Adaptive Mesh Refinement in the Context of a Spectral Evolution Scheme

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Abstract

Spectral numerical methods are known for giving faster convergence than finite difference methods, when evolving smooth quantities. In binary black hole simulations of the SpEC code this exponential convergence is clearly visible.
The same exponential dependence of the numerical error on the grid-resolution will also mean that a linear order mismatch between the grid-structure and the actual data will lead to exponential loss of accuracy.
In my talk I will show the way the Caltech-Cornell-CITA code deals with this, by use of what we call Spectral AMR. In our algorithm we monitor truncation error estimates in various regions of the grid as the simulation proceeds, and adjust the grid as necessary.

truncation error scaling

Consider the problem of numerical evolution of a smooth solution of characteristic length-scale L, using spectral methods. Let N be the number of spectral coefficients.

- For a given L, an increase of the number of coefficients N will result in an exponential convergence of the truncation error.
- Equivalently, for a given N, a decrease of the length-scale L will result in an exponential divergence of the truncation error.

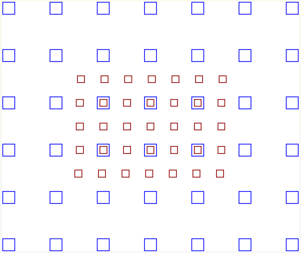
truncation error scaling

- In most problems of interest the length-scales will change dynamically. In order to preserve the truncation error at its desired level, we need to adjust the number of coefficients dynamically as well.
- In a finite-difference code lack of (proper) AMR will have a polynomial cost in the error budget of the simulation.
- In a spectral code this same issue will have an exponential cost.

AMR in finite differencing – in a nutshell

A typical algorithm:

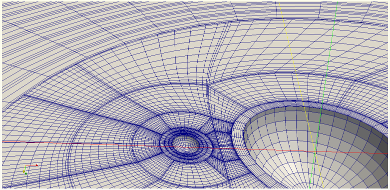
- the evolved quantities are represented on a number of 'refinement levels'
- finer grids move across the coarser ones as needed by the evolution



- the number of refinement levels may be adjusted based on various truncation error estimates

AMR in SpEC – a new approach

- single layer of grid-points



- the finer grid regions (around the excision boundaries) are kept 'at the right spot' via the control system
- truncation error estimates are constructed from the spectrum of the evolved coefficients
- If need be, rather than adding or removing an entire grid, all we do is adjust locally the number of spectral collocation points.

AMR based on projected constraints

- Assume the existence of a derivative-constraint $C_{i\alpha}$
  Here i stands for the derivative index in some x^i frame, α represents all other indices.
- Let X^a be the frame in which the spectral basis functions are defined. (Topological frame.).
  E.g., for a cylindrical grid $X^a = (\rho, \phi, z)$.
- The quantity

$$C_a := \left\| \frac{\partial x^i}{\partial X^a} C_{i\alpha} \right\|$$

  (averaged over α) can provide a measure of the amount of constraint violation along X^a.

AMR based on projected constraints

- One can use the projected constraint, C_a to decide whether a particular grid-axis needs more points.
- This approach gives us a hope to control the constraints themselves, something we always wanted.
- Downside: constraints can grow for reasons other than poor numerical accuracy.

the concept of a power monitor:

Given the spectral representation of a quantity, say

$$f(\theta, \phi) = \sum_{\ell, m} c_{\ell m} Y_{\ell m}(\theta, \phi) \tag{1}$$

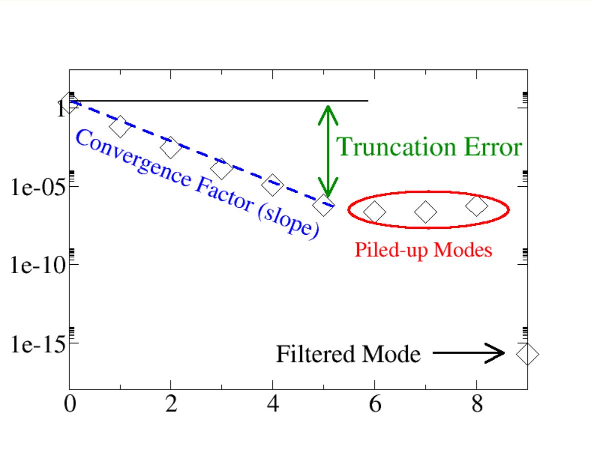
we define the associated *power monitor* as the root-mean-square power in any given ℓ mode, averaged over m, i.e,

$$P_\ell := \sqrt{\frac{1}{2\ell + 1} \sum_m |c_{\ell m}|^2} \tag{2}$$

with similar definitions to other spectral basis (Fourier, etc.)

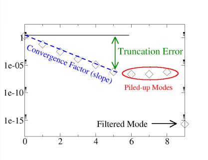
interpretation of power monitor content:

a typical power monitor:



Spectral AMR based on power monitors:

Main idea: try keeping the truncation error at some target value throughout the grid.



Complications:

- Piled-up modes make addition of another point useless.
- Power monitors are averages – the algorithm may fail if certain (e.g., tensor-) components are very different from others.

physical quantities with controlled accuracy

- one can construct accuracy measures of quantities of interest:
  - properties of the apparent horizon
    - mass, spin
    - metric on the surface
    - coordinate shape
  - gravitational wave content
    - power monitor of Ψ_4 in the wave-zone
    - power monitor of the waves at $\mathcal{I}^+$
- the truncation error requirements for the metric can be adjusted until quantities of interest have their accuracy in the desired range
  - We have code adjusting the grid based on AH.
  - AMR based on wave content is still on the to-do list
  - ideally one would want to combine these

A sample AMR run

Illustration: AMR applied to a generic binary black hole system:

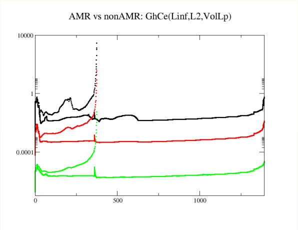
$$m_1/m_2 \approx 3, \quad a_1/m_1 = 0.3402, \quad a_2/m_2 = 0.405$$

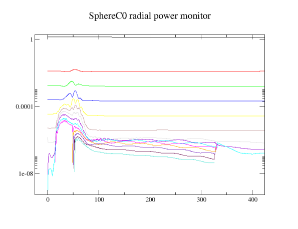
with generic spin orientations.

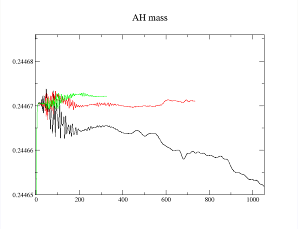
A sample AMR run

$$
\begin{aligned}
m_1/m_2 &\approx 3, \\
a_1/m_1 &= 0.3402, \\
a_2/m_2 &= 0.405,
\end{aligned}
$$

generic spin orientations







current issues:

- thus far Spectral AMR has been a human-time saver but not a CPU-time efficiency boost
- changes in the grid extents are a "shock" to the system – could work on this by time-dependent filtering
- do not have a robust algorithm for splitting/joining subdomains
- the source of "pile-up" modes is not always clear; removing them is difficult

With all of his, Spectral AMR has become an essential element for non-trivial BBH mergers in SpEC.