma}^i \equiv \tilde{\gamma}^{jk} \bar{\Gamma}^i_{jk} \]

The variables are \( \chi, \tilde{A}_{ij}, K, \tilde{\gamma}_{ij}, \alpha, \beta^i, \bar{\Gamma}^i \)
Tradional BSSN requires $\bar{\gamma} = 1$, and so $\chi = \gamma^{-1/3}$ is an object of weight $-2/3$.

- The conformal metric and trace-free extrinsic curvature weight $-2/3$
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Generalized BSSN introduces the scalar $\chi = (\tilde{\gamma}/\gamma)^{1/3}$

- The conformal metric and trace-free extrinsic curvature are usual tensors
- Must specify how the conformal metric’s determinant evolves

The GBSSN choice leads to...
GBSSN evolution system (a small sampling)

\[
\mathcal{L}_n\chi = \frac{\chi}{3} (\mathcal{L}_n\ln\bar{\gamma} + 2K),
\]
\[
\mathcal{L}_n\bar{\gamma}_{ij} = \frac{1}{3} \bar{\gamma}_{ij} \mathcal{L}_n\ln\bar{\gamma} - 2\bar{A}_{ij},
\]
\[
\mathcal{L}_nK = -\frac{1}{\alpha} D^2\alpha + \left(\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}K^2\right),
\]
\[
\mathcal{L}_n\bar{A}_{ij} = \frac{1}{3} \bar{A}_{ij} \mathcal{L}_n\ln\bar{\gamma} + K\bar{A}_{ij} - 2\bar{A}_{ik}\bar{A}^k_{\ j} + \chi \left(R_{ij} - \frac{1}{\alpha} D_i D_j \alpha\right)^{TF}
\]

- Evolution for conformal metric's determinant $\partial_t\bar{\gamma} = 0$
- 1+log and Gamma-driver evolution for the lapse and shift
GBSSN evolution system (a small sampling)

\[ \mathcal{L}_n \chi = \frac{\chi}{3} (\mathcal{L}_n \ln \bar{\gamma} + 2K), \]

\[ \mathcal{L}_n \bar{\gamma}_{ij} = \frac{1}{3} \bar{\gamma}_{ij} \mathcal{L}_n \ln \bar{\gamma} - 2 \bar{A}_{ij}, \]

\[ \mathcal{L}_n K = -\frac{1}{\alpha} D^2 \alpha + \left( \bar{A}_{ij} \bar{A}^{ij} + \frac{1}{3} K^2 \right), \]

\[ \mathcal{L}_n \bar{A}_{ij} = \frac{1}{3} \bar{A}_{ij} \mathcal{L}_n \ln \bar{\gamma} + K \bar{A}_{ij} - 2 \bar{A}_{ik} \bar{A}^k_j + \chi \left( R_{ij} - \frac{1}{\alpha} D_i D_j \alpha \right)^{TF}. \]

- Evolution for conformal metric’s determinant \( \partial_t \bar{\gamma} = 0 \)
- 1+log and Gamma-driver evolution for the lapse and shift